LOGARITHMIC BUNDLES AND LINE ARRANGEMENTS, AN APPROACH VIA THE STANDARD CONSTRUCTION

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Abstract. We propose an approach to study logarithmic sheaves $\mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}})$ associated with hyperplane arrangements $\mathcal{A}$ on the projective space $\mathbb{P}^n$, based on projective duality, direct image functors and vector bundles methods.

We focus on free line arrangements admitting a point with high multiplicity, or having low exponents, proving Terao's conjecture in this range.

Introduction

Let $\boldsymbol{k}$ be a field, and let $\mathcal{A} = (H_1, \ldots, H_m)$ be a hyperplane arrangement in $\mathbb{P}^n = \mathbb{P}^n_{\boldsymbol{k}}$, namely the H_i's are distinct hyperplanes of $\mathbb{P}^n$. The module of logarithmic derivations along the hyperplane arrangement divisor $D_{\mathcal{A}} = H_1 \cup \cdots \cup H_m$, and its sheaf-theoretic counterpart $\mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}})$ (Saito's sheaf of logarithmic vector fields) play a prominent role in the study of $\mathcal{A}$; let us only mention [Ter81, Sch00].

One main issue in the theory of arrangements is to what extent the sheaf $\mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}})$ depends on the combinatorial type of $\mathcal{A}$, defined as the isomorphism type of the lattice $L_{\mathcal{A}}$ of intersections of hyperplanes in $\mathcal{A}$. This lattice is partially ordered by reverse inclusion, and is equipped with a rank function given by codimension (cf. [OT92]). An important conjecture of Terao (reported in [OT92]) asserts that if $\mathcal{A}$ and $\mathcal{A}'$ have the same combinatorial type, and $\mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}})$ splits as a direct sum of line bundles (i.e. $\mathcal{A}$ is *free*), the same should happen to $\mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}'})$.

In this paper we study the sheaf $\mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}})$ relating it to the finite collection Z of points in the dual space $\check{\mathbb{P}}^n$ associated with $\mathcal{A}$ (we write $\mathcal{A} = \mathcal{A}_Z$ when $Z = \{z_1, \ldots, z_m\}$ satisfies $H_i = H_{z_i}$ for all i, where $H_z \subset \mathbb{P}^n$ denotes the hyperplane corresponding to a point $z \in \check{\mathbb{P}}^n$). Our first result is that $\mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}_Z})$ is obtained via the so-called standard construction from the ideal sheaf $\mathcal{I}_Z(1)$. More precisely, denoting by $\mathbb{F}$ the incidence variety $\mathbb{F} = \{(x, y) \in \mathbb{P}^n \times \check{\mathbb{P}}^n \mid x \in H_y\}$ and by p and q the projections onto $\mathbb{P}^n$ and $\check{\mathbb{P}}^n$, Theorem 1 states that:

$$\mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}_Z}) \simeq p_*(q^*(\mathcal{I}_Z(1))).$$

On the projective plane, we push this a bit further in two directions, namely to higher direct images and to higher rank bundles. This is the content of Theorem 2, where we prove that $\boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z(d)))$ is supported on points of multiplicity $d+2$ of $\mathcal{A}_Z$ and that $p_*(q^*(\mathcal{I}_Z(d)))$ is a vector bundle of rank $d+1$ and $c_1 = \binom{d+1}{2} - m$. For $d \geq 2$, this vector bundle corresponds to the derivations of higher order with poles along $D_{\mathcal{A}_Z}$; it will be studied in more detail in a forthcoming paper.

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Next we make use of the dual picture to show how to obtain special derivations i.e., sections of $\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}_Z})$ from points of high multiplicity in $\mathcal{A}_Z$. By this observation, we show that a line arrangement $\mathcal{A}_Z$ with a point of multiplicity k is free with exponents $(k, k+r)$ if and only if $c_2(\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}_Z})) = k(k+r)$, see Theorem 3. Here, by definition, $\mathcal{A}_Z$ free with exponents $(k, k+r)$ means that $\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}_Z}) \simeq \mathcal{O}_{\mathbb{P}^2}(-k) \oplus \mathcal{O}_{\mathbb{P}^2}(-r-k)$, and we write Chern classes on $\mathbb{P}^n$ as integers, with obvious meaning. Note that the second Chern class is a very weak invariant of the combinatorial type of $\mathcal{A}_Z$. For real arrangements, using a theorem of Ungar, one can push this criterion to points of multiplicity $k-1$, under the assumption that $k \leq 3r+5$, see Theorem 5.

Further on, we study the restriction of $\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}_Z})$ on a line H in $\mathbb{P}^2$. This bundle splits as $\mathcal{O}_H(-a) \oplus \mathcal{O}_H(-b)$ for some $a \leq b$. In Theorem 4 we use the blow-up of the dual plane $\check{\mathbb{P}}^2$ to show that, if H is general enough, then a is the minimal integer d such that there exists a curve of degree $d+1$ in the dual plane $\check{\mathbb{P}}^2$ passing through Z and having multiplicity d at a general point of $\check{\mathbb{P}}^2$. If the line H is not general, then d and a depend on H, and we show that $a-d$ is between zero and the number of triple points (counted with multiplicity) lying of H.

Finally, we show that Terao's conjecture holds for free line arrangements with exponents $(k, k+r)$ in the range $1 \leq k \leq 5$, $r \geq 0$. This is the content of Theorem 6, whose proof relies on Hirzebruch's inequality. As a corollary, we show that freeness is a combinatorial property for up to 12 lines (Corollary 6.3).

The paper is organised as follows. In the next section we set up the main correspondence between ideal sheaves of points in $\check{\mathbb{P}}^n$ and the sheaf of logarithmic derivations on $\mathbb{P}^n$. Section 2 contains the description of higher direct images and higher rank bundles. Section 3 is devoted to line arrangements having a point of high multiplicity. In Section 4 we show how to relate the number d_Z and the generic splitting of the sheaf of logarithmic derivations of $\mathcal{A}_Z$. In Section 5 we outline the relation of our method with the technique of deletion of one line from an arrangement, with a focus on freeness. In Section 6 we develop the above mentioned refinement for real arrangements and we prove Terao's conjecture when the lowest exponent is at most 5.

Notation. We denote the Chern classes of a coherent sheaf E on $\mathbb{P}^n$ as integers: the i^{th} Chern class will be a multiple H^i, where H is the hyperplane class. We denote by $\mathcal{I}_{X/Y}$ the ideal sheaf of a subscheme X of a scheme Y, and we suppress the notation $/Y$ when it is clear from the context). We write ω_X for the dualizing sheaf of a closed subscheme X of $\mathbb{P}^n$. The residue field at a point $x \in X$ will be denoted by $\boldsymbol{k}_x$.

Given a finite set of points Z in a projective space, we say that a line L is a *h-secant line to* Z if $|L \cap Z| \geq h$. We add the adjective *strict* if we require equality.

1. Duality and logarithmic vector fields

Let $\boldsymbol{k}$ be a field. Consider $\mathbb{P}^n = \mathbb{P}^n_{\boldsymbol{k}}$, and let $Z = \{z_1, \ldots, z_m\}$ be a finite collection of points in the dual space $\check{\mathbb{P}}^n$. Each point $y \in \check{\mathbb{P}}^n$ corresponds to a hyperplane H_y in $\mathbb{P}^n$ (and likewise we associate with $x \in \mathbb{P}^n$ a hyperplane of $\check{\mathbb{P}}^n$, denoted by L_x). So, with Z is associated the hyperplane arrangement $\mathcal{A}_Z = (H_{z_1}, \ldots, H_{z_m})$. The hyperplane arrangement divisor $D_{\mathcal{A}_Z}$ is defined as $D_{\mathcal{A}_Z} = \cup_{i=1}^m H_{z_i}$. Let f_i be a linear form defining H_{z_i} and $f = \Pi_{i=1}^m f_i$ be an equation of $D_{\mathcal{A}_Z}$.

Saito's sheaf of logarithmic vector fields $\mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}_Z})$ (see [Sai80]) is the sheafification of the module of logarithmic derivations associated with the divisor $D_{\mathcal{A}_Z}$, mod out by the Euler derivation. We will often abbreviate $\mathcal{T}_Z = \mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}_Z})$.

Our first result shows how to obtain $\mathcal{T}_Z$ from the ideal sheaf $\mathcal{I}_Z$ of Z in $\check{\mathbb{P}}^n$. Consider the flag variety:

$$\mathbb{F} = \{(x, y) \in \mathbb{P}^n \times \check{\mathbb{P}}^n \mid x \in H_y\},$$

and the projections p and q of $\mathbb{F}$ onto $\mathbb{P}^n$ and $\check{\mathbb{P}}^n$. It is well-known that $\mathbb{F} \simeq \mathbb{P}(\mathcal{T}_{\mathbb{P}^n}(-1))$.

Theorem 1. *There is a natural isomorphism of sheaves of $\mathcal{O}_{\mathbb{P}^n}$-modules:*

$$\mathcal{T}_Z \simeq p_*(q^*(\mathcal{I}_Z(1))).$$

Proof. This is somehow implicit in [FMV13, FV12]), but we give here a simplified proof.

First of all, let H be a hyperplane in $\mathbb{P}^n$. We have $\mathcal{T}_{\mathbb{P}^n}(-1)|_H \simeq \mathcal{T}_H(-1) \oplus \mathcal{O}_H$. So, $\mathrm{Hom}_H(\mathcal{T}_{\mathbb{P}^n}(-1), \mathcal{O}_H) \simeq \boldsymbol{k}$ since $\mathrm{Hom}_H(\mathcal{T}_H(-1), \mathcal{O}_H) = 0$. Therefore, there is a non-zero map $\mathcal{T}_{\mathbb{P}^n}(-1) \to \mathcal{O}_H$ unique up to a non-zero scalar, and for any choice of this scalar we have an exact sequence:

$$0 \to \mathcal{T}_{\mathbb{P}^n}(-\log H) \to \mathcal{T}_{\mathbb{P}^n}(-1) \to \mathcal{O}_H \to 0.$$

One easily sees by the Euler sequence that:

$$\mathcal{T}_{\mathbb{P}^n}(-\log H) \simeq \mathcal{O}_{\mathbb{P}^n}^n. \tag{1}$$

Letting H vary in $\mathcal{A} = \mathcal{A}_Z$, we get a map $\alpha : \mathcal{T}_{\mathbb{P}^n}(-1) \to \bigoplus_{H \in \mathcal{A}} \mathcal{O}_H$, uniquely determined by the choice of one non-zero scalar α_H for each $H \in \mathcal{A}$. For any choice of these scalars, we get the same kernel. Indeed, given $\alpha = (\alpha_H)_{H \in \mathcal{A}}$ and $\alpha' = (\alpha'_H)_{H \in \mathcal{A}}$, the automorphism of $\bigoplus_{H \in \mathcal{A}} \mathcal{O}_H$ defined by $(\frac{\alpha'_H}{\alpha_H})_{H \in \mathcal{A}}$ induces an isomorphism of $\ker(\alpha)$ onto $\ker(\alpha')$. We have thus an exact sequence:

$$0 \to \bigcap_{H \in \mathcal{A}} \mathcal{T}_{\mathbb{P}^n}(-\log H) \to \mathcal{T}_{\mathbb{P}^n}(-1) \to \bigoplus_{H \in \mathcal{A}} \mathcal{O}_H$$

Now, taking the quotient by the Euler derivation and sheafifying, [OT92, Proposition 4.8] implies:

$$\bigcap_{H \in \mathcal{A}} \mathcal{T}_{\mathbb{P}^n}(-\log H) \simeq \mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}}).$$

Let us now look at the dual side. Consider the natural exact sequence:

$$0 \to \mathcal{I}_Z(1) \to \mathcal{O}_{\check{\mathbb{P}}^n}(1) \to \mathcal{O}_Z(1) \to 0. \tag{2}$$

We apply $p_* \circ q^*$ to this sequence, and we note that, since $\mathbb{F} = \mathbb{P}(\mathcal{T}_{\mathbb{P}^n}(-1))$, by [Har77, Chapter III, Exercise 8.4] we get $p_*(q^*(\mathcal{O}_{\check{\mathbb{P}}^n}(1))) \simeq \mathcal{T}_{\mathbb{P}^n}(-1)$. Then, we get a long exact sequence:

$$0 \to p_*(q^*(\mathcal{I}_Z(1))) \to \mathcal{T}_{\mathbb{P}^n}(-1) \to p_*(q^*(\mathcal{O}_Z(1))), \tag{3}$$

Observe that, for any $t \in \mathbb{Z}$, there is a natural isomorphisms:

$$p_*(q^*(\mathcal{O}_Z(t))) \simeq \bigoplus_{z \in Z} \mathcal{O}_{H_z}. \tag{4}$$

To see this, first recall that $\mathcal{O}_Z \simeq \mathcal{O}_Z(t)$ for all t since Z has finite length. Further, $p_*(q^*(\mathcal{O}_Z(t)))$ can be seen simply as $p_*(\mathcal{O}_{q^{-1}(Z)})$ and since $q^{-1}(Z)$ is the disjoint union of the $\{H_z \mid z \in Z\}$, and clearly $p_*(\mathcal{O}_{H_z}) \simeq \mathcal{O}_{H_z}$, we get the desired isomorphism.

Then, (3) gives a map $\gamma : \mathcal{T}_{\mathbb{P}^n}(-1) \to \bigoplus_{z \in Z} \mathcal{O}_{H_z}$, which is defined by the choice of one constant γ_z for each $z \in Z$. We claim that none of these constants is zero. Indeed, restricting to one $z \in Z$, the sequence (2) becomes:

$$0 \to \mathcal{I}_z(1) \to \mathcal{O}_{\check{\mathbb{P}}^n}(1) \to \mathcal{O}_z(1) \to 0.$$

so the sequence (3) for one $z \in Z$ is:

$$0 \to p_*(q^*(\mathcal{I}_z(1))) \to \mathcal{T}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{H_z} \to 0.$$

Indeed, if the rightmost map was zero, then we would have $p_*(q^*(\mathcal{I}_z(1))) \simeq \mathcal{T}_{\mathbb{P}^n}(-1)$, and this would give $\mathrm{H}^0(\check{\mathbb{P}}^n, \mathcal{I}_z(1)) \simeq \mathrm{H}^0(\mathbb{P}^n, p_*(q^*(\mathcal{I}_z(1)))) \simeq \mathrm{H}^0(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}(-1))$ (see [Har77, Chapter II, Section 5]), while we know that these spaces have different dimensions (n and $n+1$). So the constant γ_z is non-zero hence $\ker(\alpha) \simeq \ker(\gamma)$. Summing up, we get:

$$\mathcal{T}_Z = \mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}}) \simeq \ker(\alpha) \simeq \ker(\gamma) \simeq p_*(q^*(\mathcal{I}_z(1))).$$

This finishes the proof. $\square$

2. Multiplicities, Chern classes, and higher direct images

From now on, we work on the projective plane $\mathbb{P}^2 = \mathbb{P}^2_k$. Given a point $x \in \mathbb{P}^2$, we write $\langle x^i \rangle$ for the $(i-1)^{\text{th}}$ infinitesimal neighborhood of x in $\mathbb{P}^2$. This is the subscheme cut in $\mathbb{P}^2$ by the i^{th} power of the ideal defining x. It has length $\binom{i+1}{2}$ and $c_2 = -\binom{i+1}{2}$.

Theorem 2. *Let $m \geq 1$ and $d \geq 0$ be integers, and $Z \subset \check{\mathbb{P}}^2$ be a set of m distinct points. Then the sheaf $p_*(q^*(\mathcal{I}_Z(d)))$ is a vector bundle of rank $d+1$ with $c_1 = \binom{d+1}{2} - m$ on $\mathbb{P}^2$, and we have:*

$$\boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z(d))) \simeq \bigoplus_{|L_x \cap Z| = i+d+1} \omega_{\langle x^i \rangle}. \tag{5}$$

Proof. We prove first the last assertion, concerning the first higher direct image.

First of all, we carry out some basic facts on the dualizing sheaf of infinitesimal neighborhoods. Let $x \in \mathbb{P}^2$. The dualizing sheaf $\omega_{\langle x^i \rangle}$ is defined as $\mathcal{E}xt^2_{\mathbb{P}^2}(\mathcal{O}_{\langle x^i \rangle}, \mathcal{O}_{\mathbb{P}^2}(-3))$ and is isomorphic to $\mathcal{E}xt^1_{\mathbb{P}^2}(\mathcal{I}_x^i, \mathcal{O}_{\mathbb{P}^2}(-3))$, see [Har77, Chapter III, section 7]). Since this sheaf is of finite length, it does not change under twisting by $\mathcal{O}_{\mathbb{P}^2}(t)$. We have the resolution of $\mathcal{I}_x(1)$:

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{T}_{\mathbb{P}^2}(-1) \to \mathcal{I}_x(1) \to 0.$$

Taking the i^{th} symmetric power of the map $\mathcal{T}_{\mathbb{P}^2}(-1) \to \mathcal{I}_x(1)$, we get:

$$0 \to \mathrm{Sym}^{i-1}(\mathcal{T}_{\mathbb{P}^2}(-1)) \to \mathrm{Sym}^i(\mathcal{T}_{\mathbb{P}^2}(-1)) \to \mathcal{I}_x^i(i) \to 0.$$

Dualizing this sequence we get:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-i) \to \mathrm{Sym}^i(\Omega_{\mathbb{P}^2}(1)) \to \mathrm{Sym}^{i-1}(\Omega_{\mathbb{P}^2}(1)) \to \omega_{\langle x^i \rangle} \to 0. \tag{6}$$

Next, we note that the sheaf $q^*(\mathcal{I}_Z(d))$ is flat with respect to the map p over $\mathbb{P}^2$. To check this, let $x \in \mathbb{P}^2$ and observe that the fibre of p over x is L_x. Denote by $\{y_1, \ldots, y_h\}$ the points of $L_x \cap Z$. Then, we have:

$$q^*(\mathcal{I}_Z(d))|_{L_x} \simeq \mathcal{O}_{L_x}(d-h) \oplus \bigoplus_{i=1,\ldots,h} \mathcal{O}_{y_i}. \tag{7}$$

Then, the Hilbert polynomial in the variable t of $q^*(\mathcal{I}_Z(d))|_{L_x}$ is $t+d+1$. This does not depend on x, so $q^*(\mathcal{I}_Z(d))$ is flat over $\mathbb{P}^2$ by [Har77, Chapter III, Theorem 9.9].

Now, we claim that $\boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z(d)))$ is supported at the points $x \in \mathbb{P}^2$ such that $|Z \cap L_x| \geq d+2$. To see this, let again $h = |Z \cap L_x|$ and note that $\boldsymbol{R}^2 p_*(q^*(\mathcal{I}_Z(d))) = 0$, because the relative dimension of p is 1. Then, since $q^*(\mathcal{I}_Z(d))$ is flat over $\mathbb{P}^2$, by base change (see [Har77, Chapter III, Theorem 12.11]) we have:

$$\boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z(d))) \otimes \boldsymbol{k}_x \simeq \mathrm{H}^1(L_x, \mathcal{I}_{Z \cap L_x}(d)) \simeq \mathrm{H}^1(L_x, \mathcal{O}_{L_x}(d-h)),$$

and this is non-zero if and only if $h \geq d+2$.

We have thus proved that $\boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z(d)))$ is supported at a subscheme of finite length of $\mathbb{P}^2$. We now look at each point x in its support, to check the local structure. To this purpose, we can assume that Z consists of $h \geq d+2$ points lying in the line L_x. We have then the exact sequence:

$$0 \to \mathcal{O}_{\check{\mathbb{P}}^2}(d-1) \to \mathcal{I}_Z(d) \to \mathcal{O}_{L_x}(d-h) \to 0.$$

Since $\boldsymbol{R}^1 p_*(q^*(\mathcal{O}_{\check{\mathbb{P}}^2}(d-1))) = 0$ (see [Har77, Chapter III, Exercise 8.4]), we have:

$$\boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z(d))) \simeq \boldsymbol{R}^1 p_*(q^*(\mathcal{O}_{L_x}(d-h))).$$

To compute the right-hand-side, we use the exact sequence:

$$0 \to \mathcal{O}_{\check{\mathbb{P}}^2}(d-h-1) \to \mathcal{O}_{\check{\mathbb{P}}^2}(d-h) \to \mathcal{O}_{L_x}(d-h) \to 0.$$

Since $\mathbb{F} = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2}(-1))$, again [Har77, Chapter III, Exercise 8.4] says that, applying $p_* \circ q^*$ to this exact sequence, we obtain (6) for $i = h-d-1$. This proves

$$\boldsymbol{R}^1 p_*(q^*(\mathcal{O}_{L_x}(d-h))) \simeq \omega_{\langle x^{h-d-1}\rangle}. \tag{8}$$

Letting x vary in the support of $\boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z(d)))$, we finally get formula (5).

Let us now prove the statements regarding the direct image. The incidence variety $\mathbb{F}$ is a divisor of bi-degree $(1,1)$ in $\mathbb{P}^2 \times \check{\mathbb{P}}^2$, i.e., we have an exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^2 \times \check{\mathbb{P}}^2}(-1,-1) \to \mathcal{O}_{\mathbb{P}^2 \times \check{\mathbb{P}}^2} \to \mathcal{O}_{\mathbb{F}} \to 0.$$

Denote by pr_i the projections onto the two factors of $\mathbb{P}^2 \times \check{\mathbb{P}}^2$. Tensoring the above sequence by $pr_2^*(\mathcal{I}_Z(d))$ we get:

$$0 \to pr_1^*(\mathcal{O}_{\mathbb{P}^2}(-1)) \otimes pr_2^*(\mathcal{I}_Z(d-1)) \to pr_2^*(\mathcal{I}_Z(d)) \to q^*(\mathcal{I}_Z(d)) \to 0.$$

Taking direct image by pr_1, we get a long exact sequence:

$$\begin{aligned} 0 \to & \mathrm{H}^0(\check{\mathbb{P}}^2, \mathcal{I}_Z(d-1)) \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathrm{H}^0(\check{\mathbb{P}}^2, \mathcal{I}_Z(d)) \otimes \mathcal{O}_{\mathbb{P}^2} \to p_*(q^*(\mathcal{I}_Z(d))) \to \\ \to & \mathrm{H}^1(\check{\mathbb{P}}^2, \mathcal{I}_Z(d-1)) \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathrm{H}^1(\check{\mathbb{P}}^2, \mathcal{I}_Z(d)) \otimes \mathcal{O}_{\mathbb{P}^2} \to \boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z(d))) \to 0. \end{aligned} \tag{9}$$

We also have $\mathrm{H}^2(\check{\mathbb{P}}^2, \mathcal{I}_Z(t)) = 0$ for all $t \geq -1$. Since $\boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z(d)))$ is supported on a scheme of finite length, it does not contribute to the computation of c_1. We get:

$$c_1(p_*(q^*(\mathcal{I}_Z(d)))) = \chi(\mathcal{I}_Z(d-1)) = \chi(\mathcal{O}_{\check{\mathbb{P}}^2}(d-1)) - \chi(\mathcal{O}_Z) = \binom{d+1}{2} - m.$$

The same argument gives that the rank of $p_*(q^*(\mathcal{I}_Z(d))$. Finally, the sheaf $p_*(q^*(\mathcal{I}_Z(d))$ is locally free by [Har77, Chapter III, Corollary 12.9], since $\mathrm{H}^0(L_x, \mathcal{I}_Z(d)|_{L_x})$ is constant on x in view of (7). □

Given an arrangement $\mathcal{A}$ of m lines in $\mathbb{P}^2$, according to the previous theorem and to Theorem 1, we have $c_1(\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}})) = 1-m$, while $c_2(\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}}))$ depends on the number $b_{\mathcal{A},h}$ of points of multiplicity $h \geq 3$ of $D_{\mathcal{A}}$ (we will also call them the points of multiplicity h "of $\mathcal{A}$"), according to the following lemma. Note that $c_2(\omega_{\langle x^i\rangle}) = -\binom{i+1}{2}$.

Lemma 2.1. *We have the relations:*

$$\textstyle\sum_{j\geq 2} \binom{j}{2} b_{\mathcal{A},j} = \binom{m}{2}, \tag{10}$$

$$\textstyle\sum_{j\geq 2} \binom{j}{2} b_{\mathcal{A},j+1} = \binom{m-1}{2} - c_2(\mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}})). \tag{11}$$

Proof. Let Z be the set of points in $\check{\mathbb{P}}^2$ corresponding to $\mathcal{A}$ so that $\mathcal{A} = \mathcal{A}_Z$ and $\mathcal{T}_Z = \mathcal{T}_{\mathbb{P}^n}(-\log D_{\mathcal{A}})$. First note that, for any h, $b_{\mathcal{A},h}$ is the number of strict h-secant lines to Z, i.e. the number of points $x \in \mathbb{P}^2$ such that $|L_x \cap Z| = h$. We get:

$$\sum_{j\geq 2} \binom{j}{2} b_{\mathcal{A},j} = -c_2(\boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z))),$$
$$\sum_{j\geq 2} \binom{j}{2} b_{\mathcal{A},j+1} = -c_2(\boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z(1)))).$$

Both formulas are obtained by the previous theorem. Setting $d = 0$ in the exact sequence (9), we get:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-m) \to \mathcal{O}_{\mathbb{P}^2}(-1)^m \to \mathcal{O}_{\mathbb{P}^2}^m \to \boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z)) \to 0.$$

Computing c_2, we get formula (10). Setting $d = 1$ in the exact sequence (9) and computing Chern classes, we get formula (11). $\square$

3. Line arrangements with a point of high multiplicity

Here we study freeness of line arrangements in $\mathbb{P}^2$ that admit a point having high multiplicity with respect to the exponents. Recall that a line arrangement $\mathcal{A}$ is *free with exponents* (a, b) if $\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}}) \simeq \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b)$. Of course, this implies that $c_2(\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}})) = ab$.

Theorem 3. *Let $k \geq 1$, $r \geq 0$ be integers, set $m = 2k + r + 1$, and consider a line arrangement $\mathcal{A}$ of m lines with a point of multiplicity h with $k \leq h \leq k + r + 1$. Then $\mathcal{A}$ is free with exponents $(k, k + r)$ if and only if $c_2(\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}})) = k(k + r)$.*

Remark 3.1. In the above setting, it turns out that if $h \geq k + r + 2$, then $\mathcal{A}$ cannot be free with exponents $(k, k + r)$, see the last statement of Corollary 4.3.

To prove the theorem, we will need the following lemma.

Lemma 3.2. *Let E be a rank-2 vector bundle on $\mathbb{P}^2$ and assume $c_1(E) = -r$ for some $r \geq 0$ and $c_2(E) = 0$. Then, the following are equivalent:*

i) the bundle E splits as $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-r)$,
ii) we have $\mathrm{H}^0(\mathbb{P}^2, E(-1)) = 0$,
iii) there is a line H of $\mathbb{P}^2$ such that $E|_H \simeq \mathcal{O}_H \oplus \mathcal{O}_H(-r)$.

For any line H of $\mathbb{P}^2$ we have $E|_H \simeq \mathcal{O}_H(s) \oplus \mathcal{O}_H(-r - s)$, for some integer $s \geq 0$.

Proof. Condition (i) clearly implies (ii). The equivalence of (i) and (iii) is proved in [EF80]. So it only remains to show that (ii) implies (i), which we will now do.

Let t be the smallest integer such that $\mathrm{H}^0(\mathbb{P}^2, E(t)) \neq 0$. By (ii) we know $t \geq 0$. Also, it is well-known (cf. [Bar77, Lemmas 1 and 2]) that any non-zero global section s of $E(t)$ vanishes along a subscheme W of $\mathbb{P}^2$ of codimension ≥ 2 and of length:

$$c_2(E(t)) = t(t - r) \geq 0. \tag{12}$$

We have an exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^2} \xrightarrow{s} E(t) \to \mathcal{I}_W(2t - r) \to 0.$$

So $t = 0$ would imply $X = \emptyset$ hence $\mathcal{I}_W(2t - r) \simeq \mathcal{O}_{\mathbb{P}^2}(-r)$ and E splits as $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-r)$ since $\mathrm{Ext}^1_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(-r), \mathcal{O}_{\mathbb{P}^2}) = 0$.

Then, it remains to rule out the case $t > 0$. Hence, we assume $t > 0$ i.e. $\mathrm{H}^0(\mathbb{P}^2, E) = 0$, and we look for a contradiction. By Riemann-Roch, the Euler characteristic $\chi(E)$ is positive, hence $\mathrm{H}^2(\mathbb{P}^2, E) \neq 0$, so $\mathrm{H}^0(\mathbb{P}^2, E(r-3)) \neq 0$ by Serre duality, indeed $E^* \simeq E(r)$. Therefore $t > 0$ implies $t \leq r-3$. But by (12), $t > 0$ implies $t \geq r$, a contradiction.

Let us now prove the last statement. Given a line H of $\mathbb{P}^2$, we have $E|_H \simeq \mathcal{O}_H(s) \oplus \mathcal{O}_H(-r-s)$ for some integer s, and we have to check that s is non-negative. Let us assume $s < 0$, and show that this leads to a contradiction. First, note that we may assume $s > -r$, for otherwise posing $s' = -r-s$ we have $s' \geq 0$ and we still have $E|_H \simeq \mathcal{O}_H(s') \oplus \mathcal{O}_H(-r-s')$.

Now, in case $-r < s < 0$, we have an unstable section, namely $\mathrm{H}^0(\mathbb{P}^2, E(-1)) \neq 0$ since E does not decompose as $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-r)$ (by the part we have already proved of this lemma). For all integers t, the exact sequence of restriction of $E(t)$ to H reads:

$$0 \to E(t-1) \to E(t) \to \mathcal{O}_H(t+s) \oplus \mathcal{O}_H(t-r-s) \to 0.$$

So $-r < s < 0$ implies $\mathrm{H}^0(\mathbb{P}^2, E(t-1)) \simeq \mathrm{H}^0(\mathbb{P}^2, E(t))$ for all $t \leq 0$, and this space is zero for $t \ll 0$. But this contradicts $\mathrm{H}^0(\mathbb{P}^2, E(-1)) \neq 0$. □

We will now prove our theorem.

Proof of Theorem 3. One direction if obvious. What we have to prove is that the condition on Chern classes is sufficient, so we assume $c_2(\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}})) = k(k+r)$. Let Z be the set of m points of $\check{\mathbb{P}}^2$ corresponding to $\mathcal{A}$, so that $\mathcal{A} = \mathcal{A}_Z$ and $\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}}) = \mathcal{T}_Z$.

Since $\mathcal{A}$ has a point x_0 of multiplicity $h \geq k$, on the dual side there is a line $L = L_{x_0} \subset \check{\mathbb{P}}^2$ that contains h points of Z (i.e. L is a strict h-secant to Z), and leaves out the remaining $m-h$ points of Z. Set $Z' = Z \setminus L$. Let $g = 0$ be an equation of L in $\check{\mathbb{P}}^2$.

Restricting the ideal sheaf $\mathcal{I}_Z$ to L we get the ideal sheaf of h points in $\mathbb{P}^1$, i.e. $\mathcal{O}_L(-h)$. This gives an exact sequence:

$$0 \to \mathcal{I}_{Z'} \xrightarrow{g} \mathcal{I}_Z(1) \to \mathcal{O}_L(1-h) \to 0. \tag{13}$$

We apply $p_* \circ q^*$ to this exact sequence. By Theorem 2 we have $p_*(q^*(\mathcal{O}_L(1-h))) \simeq \mathcal{O}_{\mathbb{P}^2}(1-h)$ and, setting $d = 1$ in (8), we get $\boldsymbol{R}^1 p_*(q^*(\mathcal{O}_L(1-h))) \simeq \omega_{\langle x_0^{h-2} \rangle}$.

Therefore $p_* \circ q^*$ of (13) gives:

$$\begin{aligned} 0 \to &\mathcal{O}_{\mathbb{P}^2}(h-m) \to \mathcal{T}_Z \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}^2}(1-h) \to \\ \to &\boldsymbol{R}^1 p_*(q^*(\mathcal{I}_{Z'})) \to \boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z(1))) \to \omega_{\langle x_0^{h-2} \rangle} \to 0. \end{aligned} \tag{14}$$

We know that $\boldsymbol{R}^1 p_*(q^*(\mathcal{I}_{Z'}))$ is supported at points x such that $|L_x \cap Z'| \geq 2$. The image of the map δ above is then a sub-sheaf of $\mathcal{O}_{\mathbb{P}^2}(1-h)$, whose first Chern class is $1-h$ since all the sheaves in the second row of (14) are supported in codimension ≥ 2. This means that $\mathrm{Im}(\delta) \simeq \mathcal{I}_\Gamma(1-h)$, for some finite length subscheme $\Gamma \subset \mathbb{P}^2$, and we have:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(h-m) \to \mathcal{T}_Z \to \mathcal{I}_\Gamma(1-h) \to 0. \tag{15}$$

Looking at (14), we see that the subscheme Γ parametrizes (non-strict) bisecant lines to Z' that meet L away from Z.

We apply now Lemma 3.2. If, by contradiction, the bundle $\mathcal{T}_Z \otimes \mathcal{O}_{\mathbb{P}^2}(k)$ did not split as $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-r)$, then we would have an unstable section, namely:

$$\mathrm{H}^0(\mathbb{P}^2, \mathcal{T}_Z \otimes \mathcal{O}_{\mathbb{P}^2}(k-1)) \neq 0.$$

Note that the assumption $h \leq k+r+1 = m-k$ gives $h+k-m-1 < 0$, so we have the vanishing $\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(h+k-m-1)) = 0$. So, from (15), twisted by $\mathcal{O}_{\mathbb{P}^2}(k-1)$, we deduce:

$$\mathrm{H}^0(\mathbb{P}^2, \mathcal{I}_\Gamma(k-h)) \neq 0,$$

hence clearly $k \geq h$, which implies $h = k$ so $\mathrm{H}^0(\mathbb{P}^2, \mathcal{I}_\Gamma) \neq 0$. This says that Γ is empty. But computing Chern classes via Theorem 2 in (15) twisted by $\mathcal{O}_{\mathbb{P}^2}(k-1)$ (and still with $h = k$) shows that Γ has length $c_2(\mathcal{I}_\Gamma) = r + 1$, a contradiction. $\square$

As an example of application of this description of $\mathcal{T}_Z$ as direct image, let us mention the following well-known result.

Proposition 3.3. *Let $\mathcal{A}$ be an arrangement of m lines, $k \geq 0$ be an integer, x be a point of multiplicity $k+1$ of $D_\mathcal{A}$. Set $\mathcal{A}' = \mathcal{A} \setminus \{H \in \mathcal{A} \mid x \in H\}$. Then the following are equivalent:*

i) the arrangement $\mathcal{A}$ is free with exponents $(k, m-k-1)$;
ii) any point of multiplicity $h \geq 2$ in $D_{\mathcal{A}'}$ has multiplicity $h+1$ in $D_\mathcal{A}$.

Proof. Again, we let Z be the set of m points of $\check{\mathbb{P}}^2$ corresponding to $\mathcal{A}$, so that $\mathcal{A} = \mathcal{A}_Z$ and $\mathcal{T}_{\mathbb{P}^2}(-\log D_\mathcal{A}) = \mathcal{T}_Z$. Since $\mathcal{A}$ has a point x of multiplicity $k+1$, on the dual side there is a line $L = L_x \subset \check{\mathbb{P}}^2$ that contains $k+1$ points of Z (i.e. L is a strict $k+1$-secant to Z), and leaves out the remaining $m-k-1$ points of Z. Set $Z' = Z \setminus L$. We have $\mathcal{A}' = \mathcal{A}_{Z'}$. We can then rewrite (15) as:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(k-m+1) \to \mathcal{T}_Z \to \mathcal{I}_\Gamma(-k) \to 0, \tag{16}$$

where, as above, Γ parametrizes bisecant lines to Z that meet L away from Z. Now, (ii) is equivalent to the fact that there is no such bisecant, i.e. to the fact that Γ is empty, so that (16) becomes:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(k-m+1) \to \mathcal{T}_Z \to \mathcal{O}_{\mathbb{P}^2}(-k) \to 0.$$

This is clearly equivalent to (i). $\square$

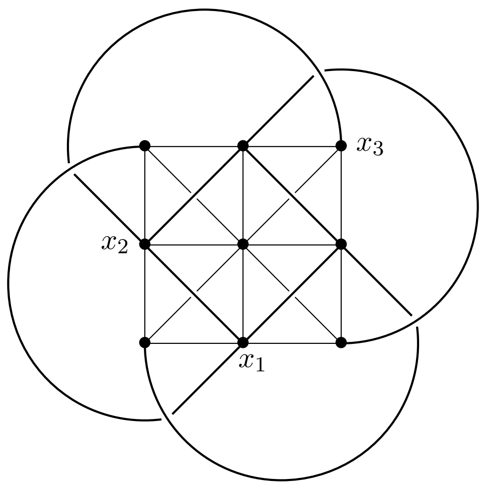

Figure 1. Hesse arrangement

Example 3.4. Theorem 3 gives a quick way to show that an arrangement $\mathcal{A}$ having the combinatorial type of the Hesse arrangement of the 12 lines passing through the 9 inflection points of a smooth complex plane cubic C is free with exponents $(4, 7)$. The pencil of cubics given by C and the Hessian of C contains 4 cubics which are unions of 3 lines, and $\mathcal{A}$ is the union of these 12 lines. In this case, any line through two inflection points passes through a third. In the picture below, the 9 points are displayed, together with the 12 lines; the circles should be though of as continuation of the diagonal lines not passing through the center, for instance x_1, x_2, x_3 are aligned.

In particular $c_2(\mathcal{T}_{\mathbb{P}^2}(-\log D_{\mathcal{A}})) = 55 - 27 = 28$ so the existence of quadruple points implies that $\mathcal{A}$ is free with exponents $(4, 7)$.

4. Blow up of the dual plane and restriction to lines

Given a line arrangement $\mathcal{A}_Z$ corresponding to a set of m points Z in $\check{\mathbb{P}}^n$, and given a line H_y corresponding to a point $y \in \check{\mathbb{P}}^2 \setminus Z$, we study here the restricted logarithmic bundle $(\mathcal{T}_Z)|_{H_y}$ in terms of the curves in the dual $\check{\mathbb{P}}^2$ containing Z and singular at y. To do this, we outline an application to our situation of the so-called *standard construction*, see [OSS80, Chapter I, Section 3.1]. We consider the blow-up $\tilde{\mathbb{P}}$ of $\check{\mathbb{P}}^2$ at a point $y \in \check{\mathbb{P}}^2 \setminus Z$. We denote by $\tilde{p}$ and $\tilde{q}$ the induced projections from $\tilde{\mathbb{P}}$ to H_y and to $\check{\mathbb{P}}^2$. Note that $\tilde{p}$ is a $\mathbb{P}^1$-bundle over H_y (in particular $\tilde{p}$ is flat). We consider the sheaf $\tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1)))$ defined over the projective line H_y. This is a vector bundle of rank 2 on H_y, and as such decomposes as a direct sum of lines bundles. We will compare this bundle to $(\mathcal{T}_Z)|_{H_y}$. Note that $\mathcal{T}_Z$ restricts to H_y as a direct sum of line bundles, so we write:

$$(\mathcal{T}_Z)|_{H_y} \simeq \mathcal{O}_{H_y}(-a_y) \oplus \mathcal{O}_{H_y}(-b_y), \tag{17}$$

for some integers $a_y \leq b_y$ with $a_y + b_y = m - 1$.

Definition 4.1. Let $y \in \check{\mathbb{P}}^2$ and let Z be a finite set of points of $\check{\mathbb{P}}^2$. We define $d_{Z,y}$ as the smallest positive integer d such that *there is a curve in* $\check{\mathbb{P}}^2$ *of degree* $d+1$ *passing through* Z *and having multiplicity* d *at* y. Equivalently, $d_{Z,y}$ is the smallest integer d such that:

$$\mathrm{H}^0(\check{\mathbb{P}}^2, \mathcal{I}_y^d \otimes \mathcal{I}_Z(d+1)) \neq 0.$$

We define d_Z as $\max_{y \in \check{\mathbb{P}}^2} d_{Z,y}$.

We also define $t_{Z,y}$ as the number of lines in $\check{\mathbb{P}}^2$ through y that are trisecant to Z. In other words, thinking of the dual side, we put:

$$t_{Z,y} = \sum_{x \in H_y \cap \mathcal{S}_Z} (\mathrm{mult}(D_{\mathcal{A}_Z}, x) - 2),$$

where $\mathcal{S}_Z$ is the singular locus of $D_{\mathcal{A}_Z}$ and $\mathrm{mult}(D_{\mathcal{A}_Z}, x)$ is the multiplicity of x as point of $\mathcal{A}_Z$, i.e., the number of lines of $\mathcal{A}_Z$ through x.

Theorem 4. *Let Z be a finite set of points of $\check{\mathbb{P}}^2$ and $y \in \check{\mathbb{P}}^2 \setminus Z$. Then we have:*

$$d_{Z,y} \leq a_y \leq d_{Z,y} + t_{Z,y}.$$

In particular, if y lies on no trisecant line to Z, we get $a_y = d_{Z,y}$.

Moreover we have the inequality $d_{Z,y} \leq m - 1 - d_{Z,y} - t_{Z,y}$ and an isomorphism:

$$\tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1))) \simeq \mathcal{O}_{H_y}(-d_{Z,y}) \oplus \mathcal{O}_{H_y}(d_{Z,y} + t_{Z,y} + 1 - m).$$

Proof. The sheaf $\tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1)))$ si a vector bundle of rank 2 on H_y, hence $\tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1))) \simeq \mathcal{O}_{H_y}(-d) \oplus \mathcal{O}_{H_y}(-e)$, for some $d \leq e$. Our first task will be to prove $d = d_{Z,y}$. The decomposition $\tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1))) \simeq \mathcal{O}_{H_y}(-d) \oplus \mathcal{O}_{H_y}(-e)$ gives an injective map $\mathcal{O}_{H_y}(-d) \to \tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1)))$. Pulling back to $\tilde{\mathbb{P}}$, since $\tilde{p}$ is flat and $\tilde{p}^*(\mathcal{O}_{H_y}) \simeq \mathcal{O}_{\tilde{\mathbb{P}}}$, we get an injection:

$$\mathcal{O}_{\tilde{\mathbb{P}}} \hookrightarrow \tilde{q}^*(\mathcal{I}_Z(1)) \otimes \tilde{p}^*(\mathcal{O}_{H_y}(d)).$$

We now push down to $\check{\mathbb{P}}^2$. Since $\tilde{\mathbb{P}}$ is the blow-up of y, i.e. the projectivization of $\mathcal{I}_y(1)$, and $\tilde{\mathbb{P}}$ maps to H_y via the linear system $|\mathcal{I}_y(1)|$ of lines through y, we get for all $t \geq 0$ an isomorphism $\tilde{q}_*(\tilde{p}^*(\mathcal{O}_{H_y}(t))) \simeq \mathcal{I}_y^t(t)$. Then applying $\tilde{p}_*$ to the previous display and using the projection formula (see [Har77, Chapter II, Exercise 5.1]) we get a map:

$$\mathcal{O}_{\check{\mathbb{P}}^2} \hookrightarrow \mathcal{I}_Z(1) \otimes \tilde{q}_*(\tilde{p}^*(\mathcal{O}_{H_y}(d))) \simeq \mathcal{I}_Z(1) \otimes \mathcal{I}_y^d(d).$$

Recall that $d_{Z,y}$ is the smallest integer t such that $\mathrm{H}^0(\check{\mathbb{P}}^2, \mathcal{I}_Z(1) \otimes \mathcal{I}_y^t(t)) \neq 0$. Then, we get $d_{Z,y} \leq d$.

Now, we prove $d_{Z,y} \geq d$. We have to check $\mathrm{H}^0(\check{\mathbb{P}}^2, \mathcal{I}_y^{d-1} \otimes \mathcal{I}_Z(d)) = 0$. Assume by contradiction that this space contains a non-zero element. This would give a section $\mathcal{O}_{\check{\mathbb{P}}^2} \hookrightarrow \mathcal{I}_Z(d)$ that vanishes with multiplicity $d-1$ at y. Pull this map back to $\tilde{\mathbb{P}}$. The resulting section vanishes with multiplicity $d-1$ along the exceptional divisor $\mathbb{E} = \tilde{q}^{-1}(y)$. In other words we have a map $\mathcal{O}_{\tilde{\mathbb{P}}^2} \hookrightarrow \mathcal{O}_{\tilde{\mathbb{P}}}((d-1)\mathbb{E}) \otimes \tilde{q}^*(\mathcal{I}_Z(d))$. Note that $\mathbb{E}$ lies in the linear system $|q^*(\mathcal{O}_{\check{\mathbb{P}}^2}(1)) \otimes \tilde{p}^*(\mathcal{O}_{H_y}(-1))|$. We can thus clear $d-1$ times the divisor $\mathbb{E}$ from our section to get a map:

$$\mathcal{O}_{\tilde{\mathbb{P}}} \hookrightarrow \tilde{q}^*(\mathcal{I}_Z(1)) \otimes \tilde{p}^*(\mathcal{O}_{H_y}(d-1)).$$

Hence, by pushing forward to H_y via $\tilde{p}$ and using projection formula, we get:

$$\mathcal{O}_{H_y}(1-d) \hookrightarrow \tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1))).$$

This is incompatible with $\tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1))) \simeq \mathcal{O}_{H_y}(-d) \oplus \mathcal{O}_{H_y}(-e)$ with $d \leq e$. We have thus proved $d = d_{Z,y}$.

We wish now to show the inequalities $d_{Z,y} \leq a_y \leq d_{Z,y} + t_{Z,y}$. To do this, we compare the vector bundles $(\mathcal{T}_Z)|_{H_y} \simeq \mathcal{O}_{H_y}(-a_y) \oplus \mathcal{O}_{H_y}(-b_y)$ and $\tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1)))$ over H_y, by exhibiting an exact sequence where they both appear. Recall that $\tilde{\mathbb{P}} = p^{-1}(H_y)$, where p is the projection map from the flag $\mathbb{F}$ to $\mathbb{P}^2$. Therefore we have an exact sequence:

$$0 \to p^*(\mathcal{O}_{\mathbb{P}^2}(-1)) \to \mathcal{O}_{\mathbb{F}} \to \mathcal{O}_{\tilde{\mathbb{P}}} \to 0,$$

with $\mathcal{O}_{\tilde{\mathbb{P}}} \simeq p^*(\mathcal{O}_{H_y})$.

Tensoring the above exact sequence by $q^*(\mathcal{I}_Z(1))$, since $y \notin Z$ and $\tilde{q}$ is flat away from y, we get an exact sequence:

$$0 \to p^*(\mathcal{O}_{\mathbb{P}^2}(-1)) \otimes q^*(\mathcal{I}_Z(1)) \to q^*(\mathcal{I}_Z(1)) \to \tilde{q}^*(\mathcal{I}_Z(1)) \to 0.$$

Taking direct image by p we get the long exact sequence:

$$0 \to \mathcal{T}_Z(-1) \xrightarrow{f} \mathcal{T}_Z \to \tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1))) \to \tag{18}$$

$$\to \boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z(1))) \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{f} \boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z(1))) \to \boldsymbol{R}^1 \tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1))) \to 0, \tag{19}$$

where here f is an equation of H_y in $\mathbb{P}^2$.

Let us note that $\boldsymbol{R}^1 \tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1)))$ is supported at trisecant lines to Z through y. Indeed, by the same reason as in Theorem 2, the sheaf $\tilde{q}^*(\mathcal{I}_Z(1))$ is flat over $\mathbb{P}^2$ with respect to the map $\tilde{p}$, so by base change over $x \in H_y$, we have:

$$\boldsymbol{R}^1 \tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1))) \otimes \boldsymbol{k}_x \simeq \mathrm{H}^1(L_x, \mathcal{I}_{Z \cap L_x}(1)),$$

and this space vanishes if and only if $|Z \cap L_x| \leq 2$.

Now we want to show that $\boldsymbol{R}^1 \tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1)))$ has length $t_{Z,y}$. From (19), we see that $\boldsymbol{R}^1 \tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1)))$ is the restriction of $\boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z(1)))$ to H_y. Next, we show that for any point $x \in H_y$ and any $i \geq 1$, we have $\omega_{\langle x^i \rangle} \otimes \mathcal{O}_{H_y} \simeq \boldsymbol{k}_x^i$. To do this, we write a slightly easier resolution of $\omega_{\langle x^i \rangle}$ than (6). Assume that the point x is defined by the vanishing of the linear forms f, g (recall that f defines H_y). Then we have:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-i-1) \to \mathcal{O}_{\mathbb{P}^2}(-1)^{i+1} \xrightarrow{(g\delta_{k,j}+f\delta_{k+1,j})_{k,j}} \mathcal{O}_{\mathbb{P}^2}^i \to \omega_{\langle x^i \rangle} \to 0.$$

Restricting the above sequence to the line H_y (i.e., setting $f = 0$), we extract a resolution:

$$0 \to \mathcal{O}_{H_y}(-1)^i \xrightarrow{(g\delta_{k,j})_{k,j}} \mathcal{O}_{H_y}^i \to \omega_{\langle x^i \rangle} \otimes \mathcal{O}_{H_y} \to 0.$$

We get $\omega_{\langle x^i \rangle} \otimes \mathcal{O}_{H_y} \simeq \boldsymbol{k}_x^i$. Varying x among the points such that $|L_x \cap Z| \geq 3$, and using the isomorphism (5) of Theorem 2, we get that $\boldsymbol{R}^1\tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1)))$ has length $t_{Z,y}$.

Let $\boldsymbol{\tau}_{Z,y}$ be the kernel of the map f among the higher direct image sheaves appearing in the sequence (19). We observe that $\boldsymbol{\tau}_{Z,y}$ has the same length as $\boldsymbol{R}^1\tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1)))$, i.e. $t_{Z,y}$. The sequence (18) becomes a diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T}_Z(-1) \arrow[r, "f"] & \mathcal{T}_Z \arrow[rr] \arrow[dr] & & \tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1))) \arrow[r] & \boldsymbol{\tau}_{Z,y} \arrow[r] & 0 \\
& & & (\mathcal{T}_Z)|_{H_y} \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

The desired exact sequence is thus:

$$0 \to \mathcal{O}_{H_y}(-a_y) \oplus \mathcal{O}_{H_y}(-b_y) \to \mathcal{O}_{H_y}(-d_{Z,y}) \oplus \mathcal{O}_{H_y}(-e) \to \boldsymbol{\tau}_{Z,y} \to 0. \tag{20}$$

Twisting by $\mathcal{O}_{H_y}(a_y)$ we see that $\mathrm{H}^0(H_y, \mathcal{O}_{H_y}(a_y - d_{Z,y})) \neq 0$ so $d_{Z,y} \leq a_y$. On the other hand if we twist the above sequence by $\mathcal{O}_{H_y}(b_y - 1)$, we get no H^1 in the leftmost term (because $a_y \leq b_y$) nor in the rightmost one (because $\boldsymbol{\tau}_{Z,y}$ has finite length), hence neither in the middle one. So $b_y - e \geq 0$. Since $t_{Z,y} = a_y - d_{Z,y} + b_y - e$, we get $a_y \geq t_{Z,y} + d_{Z,y}$. This proves the desired inequalities. Note that, if $t_{Z,y} = 0$, then $\boldsymbol{\tau}_{Z,y} = 0$ and we have:

$$\tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1))) \simeq (\mathcal{T}_Z)|_{H_y}.$$

Finally, by the sequence (20), since we have proved that $\boldsymbol{\tau}_{Z,y}$ has length $t_{Z,y}$, we get:

$$c_1(\tilde{p}_*(\tilde{q}^*(\mathcal{I}_Z(1)))) - c_1((\mathcal{T}_Z)|_{H_y}) = t_{Z,y}.$$

Then, $e = m - 1 - d_{Z,y} - t_{Z,y}$. The proof is now finished. □

Corollary 4.2. *Let Z be a finite set of $m \geq 1$ points of $\check{\mathbb{P}}^2$.*

i) The following conditions are equivalent:
- *(a) Z is contained in a line;*
- *(b) there is $y \in \check{\mathbb{P}}^2 \setminus Z$ such that $d_{Z,y} = 0$;*
- *(c) $d_Z = 0$;*
- *(d) the arrangement $\mathcal{A}_Z$ is free with exponents $(0, m-1)$.*

ii) Assume $m \geq 5$. Then the following conditions are equivalent.
- *(a) there is a line containing all but one points of Z;*
- *(b) $d_Z = 1$;*
- *(c) the arrangement $\mathcal{A}_Z$ is free with exponents $(1, m-2)$.*

Proof. Let us look at (i). Given a point $y \in \check{\mathbb{P}}^2 \setminus Z$, by definition we have $d_{Z,y} = 0$ if and only $\mathrm{H}^0(\check{\mathbb{P}}^2, \mathcal{I}_Z(1)) \neq 0$ since $\mathcal{I}_y^0 \simeq \mathcal{O}_{\check{\mathbb{P}}^2}$, i.e., if and only if Z is contained in a line. This condition does not depend on y, so $d_{Z,y} = d_Z$. So the first three conditions are clearly equivalent.

Le us check the equivalence with the fourth condition. Note that, by the isomorphism (1), the equivalence of (id) and (ia) is clear for $m = 1$. So we assume $m \geq 2$, hence $\mathrm{h}^0(\check{\mathbb{P}}^2, \mathcal{I}_Z(1)) \leq 1$. Assume that Z is contained in a line, and write down (9) for $d = 1$. Note that the cokernel of the map $\mathrm{H}^0(\check{\mathbb{P}}^2, \mathcal{I}_Z(1)) \otimes \mathcal{O}_{\mathbb{P}^2} \to \mathcal{T}_Z$ induced by (9) is a torsion-free sheaf of rank 1 and $c_1 = 1 - m$ hence isomorphic to $\mathcal{I}_\Gamma(1-m)$ where Γ is a subscheme of $\mathbb{P}^2$ of length $c_2(\mathcal{T}_Z)$. By Lemma 2.1 we easily get that $c_2(\mathcal{T}_Z) = 0$, so Γ is actually empty, and we get (id).

Conversely, if we have (id) then we have $\mathrm{H}^0(\mathbb{P}^2, \mathcal{T}_Z) \neq 0$. By (9) a global section of $\mathcal{T}_Z$ factorizes through $\mathrm{H}^0(\check{\mathbb{P}}^2, \mathcal{I}_Z(1)) \otimes \mathcal{O}_{\mathbb{P}^2}$ because there is no non-zero global section of $\mathrm{H}^1(\check{\mathbb{P}}^2, \mathcal{I}_Z) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$. Then we have (ia).

Let us now turn to (ii). We check first that (iib) implies (iia). If $d_Z = 1$, then for a general point $y \in \check{\mathbb{P}}^2$ we have $\mathrm{H}^0(\check{\mathbb{P}}^2, \mathcal{I}_{Z \cup y}(2)) \neq 0$. Hence Z is contained in at least two distinct conics C_1 and C_2 (once given C_1, choose C_2 through Z and $x \notin C_1$). Since $m \geq 5$, C_1 and C_2 have a common component, say L. Let $C_1 = L \cup L_1$ and $C_2 = L \cup L_2$ with $L_1 \neq L_2$. Moreover, the point $L_1 \cap L_2$ lies in $Z \setminus L$, for otherwise we would have $Z \subset L$ so $d_Z = 0$, a contradiction.

Of course we have (iic) $\Rightarrow$ (iib), so it only remains to see that (iia) implies (iic). Let $z \in Z$ be the point not aligned with the other points of Z. Through z, there is a strict 2-secant line L to Z. Set $Z' = Z \setminus L$. Setting $\mathcal{A} = \mathcal{A}_Z$ and $\mathcal{A}' = \mathcal{A}_{Z'}$ in Proposition 3.3, we get the result. □

Putting together the previous theorem and Lemma 3.2, we get the following result, somehow related to Yoshinaga's theorem, cf. [Yos04].

Corollary 4.3. *Let $k \geq 1$, $r \geq 0$ be integers, set $m = 2k + r + 1$, and consider a line arrangement $\mathcal{A}_Z$ associated with m points Z in $\check{\mathbb{P}}^2$ having $c_2(\mathcal{T}_Z) = k(k+r)$. Then the following are equivalent:*

i) the arrangement $\mathcal{A}_Z$ is free with exponents $(k, k+r)$;
ii) there is a line $H = H_y$ in $\mathbb{P}^2$ such that $(\mathcal{T}_Z)|_{H_y} \simeq \mathcal{O}_{H_y}(-k) \oplus \mathcal{O}_{H_y}(-k-r)$;
iii) there is a point $y \in \check{\mathbb{P}}^2 \setminus Z$ lying in no trisecant line to Z, such that $d_{Z,y} = k$;
iv) $d_Z = k$.

In particular, if Z has a h-secant line with $h \geq k + r + 2$, then $\mathcal{A}_Z$ cannot be free.

Proof. First of all, we check that d_Z is attained at a general point of $\check{\mathbb{P}}^2$. So let y_0 be a point with $d_{Z,y_0} = d_Z$ and let us show that there is a Zariski open neighborhood U of y_0 such that $d_{Z,y} = d_Z$ for all $y \in U$. Note that for any given integer d, the function $f_d : y \mapsto \dim_{\boldsymbol{k}} \mathrm{H}^0(\check{\mathbb{P}}^2, \mathcal{I}_y^d \otimes \mathcal{I}_Z(d+1))$ is upper semicontinuous. Further, $f_{d_Z - 1}(y_0) = 0$. So there is a Zariski open neighborhood U of y_0 such that, for all $y \in U$, we have $f_{d_Z-1}(y) = 0$. Then $d_{Z,y} \geq d_Z$ for all $y \in U$. On the other hand, of course $d_{Z,y} \leq d_Z$, so we have equality.

Let us now look at the equivalence of our statements. Clearly, (i) implies (ii). We write $(\mathcal{T}_Z)|_{H_y} \simeq \mathscr{O}_{H_y}(-a_y) \oplus \mathscr{O}_{H_y}(-b_y)$ with $a_y \leq b_y$. By Theorem 4, (ii) implies (iii) since $a_y = d_{Z,y}$ as soon as y lies in no trisecant to Z. Moreover, (iii) $\Rightarrow$ (i). Indeed, by (iii), we have $a_y = k$ again because y lies in no trisecant to Z. Then, by Lemma 3.2 we get (i). We have shown the equivalence of the first three conditions.

Moreover, (iv) obviously implies (iii). Finally, (i) implies (iv) since by (i) we have $a_y = k$ for all $y \in \check{\mathbb{P}}^2$, so for a general point of $y \in \check{\mathbb{P}}^2$ we get $d_{Z,y} = k$, hence $d_Z = k$.

For the last statement we argue as follows. First, since $c_2(\mathcal{T}_Z) = k(k+r)$, the arrangement $\mathcal{A}_Z$ cannot be free with other exponents than $(k, k+r)$. We now check that it cannot be free with these exponents either. Suppose that Z has a strict h-secant line L with $h \geq k+r+2$ and take a general point y in $\check{\mathbb{P}}^2$. Then we have a curve in $\check{\mathbb{P}}^2$ of degree $m-h+1 \leq k$ through Z and having multiplicity $m-h$ at y, namely the union of L and of the $m-h$ lines joining y with the $m-h$ points of $Z \setminus L$. Therefore $d_{Z,y} \leq k-1$ so $\mathcal{A}_Z$ is not free by the previous statements of this corollary. □

The way to use it will frequently be by contradiction in the follwing sense. Let $\mathcal{A}_Z$ be an arrangement with $c_2(\mathcal{T}_Z) = k(k+r)$. Then we have equivalence of the following conditions.

- $\mathcal{A}_Z$ is not free;
- for any line $H \subset \mathbb{P}^2$, there is $t > 0$ such that $(\mathcal{T}_Z)|_H \simeq \mathcal{O}_H(t-k) \oplus \mathcal{O}_H(-k-r-t)$;
- there is $t > 0$ such that $\mathrm{H}^0(\mathbb{P}^2, \mathcal{T}_Z(k-t)) \neq 0$.

In particular, if $\mathcal{A}_Z$ is not free, taking t maximal such that $\mathrm{H}^0(\mathbb{P}^2, \mathcal{T}_Z(k-t)) \neq 0$, and choosing a non-zero element of $\mathrm{H}^0(\mathbb{P}^2, \mathcal{T}_Z(k-t))$ we get an exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(t-k) \to \mathcal{T}_Z \to \mathcal{I}_W(-k-r-t) \to 0, \tag{21}$$

where $W \subset \mathbb{P}^2$ is a subscheme of finite length. In fact, a computation of c_2 shows that the length of W is $t(t+r)$. We will sometimes call a non-zero element of $\mathrm{H}^0(\mathbb{P}^2, \mathcal{T}_Z(k-t))$ an *unstable section*. Also we point out that, for any line H meeting W, we have :

$$\mathrm{H}^0(H, \mathcal{T}_Z(k-t-1)|_H) \neq 0. \tag{22}$$

Example 4.4 (Dual Hesse configuration)**.** Let $\boldsymbol{k} = \mathbb{C}$, and let $Z \subset \check{\mathbb{P}}^2$ consist of 9 points such that any bisecant line is a strict trisecant. Then $\mathcal{A}_Z$ is free with exponents $(4,4)$.

Indeed, first note that by (10) of Lemma 2.1 we get $b_{\mathcal{A}_Z,3} = 12$ and by (11) we obtain $c_2(\mathcal{T}_Z) = 16$. We choose two triangles $T = L_1L_2L_3$ and $T' = L_1'L_2'L_3'$ containing Z with $L_i \neq L_j'$ for all i,j so that $Z = T \cap T'$, and any cubic containing Z belongs to the pencil generated by T and T'. By Bertini's theorem, the general element of this pencil is smooth away from Z.

Now, if $\mathcal{A}_Z$ is not free, then by the above discussion we have an unstable section $\mathrm{H}^0(\mathbb{P}^2, \mathcal{T}_Z(4-t))$ with $t > 0$, hence a sequence of the form (21), with $W \neq \emptyset$. Assume $t = 1$ (for higher t the argument is similar). Choosing one point w of W, in view of (22) we get for any point y of L_w, the bound $a_y \leq 2$, in the notation of (17). Hence $d_{Z,y} \leq 2$ by Theorem 4. In other words, for any such y there is a cubic containing Z, singular at y. This contradicts the fact that a general cubic through Z is smooth away from Z.

The dual Hesse arrangement is given by the 9 lines corresponding to the inflection points of a smooth cubic curve C in $\check{\mathbb{P}}^2$. Its combinatorial type is the one described above. Indeed, in the Hesse pencil of C and its Hessian there are 4 triangles, which are precisely the 12 strict trisecant lines to the 9 inflection points. The 9 points appear in Figure 3.

5. Sub-arrangement obtained by deletion

A classical and useful technique in the theory of arrangements consists in considering arrangements obtained from an arrangement $\mathcal{A}$ by adding a hyperplane out of $\mathcal{A}$, or deleting one of $\mathcal{A}$, or restricting $\mathcal{A}$ to a hyperplane of $\mathcal{A}$ (see [OT92] for a comprehensive treatment). Here we provide a different approach to this technique and outline some considerations on freeness of line arrangements based on our approach. Most of the results contained in this section are certainly known to experts, and can be proved with the classical techniques of deletion.

5.1. **Deletion of a point and triple points along a line.** Let Z be a finite set of points in $\check{\mathbb{P}}^2$ and let $z \in Z$. Set $Z' = Z \setminus \{z\}$. We say that $\mathcal{A}_{Z'}$ is a sub-arrangement of $\mathcal{A}_Z$, obtained by *deletion* of z. We have the exact sequence:

$$0 \to \mathcal{I}_Z \to \mathcal{I}_{Z'} \to \mathcal{O}_z \to 0.$$

Applying $p_* \circ q^*$ to this sequence, we get:

$$0 \to \mathcal{T}_Z \to \mathcal{T}_{Z'} \xrightarrow{\beta_0} \mathcal{O}_{H_z} \xrightarrow{\beta_1} \boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z(1))) \xrightarrow{\beta_2} \boldsymbol{R}^1 p_*(q^*(\mathcal{I}_{Z'}(1))) \to 0.$$

Proposition 5.1. *We have a short exact sequence:*

$$0 \to \mathcal{T}_Z \to \mathcal{T}_{Z'} \to \mathcal{O}_{H_z}(-t_{Z,z}) \to 0. \tag{23}$$

Proof. Given a point x in $\mathbb{P}^2$, we denote again by $\langle x^i \rangle$ the $(i-1)^{\text{th}}$ infinitesimal neighborhood of x in $\mathbb{P}^2$. By Theorem 2, the sheaf $\boldsymbol{R}^1 p_*(q^*(\mathcal{I}_Z(1)))$ is the direct sum of the $\omega_{\langle x^{h-2} \rangle}$, over all points x in the singular locus of $D_{\mathcal{A}_Z}$, where we take $h = \mathrm{mult}(D_{\mathcal{A}_Z}, x)$. Therefore, the kernel of the map β_2 above describes the difference between triple points of $\mathcal{A}_Z$ and triple points of $\mathcal{A}_{Z'}$ each counted with multiplicity. By computing multiplicities, we get that the length of the support of $\ker(\beta_2)$ is precisely $t_{Z,z}$. Since $\ker(\beta_2) = \mathrm{Im}(\beta_1)$ has length $t_{Z,z}$, we get that $\ker(\beta_1) = \mathrm{Im}(\beta_0)$ has degree $-t_{Z,z}$. Summing up, $\mathrm{Im}(\beta_0)$ is a sub-sheaf of $\mathcal{O}_{H_z}$ of degree $-t_{Z,z}$ so $\mathrm{Im}(\beta_0) \simeq \mathcal{O}_{H_z}(-t_{Z,z})$. □

5.2. **Some properties of freeness of line arrangements related to deletion.** Here we give some simple relations between freeness of a given arrangements $\mathcal{A}_Z$ and the numbers $t_{Z,z}$, for $z \in Z$. Throughout the subsection, we let $k \geq 1$, $r \geq 0$ be integers, we set $m = 2k + r + 1$, and we consider a set Z of m points of $\mathbb{P}^2$ and the corresponding line arrangement $\mathcal{A}_Z$.

Proposition 5.2. *Assume $\mathcal{A}_Z$ is free with exponents $(k, k+r)$, let $z \in Z$ and set $Z' = Z \setminus \{z\}$. Then, one of the following alternatives takes place:*

i) $t_{Z,z} = k-1$ and $\mathcal{A}_{Z'}$ is free with exponents $(k-1, k+r)$;
ii) $t_{Z,z} = k+r-1$ and $\mathcal{A}_{Z'}$ is free with exponents $(k, k+r-1)$;
iii) $t_{Z,z} \geq k+r$ and $\mathcal{A}_{Z'}$ is not free.

Proof. Dualizing the exact sequence (23) (i.e., applying to it the functor $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^2}}(-, \mathcal{O}_{\mathbb{P}^2})$) and using the fact that $\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{H_Z}(-t), \mathcal{O}_{\mathbb{P}^2}) \simeq \mathcal{O}_{H_z}(t+1)$ for all integer t, we obtain an exact sequence:

$$0 \to \mathcal{T}^*_{Z'} \to \mathcal{T}^*_Z \to \mathcal{O}_{H_z}(t_{Z,z}+1) \to 0. \tag{24}$$

Here $(-)^*$ denotes the dual of a vector bundle. Since $\mathcal{T}^*_Z \simeq \mathcal{O}_{\mathbb{P}^2}(k) \oplus \mathcal{O}_{\mathbb{P}^2}(k+r)$, we have thus a a surjective map:

$$\mathcal{O}_{\mathbb{P}^2}(k) \oplus \mathcal{O}_{\mathbb{P}^2}(k+r) \twoheadrightarrow \mathcal{O}_{H_z}(t_{Z,z}+1).$$

Then, it is clear that $t_{Z,z} \geq k-1$ for otherwise there could not be an epimorphism as above. Also, it is clear that in case (i) the kernel bundle of the above map splits in the desired way, since the map above factors as:

$$\mathcal{T}^*_Z \to \mathcal{O}_{\mathbb{P}^2}(k) \twoheadrightarrow \mathcal{O}_{H_z}(k),$$

where the first map is the projection onto the direct summand $\mathcal{O}_{\mathbb{P}^2}(k)$ and the second map is the canonical surjection. The case (ii) is analogous.

Let us prove now the case (iii). We consider again the exact sequence (24). We twist it by $-t_{Z,z}-1$ and take the long exact sequence of cohomology. Since $t_{Z,z} \geq k+r$ we get $\mathrm{H}^1(\mathbb{P}^2, \mathcal{T}^*_{Z'}(-t_{Z,z}-1)) \neq 0$ which proves that $\mathcal{T}_{Z'}$ does not decompose as a direct sum of line bundles. $\square$

In the same spirit, we have the following proposition.

Proposition 5.3. *Assume $c_2(\mathcal{T}_Z) = k(k+r)$. Then:*

i) for all $z \in Z$, we have $t_{Z,z} \notin]k-1, k+r-1[$;
ii) if there is $z \in Z$ such that $t_{Z,z} = k-1$ or $t_{Z,z} = k+r-1$, then $\mathcal{A}_Z$ is free with exponents $(k, k+r)$;
iii) if there is $z \in Z$ such that $t_{Z,z} < k-1$, then $\mathcal{A}_Z$ is not free.

Moreover, if $\mathcal{A}_Z$ is not free, but has the same combinatorial type of a free arrangement, then for all $z \in Z$ we have $t_{Z,z} \geq k+r$.

Proof. Consider again the exact sequence obtained in the proof of the previous proposition (from which we borrow the notation also):

$$0 \to \mathcal{T}^*_{Z'} \to \mathcal{T}^*_Z \to \mathcal{O}_{H_z}(t_{Z,z}+1) \to 0.$$

Consider now the restriction to the line H_z of $\mathcal{T}^*_Z$. This splits as $\mathcal{O}_{H_z}(k-s) \oplus \mathcal{O}_{H_z}(k+r+s)$, for some integer $s \geq 0$ by Lemma 3.2, indeed one computes $c_2(\mathcal{T}_Z(-k)) = 0$. So we get an epimorphism:

$$\mathcal{O}_{H_z}(k-s) \oplus \mathcal{O}_{H_z}(k+r+s) \twoheadrightarrow \mathcal{O}_{H_z}(t_{Z,z}+1). \tag{25}$$

Now, in case $t_{Z,z} = k-1$ or $t_{Z,z} = k+r-1$, this forces $s = 0$, hence $\mathcal{T}_Z$ is free by Corollary 4.3. This gives (ii). By the same corollary, since $t_{Z,z} < k-1$ forces $s > 0$, we get (iii). To see (i), we note that an epimorphism of the form (25) cannot exist in this range.

To check the last statement, note that $\mathcal{A}_Z$ cannot have the combinatorial type of a free arrangement $\mathcal{A}_{Z_0}$ if $t_{Z,z} < k-1$, for necessarily we have $c_2(\mathcal{T}_{Z_0}) = k(k+r)$ and we would get $t_{Z_0,z_0} < k-1$ for some $z_0 \in Z_0$ contradicting (iii). Also, we cannot have $t_{Z,z} = k-1$ or $t_{Z,z} = k+r-1$ for any $z \in Z$ for otherwise $\mathcal{A}_Z$ would be free by (ii). Then by (i) we get $t_{Z,z} \geq k+r$ for all $z \in Z$. $\square$

6. Arrangements with a point of not as high multiplicity

We turn now our attention to line arrangements with a point of high multiplicity, but just one less than in the case of Theorem 3. In this setting, we will show that, for free arrangements with exponents $(k, k+r)$ having a point of multiplicity $k-1$, in the range $k \leq 3r+5$, freeness is a combinatorial property at least for real arrangements. The same happens for complex line arrangements in case $k \leq 5$. As an application, we see that Terao's conjecture holds for configurations of m lines in $\mathbb{P}^2_{\mathbb{C}}$ for $m \leq 12$. As far as we know, this had been checked for $m \leq 10$ lines, see [WY07]. However, Theorem 3 essentially takes care of the cases $m \leq 10$ with no need of combinatorial subtleties. On the other hand, for $m = 11, 12$ we need to describe the geometric picture that arises when the arrangement is not free.

Given an arrangement of lines $\mathcal{A}$ and a point $x \in D_{\mathcal{A}}$, we write $\mathcal{A}_x$ for the set of lines of $\mathcal{A}$ passing through x.

Proposition 6.1. *Let $\boldsymbol{k}$ be any field. Let $k \geq 1$, $r \geq 0$ be integers with $k \leq 3r+5$ and set $m = 2k+r+1$. Let $\mathcal{A}_0$ be a free arrangement with exponents $(k, k+r)$, let $\mathcal{A}$ have the same combinatorial type as $\mathcal{A}_0$ and assume that $\mathcal{A}$ has a point x of multiplicy $k-1$.*

If $\mathcal{A}$ is not free, then all singular points of $\mathcal{A} \setminus \mathcal{A}_x$ are contained in a line H. Moreover, H passes through x and does not lie in $\mathcal{A}$, and $\mathcal{A} \cup H$ is free with exponents $(k-1, k+r+2)$.

Proof. Let Z and Z_0 be the sets of points in $\mathbb{P}^2$ corresponding to $\mathcal{A}$ and $\mathcal{A}_0$, so $\mathcal{A} = \mathcal{A}_Z$ and $\mathcal{A}_0 = \mathcal{A}_{Z_0}$. Let $L = L_x \subset \check{\mathbb{P}}^2$ be the line corresponding to the $(k-1)$-tuple point x of $\mathcal{A}$. Working as in the proof of Theorem 3, we write down the exact sequence (13) for $h = k-1$, and we obtain an exact sequence of the form (15):

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2-k-r) \to \mathcal{T}_Z \to \mathcal{I}_\Gamma(2-k) \to 0. \tag{26}$$

Just like in Theorem 3, we have that $\mathcal{A}$ fails to be free precisely when $\mathrm{H}^0(\mathbb{P}^2, \mathcal{I}_\Gamma(1)) \neq 0$. So by assumption the subscheme $\Gamma \subset \mathbb{P}^2$ is contained in a line H, which is the line we need.

Let us now show that H has the required properties. Recall from the proof of Theorem 3 that Γ is the locus of singular points of $\mathcal{A} \setminus \mathcal{A}_x$. Dually Γ is the set of bisecant lines to $Z' = Z \setminus L$ that meet L away from Z. The first statement of the lemma is thus proved. Let $w \in \check{\mathbb{P}}^2$ correspond to H, so $H = H_w$. Computing Chern classes, we get that Γ is a subscheme of length $2r+4$ of $\mathbb{P}^2$.

Let us prove that the point w does not lie in Z, i.e., $H \notin \mathcal{A}$. The fact that Γ sits in H means that the bisecant lines to Z' that meet L away from Z all meet at w. If w belongs to Z, then this is a combinatorial property that must also hold for Z_0, namely the subscheme Γ_0 associated to Z_0 should be contained in a line H_{w_0} corresponding to the meeting point w_0. But $\mathcal{A}_0$ is free so by Lemma 3.2 we have $\mathrm{H}^0(\mathbb{P}^2, \mathcal{T}_{Z_0}(k-1)) = \mathrm{H}^0(\mathbb{P}^2, \mathcal{I}_{\Gamma_0}(1)) = 0$. Hence Γ_0 lies in no line.

Now we consider the set of points $\tilde{Z} = Z \cup \{w\}$ and the corresponding arrangement $\tilde{\mathcal{A}}$. We first want to show that $t_{\tilde{Z},w} = k+2r+2$. Restricting to H the surjection $\mathcal{T}_Z \to \mathcal{I}_\Gamma(2-k)$ we get $(\mathcal{T}_Z)|_H \twoheadrightarrow \mathcal{O}_H(-k-2r-2)$. Since $c_1((\mathcal{T}_Z)|_H) = -2k-r$, we get:

$$(\mathcal{T}_Z)|_H \simeq \mathcal{O}_H(-k-2r-2) \oplus \mathcal{O}_H(r+2-k). \tag{27}$$

Of course $\mathcal{A}$ is obtained from $\tilde{\mathcal{A}}$ by deletion of w, so we get an exact sequence like (23):

$$0 \to \mathcal{T}_{\tilde{Z}} \to \mathcal{T}_Z \to \mathcal{O}_{H_z}(-t_{\tilde{Z},w}) \to 0. \tag{28}$$

By our interpretation of Γ, the value of $t_{\tilde{Z},w}$ is at least the length of Γ, i.e., $t_{\tilde{Z},w} \geq 2r+4$. By our numerical assumption, this implies $t_{\tilde{Z},w} > k-r-2$. So using (i) of Proposition 5.3 and (27) we get $t_{\tilde{Z},w} = k+2r+2$.

Let us now see that w lies in L, i.e., $x \in H$. Assuming the contrary, and let s be the number (with multiplicity) of bisecant lines to Z passing through w and one point of $Z \cap L$. We have $t_{\tilde{Z},w} = 2r+s+4$. Since Γ has length $2r+4$, there are at least $2r+5$ points of Z that contribute to Γ (there are precisely $2r+5$ such points in case they are all aligned, and even more points in case they lie on several lines). So there are at most $2k+r+1-|Z \cap L|-(2r+5) = k-r-3$ points of Z that contribute to the s bisecant lines above. So $s \leq k-r-3$. Hence $t_{\tilde{Z},w} \leq k+r+1$, which contradicts $t_{\tilde{Z},w} = k+2r+2$.

Finally, let us check that $\tilde{\mathcal{A}}$ is free with exponents $(k-1, k+r+2)$. To do this, recall that $\mathrm{H}^0(\mathbb{P}^2, \mathcal{I}_\Gamma(1)) \neq 0$ gives $\mathrm{H}^0(\mathbb{P}^2, \mathcal{T}_Z(k-1)) \neq 0$ and look at the associated map $\mathcal{O}_{\mathbb{P}^2}(1-k) \to \mathcal{T}_Z$. Note that the map $\mathcal{O}_{\mathbb{P}^2}(-k-r-2) \to \mathcal{T}_Z$ of (26) does not factor through $\mathcal{O}_{\mathbb{P}^2}(1-k) \to \mathcal{T}_Z$, for in that case its cokernel would have torsion. This gives:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-k-r-2) \oplus \mathcal{O}_{\mathbb{P}^2}(1-k) \to \mathcal{T}_Z \to \mathcal{O}_{H_w}(-k-2r-2) \to 0.$$

Since any non-zero map $\mathcal{T}_Z \to \mathcal{O}_{H_w}(-k-2r-2)$ gives the same kernel, comparing the above display and (28), we get $\mathcal{O}_{\mathbb{P}^2}(-k-r-2) \oplus \mathcal{O}_{\mathbb{P}^2}(1-k) \simeq \mathcal{T}_{\tilde{Z}}$. □

6.1. **Real arrangements with a point of high multiplicity.** Here, we show that freeness of real arrangements with exponents $(k, k+r)$ having a point of multiplicity $k-1$ is combinatorial in our range $k \leq 5+3r$, by proving that the alignment of the previous proposition is impossible over $\mathbb{R}$.

Theorem 5. *Assume $\boldsymbol{k}$ is a subfield of $\mathbb{R}$, let $k \geq 1$, $r \geq 0$ be integers, and set $m = 2k+r+1$. Suppose $k \leq 5+3r$. Let $\mathcal{A}$ be an arrangement of m lines with a point of multiplicity $k-1$, having the same combinatorial type of a free arrangement with exponents $(k, k+r)$. Then $\mathcal{A}$ is also free with exponents $(k, k+r)$.*

Proof. If $\mathcal{A}$ is free, then its exponents are necessarily $(k, k+r)$, so we have only to prove that $\mathcal{A}$ is free. Let us assume that $\mathcal{A}$ is not free, and see that this leads to a contradiction.

We let again Z be the set of points of $\check{\mathbb{P}}^2$ corresponding to $\mathcal{A}$ so $\mathcal{A} = \mathcal{A}_Z$, and we let L be a line of $\check{\mathbb{P}}^2$ containing precisely $k-1$ points of Z, corresponding to the $(k-1)$-tuple point of $\mathcal{A}$. Set $Z' = Z \setminus L$. By the previous proposition, there is a point $w \in L \setminus Z$ which is the intersection point of all strict h-secant lines (for $h \geq 2$) to Z' that are strict h-secant to Z too (i.e. they are not $(h+1)$-secant to Z).

Set $Z'' = \{w\} \cup (Z \cap L)$. We have $k+r+2$ points in $\check{\mathbb{P}}^2$ (the points of Z'), and the set Z'' of k points in L, such that any bisecant line to Z' cuts L along Z''. If we let now L be the line at infinity in $\check{\mathbb{P}}^2$, we see that Z' is a set of $k+r+2$ points of an affine 2-dimensional space, that determines at most k directions. But, since we are working over $\mathbb{R}$, the set Z' should determine at least $k+r+1 \geq k+1$ directions, according to Ungar's theorem, see [Ung82]. This is a contradiction. □

6.2. **Combinatorial nature of freeness for low exponents.** Here we show that freeness of arrangements with exponents $(k, k+r)$ is combinatorial when $k \leq 5$, by proving that the combinatorics of Proposition 6.1 is actually impossible for $k \leq 5$.

Theorem 6. *Assume $\boldsymbol{k}$ is a subfield $\mathbb{C}$. Let $0 \leq k \leq 5$ and $r \geq 0$ be integers, and $\mathcal{A}$ be a line arrangement, having the same combinatorial type of a free arrangement with exponents $(k, k+r)$. Then $\mathcal{A}$ is also free with exponents $(k, k+r)$.*

We fix again our notation: we consider the finite set of point Z in $\check{\mathbb{P}}^2$ corresponding to $\mathcal{A}$ so that $\mathcal{A} = \mathcal{A}_Z$. We also consider another finite set of point $Z_0 \subset \check{\mathbb{P}}^2$, such that $\mathcal{A}_0 = \mathcal{A}_{Z_0}$ is free with exponents $(k, k+r)$, and has the same combinatorial type as $\mathcal{A}$.

We will need *Hirzebruch's inequality* (see [Hir83]), in the "improved" version:

$$b_{\mathcal{A},2} + \tfrac{3}{4} b_{\mathcal{A},3} \geq m + \textstyle\sum_{h \geq 5} (2h-9) b_{\mathcal{A},h}, \tag{29}$$

valid for arrangements $\mathcal{A}$ of m complex projective lines with $b_{\mathcal{A},m} = b_{\mathcal{A},m-1} = b_{\mathcal{A},m-2} = 0$.

Lemma 6.2. *Assume that $\mathcal{A}$ is free with exponents $(k, k+r)$, with $r \geq 0$, $k \geq 1$, that $\mathcal{A}$ has points of multiplicity 3 at most, and that not all lines of $\mathcal{A}$ pass through a point. Then the possible pairs $(k, k+r)$ are $(1,1)$, $(1,2)$, $(2,2)$, $(2,3)$, $(3,3)$, $(3,4)$ or $(4,4)$. In the last case, $\mathcal{A}$ has the combinatorial type of the dual Hesse arrangement.*

Proof. The set $Z \subset \check{\mathbb{P}}^2$ of points corresponding to $\mathcal{A}$ has no alignment of 4 points. By Corollary 4.2, Z is non-degenerate, since $k \geq 1$. Since $b_{\mathcal{A},t} = 0$ for $t \geq 4$, the relations (10) and (11) allow to compute $b_{\mathcal{A},2}$ and $b_{\mathcal{A},3}$:

$$b_{\mathcal{A},2} = -k(k+r-4) - r(r-2), \qquad b_{\mathcal{A},3} = k(k+r-1) + \frac{r(r-1)}{2}.$$

Of course $b_{\mathcal{A},2} \geq 0$, so if $r \geq 2$ we have $k+r \leq 4$ hence $(k, k+r)$ equals $(1,3)$, $(1,4)$, or $(2,4)$. In view of (ii) of Corollary 4.2, in cases $(1,3)$ and $(1,4)$, the set Z should then have a 4-secant line, which contradicts the hypothesis. A quick calculation shows that $(2,4)$ contradicts (29).

We are left with $r = 0$ and $k \leq 4$ or $r = 1$ and $k \leq 3$, which are the cases allowed by our statement. In case $r = 0$, $k = 4$, we get $b_{\mathcal{A},3} = 12$ and $b_{\mathcal{A},2} = 0$, which is the combinatorial type of the arrangement of 9 lines dual to the 9 inflection points of a smooth cubic curve as in Example 4.4. □

It is worth noting that, in the setup of the previous lemma, there are more cases in positive characteristic. For instance, if $\boldsymbol{k} = \mathbb{Z}/2\mathbb{Z}$, the case $(k, k+r) = (2,4)$ corresponds to the 7 points of $\mathbb{P}^2_{\boldsymbol{k}}$. In this case $b_{\mathcal{A},2} = 0$ and $b_{\mathcal{A},3} = 7$, and $\mathcal{A}$ is free with exponents $(2,4)$.

Proof of Theorem 6. We know that, if $\mathcal{A}$ is free, then its exponents are $(k, k+r)$, so we only have to check freeness of $\mathcal{A}$. We take up our usual dual notation. If Z is degenerate, the statement is clear by Corollary 4.2. If Z is non-degenerate, then, in order for $\mathcal{A}_0$ to be free, the set of points Z_0 must have at least a trisecant line, so the same must happen to Z. If there is no 4-secant line to Z, then by the previous lemma we have $k \leq 3$ or $k = 4$. In the former case, the existence of trisecant lines to Z forces $\mathcal{A}$ to be free by Theorem 3. In the latter case, we are done by Example 4.4.

We move forward to the case when the set of points Z has 4-secant lines: let $L \subset \check{\mathbb{P}}^2$ be one of them. Again by Theorem 3, we can assume that any 4-secant line to Z is strict, and that k is at least 5 so in fact $k = 5$. So we assume that $\mathcal{A}$ is not free and we seek a contradiction. Note that in this range we can use Proposition 6, so there is a point $w \in L \setminus Z$ which is the intersection point of all strict h-secant lines (for $h \geq 2$) to $Z' = Z \setminus L$ that are strict h-secant to Z too. We let $\tilde{Z} = Z \cup \{w\}$.

We call Γ the set of these lines, according to the notation set up in Proposition 6. Recall that, according to the proof of this proposition, Γ appears as a subscheme of $\mathbb{P}^2$ of length $2r+4$ obtained by reduction step through the 4-secant line L. Denote by s_i the number of strict i-secants to $Z \setminus L$ though w. We have:

$$s_1 + 2s_2 + 3s_3 + 4s_4 = r + 7, \qquad s_2 + 2s_3 + 3s_4 = 2r + 4.$$

We get $s_2 + s_3 + s_4 < 3 - r$ so $r \leq 2$. One sees easily that the case $r = 2$ is in fact impossible.

Let us look at the case $r = 0$. There are three subcases to look at, corresponding to the values of $(s_1, \ldots, s_4)$, namely $(1,0,2,0)$, or $(0,2,1,0)$, or $(1,1,0,1)$. In all of them $\tilde{Z}$ has no h-secants with $h \geq 6$ and all 5-secants to $\tilde{Z}$ comes from a 4-secant to Z through w. So we have 1 or 2 strict 5-secants to $\tilde{Z}$. If $b_{\tilde{\mathcal{A}},5} = 2$, by Lemma 2.1 we get $b_{\tilde{\mathcal{A}},2} = 3b_{\tilde{\mathcal{A}},4} + 1$ and $b_{\tilde{\mathcal{A}},3} = 15 - 3b_{\tilde{\mathcal{A}},4}$. So $0 \leq b_{\tilde{\mathcal{A}},4} \leq 5$, by formula (29) we actually get:

$$b_{\tilde{\mathcal{A}},5} = 2 \Longrightarrow 3 \leq b_{\tilde{\mathcal{A}},4} \leq 5. \tag{30}$$

If $b_{\tilde{\mathcal{A}},5} = 1$, then $b_{\tilde{\mathcal{A}},2} = 3b_{\tilde{\mathcal{A}},4} - 7$ and $b_{\tilde{\mathcal{A}},3} = 21 - 3b_{\tilde{\mathcal{A}},4}$ so $3 \leq b_{\tilde{\mathcal{A}},4} \leq 7$. By (29) we get:

$$b_{\tilde{\mathcal{A}},5} = 1 \Longrightarrow 6 \leq b_{\tilde{\mathcal{A}},4} \leq 7. \tag{31}$$

We want thus to bound the number of 4-secants to $\tilde{Z}$.

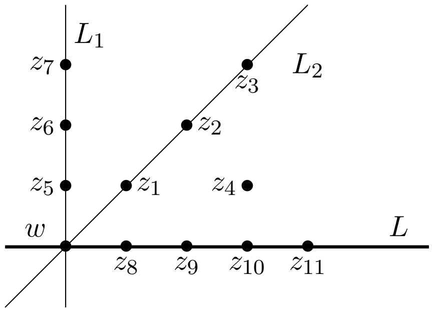

FIGURE 2. Case (i)

i) Γ *consists of* 2 *strict* 3*-secants* L_1 *and* L_2 *to* Z *passing through* w. In this case we have a point $z \in Z$ lying off the lines of Γ ($z = z_4$ in the figure above). We see that $b_{\tilde{\mathcal{A}},5} = 1$ and $b_{\tilde{\mathcal{A}},4} \leq 5$ since any strict 4-secant to $\tilde{Z}$ besides the 4-secants L_1 and L_2 through w must pass through the triples of points on L_i. This contradicts the inequality (31).

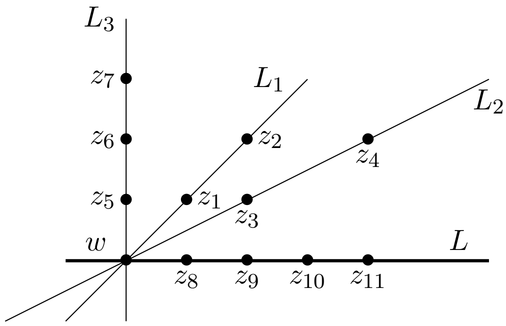

FIGURE 3. Case (ii)

ii) Γ *consists of* 2 *strict bisecant lines* L_1, L_2 *and one strict* 3*-secant line* L_3 *to* Z *passing through* w. Let the $\{z_{2i-1}, z_{2i}\} = L_i \cap Z$ for $i = 1, 2$. In this case, again $b_{\tilde{\mathcal{A}},5} = 1$ and $b_{\tilde{\mathcal{A}},4} \leq 5$ since all strict 4-secant lines to $\tilde{Z}$ are L_3 and, at most, the lines $[z_1, z_3]$, $[z_1, z_4]$, $[z_2, z_3]$, $[z_2, z_4]$. Again we contradict (31).

iii) Γ *consists of one strict bisecant line* L_1 *and one strict* 4*-secant line* L_2 *to* Z *passing through* w. In this case we have $b_{\tilde{\mathcal{A}},5} = 2$. We have one point z lying off of $L_1 \cup L_2 \cup L$, and any strict 4-secant line to $\tilde{Z}$ passes through z and $Z \cap L_1$, so that $b_{\tilde{\mathcal{A}},4} \leq 2$. This contradicts (30).

The case $r = 0$ is thus settled (i.e. $m = 11$ lines). Let us look at $r = 1$, i.e. $m = 12$. In this case, Γ has length 6, $\tilde{Z}$ has no 6-secants and there are 8 points in $\tilde{Z} \setminus L$. The only possible configuration consists of 3 strict 5-secant lines to $\tilde{Z}$ meeting at w, among which is L, i.e. we must have $(s_1, \ldots, s_4) = (0, 0, 0, 2)$. Moreover, any bisecant line to $\tilde{Z}$ is in fact a trisecant line. This contradicts a lemma due to Kelly, [Kel86, Lemma 2].

For $r \geq 2$, Γ has length $2r+4$, $\tilde{Z}$ has no 6-secants and there are $r+7$ points in $\tilde{Z} \setminus L$. This is impossible. $\square$

Since free arrangements of $m \leq 12$ lines have exponents $(k, k+r)$ with $r \geq 0$ and $k \leq 5$, we get the following corollary.

Corollary 6.3. *Terao's conjecture holds for up to* 12 *lines in* $\mathbb{P}^2_{\mathbb{C}}$.

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