BUNDLES OVER THE FANO THREEFOLD V_5

DANIELE FAENZI

Abstract. Using an explicit resolution of the diagonal for the variety V_5, we provide cohomological characterizations of the universal and quotient bundles. A splitting criterion for bundles over V_5 is also proved.

The presentation of semistable aCM bundles is shown, together with a resolution–theoretic classification of low rank aCM bundles.

1. Introduction

According to the classification of Fano threefolds with second Betti number equal to 1 (see [Isk78] and [Isk77]), there are 4 Fano threefolds with $b_2 = 1$ and $b_3 = 0$. Namely the projective space $\mathbb{P}^3$, the quadric Q_3, the degree-5 *del Pezzo* threefold V_5 and the genus-12 variety V_{22}. The cohomology ring over $\mathbb{C}$ of any of these is isomorphic to the cohomology ring of $\mathbb{P}^3$ i.e. $\mathbb{C}[h]/h^4 = \oplus \mathrm{H}^{p,p}(X)$, where h is the hyperplane divisor, so we say that these are the cases with *trivial* Hodge structure.

The cohomology rings are nonisomorphic over $\mathbb{Z}$ by the degree of h^2 with respect to the generator of $\mathrm{H}^4(X)$, the Poincaré dual of a straight line. This degree is the degree of the threefold: respectively 1, 2, 5, 22. The intermediate Jacobian is trivial and the Chow ring is isomorphic to the cohomology ring.

All of these varieties are rational. The variety V_{22} is the only one with nontrivial moduli space and this moduli space admits a *special point* corresponding to a distinguished threefold U_{22}. The variety U_{22} is called the Mukai–Umemura threefold.

Any of the varieties $\mathbb{P}^3$, Q_5, V_5 and U_{22} admits a quasi–homogeneous structure for the group $\mathsf{SL}(2)$ (equivalently for $\mathsf{PGL}(2)$ or $\mathsf{SO}(3)$), i.e. it is the (smooth) closure of an orbit by the action of this group. We will briefly resketch this in section (2).

However here we will concentrate on V_5 and on vector bundles over it. In section (3) we review some facts about bundles over V_5, while in section (4) we will give an explicit resolution of the diagonal, see Theorem 4.1. This is constructed rather directly, but, unlike the case of projective spaces and Grassmannians, cfr. [Bei78] and [Kap88], it is not given by a Koszul complex. Results for V_5 are obtained also by Orlov in [Orl91], while Canonaco in [Can00] provides similar results for weighted projective spaces

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and Kawamata in [Kaw02] exhibits an infinite resolution for general projective varieties.

In section (5) we prove a splitting criterion making use of the canonical resolution, and cohomological characterizations of universal and quotient bundles. Then we concentrate on bundles with the property that $\mathrm{H}^p(V_5, E \otimes \mathscr{O}(t)) = 0$ for any $t \in \mathbb{Z}$ and $0 < p < 3 = \dim(V_5)$. We call a vector bundle E with this property *aCM* for *arithmetically Cohen–Macaulay*, following standard terminology. We will say also that E has *no intermediate cohomology*.

We describe moduli spaces of aCM bundles over V_5 using resolutions and prove that semistable aCM bundles admit a peculiar presentation that may lead to classification at least in low rank, see section (6). In particular we classify these bundles up to rank 3.

The manifold V_5 is a Fano threefold of *index two*, i.e. $\omega_{V_5} \simeq \mathscr{O}_{V_5}(-2)$, where $\mathscr{O}_{V_5}(1)$ is the minimal very ample line bundle on V_5. This is clear since V_5 is obtained cutting $\mathbb{G}(\mathbb{P}^1, \mathbb{P}^4)$ with a generic $\mathbb{P}^6 \subset \mathbb{P}^9$.

$$V_5 = \mathbb{G}(\mathbb{P}^1, \mathbb{P}^4) \cap \mathbb{P}^6 \subset \mathbb{P}(\wedge^2 V) \tag{1}$$

A quadric section of V_5 is a $K3$ surface, whose linear section is a *canonical* curve of genus 6, that is a curve embedded by the canonical linear system. Such curve is *generic*. This is related to the classification of Fano threefolds and to the moduli space of $K3$ surfaces, see [Muk88].

From the point of view of birational geometry, V_5 is the contraction of the proper preimage of a 2-dimensional quadric in Q_3 under the blow up of Q_3 with center at a rational normal cubic. For a more detailed description of many properties of this and other threefolds, one may consult [IP99].

2. Quasi–homogeneous structure

The linear orbit of a d-tuple of points in $\mathbb{P}^1 = \mathbb{P}^1_k = \mathbb{P}(Y)$ is the orbit of a set of d points in $\mathbb{P}^1$ under the standard action of $\mathsf{SL}(2) = \mathsf{SL}(Y)$. The orbit sits in $\mathbb{P}(\mathrm{Sym}^d Y)$ where $\mathsf{SL}(2)$ naturally acts by the weight-d representation, denoted by Y_d. We have $Y_d \simeq \mathrm{Sym}^d Y$.

The orbit has dimension at most 3, and there are few cases in which its closure is smooth. In this case the closure is actually a Fano threefold with $b_3 = 0$ and the possible cases are

(1) $\mathbb{P}^3 \simeq \mathbb{P}(Y_3)$ orbit of $x^3 + y^3$;
(2) $Q_3 \subset \mathbb{P}(Y_4)$ orbit of $x^4 + xy^3$;
(3) $V_5 \simeq \mathbb{G}(k^2, Y_4) \cap \mathbb{P}(Y_6)$ orbit of $x^5y - xy^5$;
(4) $U_{22} \subset \mathbb{P}(Y_{12})$ orbit of $xy(x^{10} + 11x^5y^5 - y^{10})$.

This is studied in detail in [AF93], [MU83], [Muk92]. Clearly the stabilizer is a finite group and looking at isometries of the roots of such polynomials one checks that it is isomorphic respectively to S_3, A_4, S_4, A_5.

Proposition 2.1 (Mukai–Umemura)**.** *The manifold V_5 is isomorphic to the closure of $\mathsf{SL}(2) \cdot (x^5y - xy^5)$ in $\mathbb{P}(Y_6) = \mathbb{P}^6$. The boundary divisor has an open part given by $\mathsf{SL}(2) \cdot x^5y$ whose complementary is the degree-6 rational normal curve $\mathsf{SL}(2) \cdot x^6$.*

On the other hand, let B be a 3-dimensional vector space, $\mathbb{P}^2 = \mathbb{P}(B)$ and the dual plane $\check{\mathbb{P}}^2 = \mathbb{P}(B^*)$. Denote by $R = \operatorname{Sym} B$ the coordinate ring, and let $S = \operatorname{Sym}^* B$ act by apolarity on R. Write z_0, z_1, z_2 for the generators of B and $\partial_0, \partial_1, \partial_2$ for those of B^*. Let $F \in \operatorname{Sym}^2 B$ be the equation of a smooth conic in $\mathbb{P}(B)$ and $V = \operatorname{Sym}^2 B/(F)$.

Proposition 2.2 (Mukai). *Let* $\mathrm{VPS}(3, \mathrm{V}(F))$ *be the variety of polar* 3*-sides to the conic* $\mathrm{V}(F)$*, i.e. the closure in* $\mathrm{Hilb}_3(\check{\mathbb{P}}^2)$ *of the set*

$$\{(f_0, f_1, f_2) \in \mathrm{Hilb}_3(\check{\mathbb{P}}^2) | f_0^2 + f_1^2 + f_2^2 = F\}$$

Then the variety V_5 *is isomorphic to* $\mathrm{VPS}(3, \mathrm{V}(F))$*.*

The conic F endows B with a natural $\mathsf{SO}(3)$-action, and we consider the dual conic F^{-1} as an element of $\operatorname{Sym}^2 B^*$. Taking 2×2 minors of the natural evaluation $B^* \otimes \operatorname{Sym}^2 B \longrightarrow B$ provides a map $\wedge^2 B^* \simeq B \to \wedge^2 V^*$ and the linear section (1) in $\mathbb{P}(\wedge^2 V)$ corresponds to the net of alternating forms $\sigma\colon \wedge^2 B^* \simeq B \to \wedge^2 V^*$ described by the following

$$\wedge^2 B^* \otimes \wedge^2 V \longrightarrow \operatorname{Sym}^2 B \xrightarrow{\quad F^{-1} \quad} k$$

$$\partial_1 \wedge \partial_2 \otimes f_0 \wedge f_1 \longmapsto F^{-1}(\partial_1 \wedge \partial_2(f_0 \wedge f_1))$$

The standard $2:1$ covering $\phi\colon \mathsf{SL}(2) \longrightarrow \mathsf{SO}(3)$ gives B a structure of weight-2 representation, and we get $B \simeq Y_2$ and $V \simeq Y_4$. Moreover the arrow $\sigma\colon B \longrightarrow \wedge^2 V^*$ giving the net of alternating forms is equivariant, i.e. is just the kernel of the projection $\wedge^2 V^* \longrightarrow \operatorname{Sym}^6 Y \simeq Y_6$. So $\mathbb{P}^6 \simeq \mathbb{P}(Y_6)$.

If we choose F to be given by the identity matrix, then $(z_0^2, z_1^2, z_2^2)/(F)$ represents a polar triangle of $C = \mathrm{V}(F)$. Its $\mathsf{SO}(3)$-orbit is $\mathsf{SO}(3) \cdot z_0^2 \wedge z_1^2$ in $\wedge^2 V$ and under a ϕ-equivariant projection $z_0^2 \wedge z_1^2$ corresponds to $(x^5 y - x y^5)$.

Now let us define the variety $\mathbb{G}(2\times 3; B^*)$ given by classes of 2×3 matrices over the space B^*, that is

$$\mathbb{G}(2\times 3; B^*) = \mathbb{P}(\mathrm{M}_{2,3}(B^*))// \mathsf{SL}(2) \times \mathsf{SL}(3)$$

The variety $\mathbb{G}(2\times 3; B^*)$ compactifies the open subset $\mathrm{Hilb}_3(\check{\mathbb{P}}^2)^\circ$ of triples of distinct points and is endowed with the natural bundles $\tilde{Q}^*$ and $\tilde{U}$ respectively of rank 3 and 2. This variety has been deeply studied by Drezet in [Dre88], where he proves ([Dre88, Theorem 4]) that such compactification actually provides a morphism

$$\mathrm{Hilb}_3(\check{\mathbb{P}}^2) \longrightarrow \mathbb{G}(2\times 3, B^*)$$

which is the blow up of $\mathbb{G}(2\times 3, B^*)$ at $\mathbb{P}^2 \subset \mathbb{G}(2\times 3, B^*)$. The bundles $\tilde{Q}^*$ and $\tilde{U}$ extend the sheaves over $\mathrm{Hilb}_3(\check{\mathbb{P}}^2)^\circ$ whose fiber over Z is given respectively by the ideal generators and the first order syzygies of the subscheme $Z \subset \check{\mathbb{P}}^2$. It follows that $\mathrm{H}^0(\tilde{Q})^* \simeq \operatorname{Sym}^2 B^*$ and $\mathrm{H}^0(\tilde{U})^* \simeq \ker(m : \operatorname{Sym}^2 B^* \otimes B^* \to \operatorname{Sym}^3 B^*)$, where m is the multiplication in S, the coordinate ring of $\check{\mathbb{P}}^2$. The restrictions of $\tilde{U}$ and $\tilde{Q}$ to V_5 coincide with U and Q, defined as restrictions from $\mathbb{G}(k^2, V)$.

Lemma 2.3. *Let* $F \in \operatorname{Sym}^2 B$ *be a smooth conic and consider* F *as a function* $F\colon \operatorname{Sym}^2 B^* \to k$*. Then* V_5 *is isomorphic to the following subset* *of* $\mathrm{Hilb}_3(\check{\mathbb{P}}^2)$

$$\{Z \in \mathrm{Hilb}_3(\check{\mathbb{P}}^2) |\, \mathrm{H}^0(I_Z(2)) \subset \ker F\}$$

Moreover V_5 *is isomorphic to the zero locus of the section* $F \in \mathrm{H}^0(\mathbb{G}(2 \times 3; B^*), \tilde{Q}))$

Proof. One may check this when $F = z_0^2 + z_1^2 + z_2^2$, taking the polar triangle $Z \in \mathrm{Hilb}_3(\check{\mathbb{P}}^2)$ given by the three lines $(\mathrm{V}(z_0), \mathrm{V}(z_1), \mathrm{V}(z_2))$ and then using the action of $\mathsf{SO}(3)$, indeed the section F is invariant under $\mathsf{SO}(3)$. The ideal I_Z of such a point in $S = \mathrm{Sym}\, B^*$ is generated by $(\partial_0\partial_1, \partial_0\partial_2, \partial_1\partial_2)$. Clearly such generators lie in $\ker F$. Notice also that the section F vanishes in the expected codimension. Furthermore the total spaces of Q and U^* in this framework are

$$\mathrm{H}^0(V_5, Q)^* \simeq V^* \simeq \ker(F\colon\ \mathrm{Sym}^2 B^* \to k)$$
$$\mathrm{H}^0(V_5, U^*)^* \simeq V \simeq \ker(m\colon V^* \otimes B^* \to \mathrm{Sym}^3 B^*)$$

We can identify the kernel of m with $V \simeq (V^*)^*$ because, denoting by J_F the ideal generated by V^*, S/J_F is a codimension three Gorenstein ring (the *apolar* ring to F), whose resolution is given by the structure theorem in [BE77]. Such resolution provides

$$V^* \simeq \mathrm{Tor}_1^S(R/J_F, k)_2 \simeq \mathrm{Tor}_2^S(R/J_F, k)_3^* \simeq (\ker m)^*$$

Notice also that all the arrows in this setup are $\mathsf{SL}(2)$-invariant. □

3. Universal vector bundles

Here we consider bundles over the variety V_5. An exceptional collection of bundles over V_5 is known after [Orl91], where helix conditions are also proved. We implicitly refer to [Bon90], [Nog94] for implications of this. We will resketch Orlov's exceptional collection here.

Given two bundles E and F define $i_{E,F}$ as the dual evaluation $i_{E,F} : E \to \mathrm{Hom}(E,F)^* \otimes F$ and $\mathrm{ev}_{E,F}$ the evaluation $\mathrm{ev}_{E,F} : \mathrm{Hom}(E,F) \otimes E \to F$. Moreover, denote by $c_i \in \mathbb{Z}$ the i-th Chern class of a sheaf E, meaning $c_i(E) = c_i\xi_i$, where $\xi_1 = c_1(\mathscr{O}_{V_5}(1))$ and ξ_2 (resp. ξ_3) is the class of a line (resp. of a point) in V_5.

Definition 3.1. Given two bundles E and F, if $i_{E,F}$ is injective we define the right mutation $\mathrm{R}_F(E) = \mathrm{coker}(i_{E,F})$. If $\mathrm{ev}_{E,F}$ is surjective we define the left mutation $\mathrm{L}_E(F) = \ker(\mathrm{ev}_{E,F})$. We refer the reader to [Gor90], [Rud90], [Dre86] for more general definitions and essential properties of mutations.

We consider the following vector bundles over V_5. The bundle U is the universal subbundle with $\mathrm{rk}(U) = 2$, $c_1(U) = -1$ and $c_2(U) = 2$. The bundle Q is the universal quotient bundle, $\mathrm{rk}(Q) = 3$, $c_1(Q) = 1$, $c_2(Q) = 3$, $c_3(Q) = 1$.

Definition 3.2. Define the collection $\big(G_3, \dots, G_0\big)$ as $\big(\mathscr{O}(-1), U, Q^*, \mathscr{O}\big)$. Define also the dual collection $\big(G^3, \dots, G^0\big)$ as $\big(\mathscr{O}(-1), \wedge^2 Q^*, U, \mathscr{O}\big)$.

The following lemma is straightforward.

Lemma 3.3. *We have the exact sequence with* $\mathsf{SL}(2)$*-equivariant maps*

$$0 \longrightarrow U \xrightarrow{i_{U,\mathscr{O}}} V \otimes \mathscr{O} \longrightarrow Q \longrightarrow 0 \tag{2}$$

and we have the following isomorphisms of $\mathsf{SL}(2)$*-modules*

$$\mathrm{H}^0(\mathscr{O}(1)) \simeq Y_6 \qquad \mathrm{H}^0(U^*) \simeq V^* \simeq Y_4 \qquad \mathrm{Hom}(U, Q^*) \simeq B \simeq Y_2 \tag{3}$$

Moreover we have the descriptions of the Hilbert scheme of lines and conics in V_5

$$\mathrm{Hilb}_{2t+1}(V_5) \simeq \mathbb{P}(\mathrm{H}^0(U^*)) \simeq \mathbb{P}(Y_4) = \mathbb{P}^4 \tag{4}$$

$$\mathrm{Hilb}_{t+1}(V_5) \simeq \mathbb{P}(B) = \mathbb{P}^2 \tag{5}$$

The vector spaces in (3) inherit an invariant duality, hence we will deliberately confuse them with their dual from now on. Notice that the net σ of alternating forms on the space V is just a 3-subspace of $\mathrm{H}^0(\wedge^2 U^*_{\mathbb{G}}) \simeq \mathrm{H}^0(\mathscr{O}_{\mathbb{G}}(1)) \simeq \wedge^2 V^*$. On other hand, given an element b in $B \simeq \mathrm{Hom}(U, Q^*)$, the homomorphism $b : U \to Q^*$ takes $u \in U$ to $\sigma_b(u, -) \colon Q \to k$ with σ described above. It is elementary to check the degeneration locus of this homomorphism and write the exact sequence

$$0 \longrightarrow U \xrightarrow{b} Q^* \longrightarrow I_L \longrightarrow 0 \tag{6}$$

where L is a line in V_5. We thus recover (5), actually known to Castelnuovo, [Cas91].

Proposition 3.4. *There are* $\mathsf{SL}(2)$*-equivariant exact sequences*

$$0 \longrightarrow Q(-1) \xrightarrow{i_{Q(-1),U}} Y_2 \otimes U \xrightarrow{\mathrm{ev}_{U,Q^*}} Q^* \longrightarrow 0 \tag{7}$$

$$0 \longrightarrow \mathscr{O}(-1) \xrightarrow{i_{\mathscr{O}(-1),U}} Y_4 \otimes U \longrightarrow E_9 \longrightarrow 0 \tag{8}$$

$$0 \longrightarrow E_9 \longrightarrow Y_4 \otimes Q^* \longrightarrow \Omega^1_{\mathbb{P}^6}(1)_{|V_5} \longrightarrow 0 \tag{9}$$

where E_9 *is a rank-9 vector bundle. The sequences fit together to the helix below*

$$\begin{array}{ccccccccc}
\mathscr{O}(-1) & \hookrightarrow & Y_4 \otimes U & \longrightarrow & Y_4 \otimes Q^* & \longrightarrow & Y_6 \otimes \mathscr{O} & \twoheadrightarrow & \mathscr{O}(1) \\
 & & & \searrow \quad \nearrow & & \searrow \quad \nearrow & & & \\
 & & & E_9 & & \Omega^1_{\mathbb{P}^6}(1)_{|V_5} & & &
\end{array}$$

Proof. First recall that $\wedge^2 Q^* \simeq Q(-1)$ and $B \simeq \mathrm{Hom}(U, Q^*) \simeq Y_2$. Clearly the evaluation $Y_2 \otimes U \longrightarrow Q^*$ is $\mathsf{SL}(2)$-equivariant and $\wedge^2 Q^* \simeq Q(-1)$ so by self duality we have the exact sequence with invariant maps (7) where the first map in coincides with $i_{Q(-1),U}$.

First notice that $\mathrm{Ext}^1(Q, U(1)) = 0$. Indeed one can compute $\mathrm{H}^1(V_5, Q^* \otimes U(1)) = 0$ taking global sections in the Koszul complex of $V_5 \subset \mathbb{G}(k^2, V)$ and using $\mathrm{H}^p(\mathbb{G}(k^2, V), Q^* \otimes U(2-p)) = 0$ for $1 \leq p \leq 4$.

Then applying $\mathrm{Hom}(-, U(1))$ to the sequence (2), and using the identifications $\mathrm{Hom}(U, U(1)) \simeq \mathrm{Hom}(U(-1), U)$ and $\mathrm{Hom}(Q, U(1)) \simeq \mathrm{Hom}(Q, U^*) \simeq \mathrm{Hom}(U, Q^*) \simeq B$ we get

$$0 \longrightarrow B \xrightarrow{\sigma} V^* \otimes V^* \longrightarrow \mathrm{Hom}(U(-1), U) \longrightarrow 0 \tag{10}$$

Since there is a unique (up to scalar) equivariant map $B \to V^* \otimes V^*$, it coincides with $B \xrightarrow{\sigma} \wedge^2 V^* \to V^* \otimes V^*$, so we denote it by σ by abuse of notation. Since $Y_4 \otimes Y_4 \simeq Y_8 \oplus Y_6 \oplus Y_4 \oplus Y_2 \oplus Y_0$, the group $\mathrm{Hom}(U(-1), U)$ is a rank-22 vector space isomorphic to $Y_8 \oplus Y_6 \oplus Y_4 \oplus Y_0$. We get a rank-42 bundle by the following right mutation (i.e. $E_{42} = \mathrm{R}_U(U(-1))$)

$$0 \longrightarrow U(-1) \longrightarrow \mathrm{Hom}(U(-1), U) \otimes U \longrightarrow E_{42} \longrightarrow 0 \tag{11}$$

Mutating again gives the following the exact sequence (8). One computes $\mathrm{Hom}(E_9, Q^*) \simeq V \simeq Y_4$ and sees that E_{42} is also the left mutation of Q^* with respect to E_9, i.e. we have an exact sequence

$$0 \longrightarrow E_{42} \longrightarrow Y_4 \otimes E_9 \longrightarrow Q^* \longrightarrow 0 \tag{12}$$

Finally denote by E_6 the bundle given by right mutating E_9 with respect to Q^*

$$0 \longrightarrow E_9 \longrightarrow Y_4 \otimes Q^* \longrightarrow E_6 \longrightarrow 0$$

Since $\Omega^1_{\mathbb{P}^6}(1)_{|V_5}$ is obtained as the kernel

$$0 \longrightarrow \Omega^1_{\mathbb{P}^6}(1)_{|V_5} \longrightarrow \mathrm{Hom}(\mathscr{O}, \mathscr{O}(1)) \otimes \mathscr{O} \simeq Y_6 \otimes \mathscr{O} \longrightarrow \mathscr{O}(1) \longrightarrow 0$$

we get $E_6 \cong \Omega^1_{\mathbb{P}^6}(1)_{|V_5}$ thus obtaining (9). The cycle of left mutations in the exceptional collection $\big(U, Q^*, \mathscr{O}, \mathscr{O}(1)\big)$ then gives the helix condition for $\mathscr{O}(1)$, proving the proposition. This appears also in Orlov's paper [Orl91]. □

4. Resolution of the diagonal

We will write a resolution of the diagonal over V_5 in terms of the collection of Definition 3.2. We will make use of the terminology of derived categories and derived functors, and we refer to [GM96] for definition and basic properties of these categories. So let $\mathrm{D}^b(V_5)$ be the bounded derived category of coherent sheaves on V_5. An object in $\mathrm{D}^b(V_5)$ is represented by a complex $\mathscr{K}$ with finite nonzero cohomology. We write i-th term of $\mathscr{K}$ as $\mathscr{K}^i$ and we can shift $\mathscr{K}$ setting $\mathscr{K}[p]^j = \mathscr{K}^{j+p}$. Also we set $\mathscr{K}^* = \mathbf{R}\mathscr{H}om(\mathscr{K}, \mathscr{O})$.

Theorem 4.1. *The variety V_5 admits the following resolution of the diagonal:*

$$0 \longrightarrow \mathscr{O}(-1,-1) \longrightarrow U \boxtimes \wedge^2 Q^* \longrightarrow Q^* \boxtimes U \longrightarrow \mathscr{O} \longrightarrow \mathscr{O}_\Delta \longrightarrow 0$$

where the arrows are determined as the unique (up to scalar) morphisms invariant for the $\mathsf{SL}(2)$-action. Denote by $d_i : G_{i+1} \boxtimes G^{i+1} \to G_i \boxtimes G^i$ the $-i$-th differential $\mathscr{C}_\Delta^{-i-1} \to \mathscr{C}_\Delta^{-i}$ in the complex $\mathscr{C}_\Delta$ over the diagonal. Then we have

$$\mathscr{C}_\Delta^*[3] \simeq \tau^*(\mathscr{C}_\Delta(-1))$$

where τ is the involution that interchanges factors in $V_5 \times V_5$.

Proof. First notice that such morphisms are indeed unique up to scalar, since they are nonzero and

$$\mathrm{Hom}(\mathscr{O}(-1,-1), U \boxtimes \wedge^2 Q^*) \simeq V^* \otimes V \simeq \mathrm{End}(Y_4)$$
$$\mathrm{Hom}(U \boxtimes \wedge^2 Q^*, Q^* \boxtimes U) \simeq B^* \otimes B \simeq \mathrm{End}(Y_2)$$
$$\mathrm{Hom}(Q^* \boxtimes U, \mathscr{O}) \simeq V \otimes V^* \simeq \mathrm{End}(Y_4)$$

In each case the differential we have defined is the identity element respectively over V or B or V, which is the unique $\mathsf{SL}(2)$-invariant element in the endomorphism groups, since the tensor product decomposition of irreducible $\mathsf{SL}(2)$-modules contains a unique trivial summand. On the other hand

$$\operatorname{Hom}(\mathscr{O}(-1,-1), Q^* \boxtimes U) \simeq Y_6 \otimes Y_4 \qquad \operatorname{Hom}(U \boxtimes \wedge^2 Q^*, \mathscr{O}) \simeq Y_4 \otimes Y_6$$

These modules contain no nontrivial invariant element, hence the composition of invariant maps must be zero.

To prove that the complex is actually exact, we need a more explicit description of such maps. We will provide a fiberwise description over a point $(p, q) \in V_5 \times V_5$. However, the duality of the complex tells us that we need only to prove the exactness in $\mathscr{O}$ and $U \boxtimes \wedge^2 Q^*$. Recall from Lemma 3.3 and Proposition 3.4 the maps $i_{U,\mathscr{O}}$ and $i_{\mathscr{O}(-1),U}$ and consider $i_{Q(-1),U}$ $i_{Q^*,\mathscr{O}}$ and $i_{\mathscr{O}(-1),Q(-1)}$. Then we have the following commutative diagrams

$$d_0 : Q^* \boxtimes U \longrightarrow \mathscr{O} \qquad \begin{array}{ccc} Q_q^* \otimes U_p & & \\ \downarrow{\scriptstyle i_{Q^*,\mathscr{O}} \boxtimes i_{U,\mathscr{O}}} & \overset{d_0}{\searrow} & \\ V \otimes V^* & \xrightarrow[\chi]{} & k \end{array}$$

$$d_1 : U \boxtimes \wedge^2 Q^* \longrightarrow Q^* \boxtimes U \qquad \begin{array}{ccc} U_p \otimes \wedge^2 Q_q^* & & \\ \downarrow{\scriptstyle 1 \boxtimes i_{Q(-1),U}} & \overset{d_1}{\searrow} & \\ U_p \otimes B \otimes U_q & \xrightarrow[\mathrm{ev}_{U,Q^*} \otimes 1]{} & Q_q^* \otimes U_p \end{array}$$

$$d_2 : \mathscr{O}(-1,-1) \longrightarrow U \boxtimes \wedge^2 Q^* \qquad \begin{array}{ccc} \mathscr{O}_p(-1) \otimes \mathscr{O}_q(-1) & & \\ \downarrow{\scriptstyle i_{\mathscr{O}(-1),U} \boxtimes i_{\mathscr{O}(-1),Q(-1)}} & \overset{d_2}{\searrow} & \\ U_p \otimes V \otimes V^* \otimes \wedge^2 Q_q^* & \xrightarrow[1 \otimes \chi \otimes 1]{} & U_p \otimes \wedge^2 Q_q^* \end{array}$$

where the morphism χ here is the natural evaluation $V^* \otimes V \to k$. The definition of d_0 immediately tells that the complex is exact in $\mathscr{O}$ since the evaluation vanishes only when $U_p = Q_q^*$ i.e. for $p = q$. To prove the other step it is useful to give another description of d_2

$$\begin{array}{ccc} \mathscr{O}_p(-1) \otimes \mathscr{O}_q(-1) & \overset{d_2}{\dashrightarrow} & U_p \otimes \wedge^2 Q_q^* \\ \downarrow{\scriptstyle i_{\mathscr{O}(-1),U} \boxtimes i_{\mathscr{O}(-1),U}} & & \downarrow{\scriptstyle 1 \boxtimes i_{Q(-1),U}} \\ U_p \otimes V \otimes V \otimes U_q & \xrightarrow[1 \otimes \sigma^* \otimes 1]{} & U_p \otimes B \otimes U_q \end{array}$$

where $\sigma^* \colon V \otimes V \longrightarrow B \simeq B^*$ is the transpose of the map σ defined in section (2), composed with $V \otimes V \longrightarrow \wedge^2 V$.

All such definitions coincide up to scalar with the previous ones by uniqueness of the invariant map since they produce nonzero maps. From these descriptions one readily gets

$$\ker(d_1) \simeq \wedge^2 Q_p^* \otimes U_q \cap U_p \otimes \wedge^2 Q_q^* \subset U_p \otimes B \otimes U_q$$

On the other hand the image of d_2 is the image in $U_p \otimes B \otimes U_q$ of $1 \otimes \sigma \otimes 1$ restricted to $\mathscr{O}_p(-1) \otimes \mathscr{O}_q(-1)$. This coincides with $\ker(d_1)$ if

$$\begin{aligned} \wedge^2 Q_p^* \otimes U_q &= \operatorname{Im}(1 \otimes \sigma^* \otimes 1) && \text{restricted to } \mathscr{O}_p(-1) \otimes V \otimes U_q \\ U_p \otimes \wedge^2 Q_q^* &= \operatorname{Im}(1 \otimes \sigma^* \otimes 1) && \text{restricted to } U_p \otimes V \otimes \mathscr{O}_q(-1) \end{aligned}$$

Consider the first equality (they are clearly symmetric) and notice that we may factor out the identity over U_q and thus reduce it to

$$\wedge^2 Q^* = \operatorname{Im}(1 \otimes \sigma^*) \qquad \text{restricted to } \mathscr{O}(-1) \otimes V$$

This is provided by the commutativity of the following diagram, since the only equivariant morphism $\mathscr{O}(-1) \otimes V \to \wedge^2 Q^*$ is the map given by the universal quotient (twisted by -1), which is of course surjective.

(13)
$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U(-1) & \xrightarrow{i_{U,\mathscr{O}}} & V \otimes \mathscr{O}(-1) & \xrightarrow{\mathrm{ev}_{\mathscr{O},Q}} & \wedge^2 Q^* & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle 1 \otimes i_{\mathscr{O}(-1),U}} & & \downarrow{\scriptstyle i_{Q(-1),U}} & & \\
 & & & & V \otimes V \otimes U & \xrightarrow[\sigma^* \otimes 1]{} & B \otimes U & \longrightarrow & 0 \\
 & & & & & & \downarrow{\scriptstyle \mathrm{ev}_{U,Q^*}} & & \\
 & & & & & & Q^* & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

□

The immediate consequence of this theorem can be formulated at the level of derived categories. Indeed the functor $\mathbf{R} p_{2*}(\mathscr{C}_\Delta \overset{\mathbf{L}}{\otimes} p_1^*(-))$ associated to the complex $\mathscr{C}_\Delta$ resolving the diagonal (where p_1, p_2 are the two projections $V_5 \times V_5 \to V_5$) is isomorphic the identity in $\mathrm{D}^b(V_5)$. Then for any sheaf F we have a complex $\mathscr{C}_F$, exact except in cohomological degree 0, whose cohomology is F. The i-th term of $\mathscr{C}_F$ is the sum $\oplus_j \mathrm{H}^{j+i}(F \otimes G^j) \otimes G_j$. In order to construct the i-th term of $\mathscr{C}_F$ it is thus sufficient to consider i-th diagonal (where the 0-th diagonal is the middle one and the 3rd (resp. -3rd) is the upper–right (resp. lower–left) corner) of the following cohomology matrix

(14)
$$\begin{pmatrix} \mathrm{h}^3(F \otimes \mathscr{O}(-1)) & \mathrm{h}^3(F \otimes Q(-1)) & \mathrm{h}^3(F \otimes U) & \mathrm{h}^3(F) \\ \mathrm{h}^2(F \otimes \mathscr{O}(-1)) & \mathrm{h}^2(F \otimes Q(-1)) & \mathrm{h}^2(F \otimes U) & \mathrm{h}^2(F) \\ \mathrm{h}^1(F \otimes \mathscr{O}(-1)) & \mathrm{h}^1(F \otimes Q(-1)) & \mathrm{h}^1(F \otimes U) & \mathrm{h}^1(F) \\ \mathrm{h}^0(F \otimes \mathscr{O}(-1)) & \mathrm{h}^0(F \otimes Q(-1)) & \mathrm{h}^0(F \otimes U) & \mathrm{h}^0(F) \end{pmatrix}$$

We say that the collection $(G_3, \ldots, G_0)$ of Definition 3.2 is a *basis* for $\mathrm{D}^b(V_5)$ and that $(G^3, \ldots, G^0)$ is its *dual basis*. Orlov in [Orl91] proves that the exceptional sequence $(U(-1), \mathscr{O}(-1), U, \mathscr{O})$ generates $\mathrm{D}^b(V_5)$ since it satisfies helix conditions.

Corollary 4.2. *Any coherent sheaf F on V_5 is isomorphic to the cohomology of a complex $\mathscr{C}_F^\bullet$ (dually, of a complex $\mathscr{D}_F^\bullet$) whose terms are given by*

$$\mathscr{C}_F^k = \bigoplus_{i-j=k} \mathrm{H}^i(F \otimes G^j) \otimes G_j \qquad \mathscr{D}_F^k = \bigoplus_{i-j=k} \mathrm{H}^i(F \otimes G_j) \otimes G^j$$

The duality in Theorem 4.1 implies $\mathscr{D}_F^\bullet \simeq \mathscr{C}_F^\bullet(-1)^*$.

Remark 4.3. The construction of V_5 as subvariety of $\mathbb{G}(2 \times 3; B^*)$ allows to prove Theorem 4.1 without making use of the $\mathsf{SL}(2)$-action. The diagram (13) can be completed in the following, that summarizes the various mutations (11), (8), (12).

$$\begin{array}{ccccc}
U(-1) & \longrightarrow & V \otimes \mathscr{O}(-1) & \longrightarrow & \wedge^2 Q^* \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(U(-1), U) \otimes U & \longrightarrow & V \otimes V \otimes U & \longrightarrow & B \otimes U \\
\downarrow & & \downarrow & & \downarrow \\
E_{42} & \longrightarrow & V \otimes E_9 & \longrightarrow & Q^*
\end{array}$$

Rows and columns are exact and zeroes all around the diagram are omitted for brevity. The arrows in the central row are just the the sequence (10) tensorized with 1_U and all maps are invariant evaluations.

5. Splitting criterion

We have the following well–known splitting criterion (cfr. [Ott89]).

Proposition 5.1. *Let X be a connected projective variety embedded by $\mathscr{O}_X(1)$ and containing a straight line L with ideal sheaf I_L. Let E be a vector bundle on X. Then E splits as a sum of line bundles if:*

$$\mathrm{H}^1(E \otimes I_L \otimes \mathscr{O}_X(1)^{\otimes t}) = 0 \quad \textit{for any } t \in \mathbb{Z} \tag{15}$$

Proof. Restricting E to L gives

$$0 \longrightarrow E \otimes I_L \longrightarrow E \longrightarrow E_{|L} \longrightarrow 0$$

Let F be the split bundle on X such that $F_{|L} \simeq E_{|L}$. Then consider the exact sequence

$$0 \longrightarrow F^* \otimes E \otimes I_L \longrightarrow F^* \otimes E \longrightarrow F^*_{|L} \otimes E_{|L} \longrightarrow 0$$

Condition (15) gives $\mathrm{H}^1(F^* \otimes E \otimes I_L) = 0$, hence surjectivity in the map

$$\mathrm{H}^0(F^* \otimes E) \longrightarrow \mathrm{H}^0(F^*_{|L} \otimes E_{|L})$$

so the identity of $F_{|L}$ lifts to a morphism of F to E. The determinant of such a morphism is a constant since it lies in $\mathrm{H}^0(\mathscr{O})$. This constant is nonzero by restriction to L, i.e. E is indeed isomorphic to F. □

Proposition 5.2. *A vector bundle E on V_5 splits if and only if, for any $t \in \mathbb{Z}$*

i) $\mathrm{H}^1(E \otimes Q^*(t)) = 0$
ii) $\mathrm{H}^2(E \otimes U(t)) = 0$

Proof. A sum of line bundles obviously satisfies the conditions since $\mathrm{h}^2(U(t)) = 0 = \mathrm{h}^1(Q^*(t))$ for any t.

Viceversa, consider the ideal sheaf of a line in V_5 as the cokernel of a map $b : U \to Q^*$ as in the exact sequence (6). Then the hypothesis implies the condition of Proposition 5.1. □

This proposition can be thought of as an analog of the Horrocks splitting criterion on the projective space $\mathbb{P}^n$ (see [Hor64]), namely that a bundle splits if and only if its cohomology modules other than 0 and n vanish in all degrees. Next we will show how to deduce a generalization of Proposition 5.2 from the Corollary 4.2. By applying convenient mutations to the exceptional collection $\big(\mathscr{O}(-1), U, Q^*, \mathscr{O}\big)$, we will find another resolution of the diagonal over $V_5 \times V_5$, suitable to deduce the splitting criterion.

Lemma 5.3. *The exceptional collection* $\big(G_3, \dots, G_0\big) = \big(\mathscr{O}(-1), U, Q^*, \mathscr{O}\big)$ *can be mutated to the exceptional collection*

$$\big(U(-1), Q^*(-1), \mathscr{O}(-1), \mathscr{O}\big)$$

Proof. This is elementary. The first step is

$$\big(\mathscr{O}(-1), U, Q^*, \mathscr{O}\big) \mapsto \big(\mathscr{O}(-1), \wedge^2 Q^*, U, \mathscr{O}\big)$$

by mutating U and Q^*. This is nothing but passing to the dual collection. Next we mutate $\mathscr{O}(-1)$ and $\wedge^2 Q^*$ to get

$$\big(\mathscr{O}(-1), \wedge^2 Q^*, U, \mathscr{O}\big) \mapsto \big(U(-1), \mathscr{O}(-1), U, \mathscr{O}\big)$$

finally we mutate $\mathscr{O}(-1)$ and U and get

$$\big(U(-1), \mathscr{O}(-1), U, \mathscr{O}\big) \mapsto \big(U(-1), Q^*(-1), \mathscr{O}(-1), \mathscr{O}\big)$$

□

Corollary 5.4. *The variety V_5 admits the mutated resolution of the diagonal*

$$0 \to U(-1) \boxtimes U \to Q^*(-1) \boxtimes Q^* \to \mathscr{O}(-1) \boxtimes \Omega^1_{\mathbb{P}^6}(1)_{|V_5} \to \mathscr{O} \to \mathscr{O}_\Delta \to 0$$

Proof. It suffices to perform the left mutations of the collection $\big(G_3, \dots, G_0\big)$ as in Lemma 5.3 and the appropriate right mutations in the dual collection $\big(G^3, \dots, G^0\big)$. Notice that the first step in Lemma 5.3 allows to switch the basis $\big(G_3, \dots, G_0\big)$ with the dual basis. We need the following right mutations

$$\big(\mathscr{O}(-1), U, Q^*, \mathscr{O}\big) \mapsto \big(U, E_9, Q^*, U, \mathscr{O}\big)$$

and

$$\big(U, E_9, Q^*, U, \mathscr{O}\big) \mapsto \big(U, Q^*, E_6, \mathscr{O}\big)$$

But we have seen in section (3) that $E_6 \simeq \Omega^1_{\mathbb{P}^6}(1)_{|V_5}$. □

This allows to give an algebraic proof and a generalization of the splitting criterion 5.2, in the spirit of [AO91].

Corollary 5.5. *Let E be a sheaf on V_5 such that*

$$\mathrm{h}^1(Q^* \otimes E(t)) = \mathrm{h}^2(U \otimes E(t)) = 0 \qquad \forall\, t \in \mathbb{Z}$$

Then E is isomorphic to a sum of line bundles plus a sheaf E' with $\dim(\operatorname{supp}(E')) = 0$

Proof. By Corollary 5.4, the mutated cohomology table of a given bundle E reads

$$\begin{pmatrix} \star & \star & \star & \star \\ \mathrm{H}^2(U\otimes E(-1)) & \star & \star & \star \\ \star & \mathrm{H}^1(Q^*\otimes E(-1)) & \star & \star \\ \star & \star & \mathrm{H}^0(E(-1)) & \mathrm{H}^0(E) \end{pmatrix}$$

Suppose that the Hilbert polynomial of E is not constant. Then there exists a t_0 such that $\mathrm{h}^0(E(t_0)) \neq 0$ and $\mathrm{h}^0(E(t_0-1)) = 0$. This and the conditions of the hypothesis imply that all terms in the -1 diagonal are 0 for $E(t_0)$. Then $\mathrm{H}^0(E(t_0))\otimes\mathscr{O}$ is a direct summand of $E(t_0)$.

By induction on the leading term of the Hilbert polynomial of E, E is a sum of line bundles plus a sheaf E' whose Hilbert polynomial is constant. Then E' is supported at a finite set of points. □

By mutating again we get the collection $\big(\wedge^2 Q^*(-1), U(-1), \mathscr{O}(-1), \mathscr{O}\big)$. However this gives the same splitting principle since $\mathrm{h}^1(U\otimes E) = \mathrm{h}^2(U(-1)\otimes E^*)$ and $\mathrm{h}^1(Q^*\otimes E) = \mathrm{h}^2(\wedge^2 Q^*(-1)\otimes E^*)$ and obviously E splits if and only if E^* does. The following proposition is in the spirit of [AG99], namely one characterizes U or $\mathscr{O}$ by certain cohomology vanishing.

Proposition 5.6. *Let E be an indecomposable sheaf over V_5 such that* $\dim(\mathrm{supp}(E)) > 0$ *and*

i) $\mathrm{H}^1(U\otimes E(t)) = \mathrm{H}^2(U\otimes E(t)) = 0$ *for all* $t\in\mathbb{Z}$

ii) $\mathrm{H}^1(E(t)) = \mathrm{H}^2(E(t)) = 0$ *for all* $t\in\mathbb{Z}$

Then E is isomorphic to either $U(a)$ or to $\mathscr{O}(a)$ for some $a\in\mathbb{Z}$.

Proof. In case E is a bundle one can prove this with a technique analogous to [AG99]. However here we show a different method. Let us consider the cohomology table of such E for the *dual basis* $\big(G^3,\dots,G^0\big)$, i.e. we consider the complex $\mathscr{D}_E^\bullet$ of Corollary 4.2.

$$\begin{pmatrix} \star & \star & \mathrm{h}^2(Q^*\otimes E) & \star \\ 0 & 0 & \mathrm{h}^2(Q^*\otimes E) & 0 \\ 0 & 0 & \mathrm{h}^1(Q^*\otimes E) & 0 \\ \star & \star & \mathrm{h}^0(Q^*\otimes E) & \mathrm{h}^0(E) \end{pmatrix}$$

Then, the bundle $\mathrm{H}^1(Q^*\otimes E)\otimes U$ is a direct summand of $\mathscr{D}_E^0$ mapping to zero in $\mathscr{D}_E^1$ and with zero differential from $\mathscr{D}_E^{-1}$. Therefore E contains such bundle as a direct summand. Since this happens for any twist of E, we conclude that $\oplus_t \mathrm{H}^1(Q^*\otimes E(t))\otimes U(t)$ is a direct summand of E.

By factoring out such summand, we can suppose $\mathrm{H}^1(Q^*\otimes E(t)) = 0$ for all t. But this condition, together with $\mathrm{H}^2(U\otimes E(t)) = 0$ for all t, implies that E splits by Corollary 5.5.

This proves one statement if E is indecomposable. The converse follows taking cohomology of the symmetrized square of (2)

$$0 \longrightarrow \mathrm{Sym}^2 U \longrightarrow V\otimes U \longrightarrow \wedge^2 V \longrightarrow \wedge^2 Q \longrightarrow 0$$

and using the facts

$$\mathrm{Sym}^2 U \oplus \mathscr{O}(-1) \simeq U \otimes U \qquad \mathrm{Sym}^2 U^* \simeq \mathrm{Sym}^2 U(2)$$

Then $\mathrm{Sym}^2 U$ is aCM as well as $U \otimes U$. For the twist $U \otimes U^*$ it just says that U is exceptional. □

Similar considerations lead to the following two propositions.

Proposition 5.7. *An indecomposable sheaf E over V_5 such that $\dim(\mathrm{supp}(E)) > 0$ is either $\mathscr{O}(a)$ or $Q^*(a)$ if and only if it has no intermediate cohomology and*

$$\mathrm{H}^1(Q \otimes E(t) = \mathrm{H}^2(Q \otimes E(t)) = 0 \qquad \forall t \in \mathbb{Z}$$

Proposition 5.8. *Let E be an indecomposable sheaf over V_5 such that $\dim(\mathrm{supp}(E)) > 0$ and*

i) $\mathrm{H}^1(U \otimes E(t)) = 0$ *for all* $t \in \mathbb{Z}$
ii) $\mathrm{H}^1(E(t)) = \mathrm{H}^2(E(t)) = 0$ *for all* $t \in \mathbb{Z}$

Then E is either $U(a)$ or $\mathscr{O}(a)$ or $Q(a)$ for some $a \in \mathbb{Z}$.

Proof. By looking at the cohomology table as in (5.6), we remove the direct summand $\oplus_t \mathrm{H}^1(Q^* \otimes E(t)) \otimes U(t)$. Then using the dual of the universal sequence (2) tensorized by $E(t)$ we get $\mathrm{h}^2(Q^* \otimes E(t)) = 0$ for all t by the hypothesis.

Now looking again at the cohomology table we see that the module $\oplus_t \mathrm{H}^2(U \otimes E(t)) \otimes Q(t-1)$ is a direct summand of E, since the only cokernel could sit in $\mathrm{H}^2(Q^* \otimes E) \otimes U(t)$. Again factoring out such summand we get that the remaining part splits by Corollary 5.5. □

6. aCM semistable bundles

In this section we focus on *aCM* bundles i.e. with the condition $\mathrm{H}^p(V_5, E \otimes \mathscr{O}(t)) = 0$ for any $t \in \mathbb{Z}$ and $0 < p < 3 = \dim(V_5)$.

Theorem 6.1. *An indecomposable aCM sheaf E over V_5 with $\dim(\mathrm{supp}(E)) > 0$ fits into the following exact sequence*

$$0 \longrightarrow E \longrightarrow F_U \oplus F_Q \oplus F_{\mathscr{O}} \longrightarrow G_U \longrightarrow 0$$

where the bundles F_U, F_Q, $F_{\mathscr{O}}$ and G_U are given by

$$G_U = \bigoplus_{t \in \mathbb{Z}} \mathrm{H}^1(U \otimes E(t)) \otimes U(1-t)$$

$$F_U = \bigoplus_{t \in \mathbb{Z}} \mathrm{H}^1(Q^* \otimes E(t)) \otimes U(t)$$

$$F_Q = \bigoplus_{t \in \mathbb{Z}} \mathrm{H}^2(U \otimes E(t)) \otimes Q(t-1)$$

$$F_{\mathscr{O}} = \bigoplus_{t \in I} \mathrm{H}^0(E(t)) \otimes \mathscr{O}(t) \qquad I = \{t \in \mathbb{Z} |\, \mathrm{h}^0(E(t)) \neq 0 = \mathrm{h}^0(E(t-1))\}$$

Proof. Since $\mathrm{H}^1(U \otimes E(t)) = \mathrm{Ext}^1(U(1-t), E)$ we have a universal extension

$$0 \longrightarrow E \longrightarrow F \longrightarrow \oplus_t \mathrm{H}^1(U \otimes E(t)) \otimes U(1-t) \longrightarrow 0$$

Now F is still aCM because E and U are. Furthermore, since $U \otimes U$ is aCM and we have taken *all extensions* between E and U we get

$$\mathrm{H}^1(F \otimes U(t)) = 0 \qquad \text{for all } t \in \mathbb{Z}$$

Thus the result follows by (5.8). To compute the expression of G_U, F_U, F_Q and $F_{\mathscr{O}}$ it suffices to trace back the proof of Proposition (5.8). $\square$

To sharpen the analysis, we now assume that E is a *semistable bundle* (in the sense of Mumford–Takemoto, with respect to the hyperplane divisor). Recall that for any coherent sheaf F one defines $\mathrm{rk}(F) = \mathrm{rk}(F^\circ)$ where F° is the restriction of F to a Zariski open set where F is locally free. Recall also the definition of slope $\mu(F) = c_1(F)/\mathrm{rk}(F) \in \mathbb{Q}$, writing $c_1(F)$ as an integer as in section (3).

Definition 6.2. A vector bundle E on V_5 is semistable (resp. stable) with respect to $\mathscr{O}(1)$ if, for any coherent subsheaf $F \subset E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$ we have $\mu(F) \leq \mu(E)$ (resp. we have $\mu(F) < \mu(E)$).

For details concerning semistable sheaves we refer for instance to [HL97]. We first *normalize* that E, that is twist it until

$$-r < c_1(E) \leq 0$$

Let us now read the cohomology table (14) of such E. First notice that in the cohomology table of E the last column is zero. Indeed $\mathrm{h}^0(E(-1)) = 0$ and $\mathrm{h}^3(E) = \mathrm{h}^0(E^*(-2)) = 0$ by semistability. In the first column, the only nonzero term can be $\mathrm{h}^0(E)$. Now recall that the tensor product of semistable bundles is semistable (if they are both stable bundles one can use Hermite–Einstein metrics, otherwise see [Mar81, Theorem 1.14]). Thus $E \otimes U$ and $E \otimes Q$ are semistable, so we get

$$\begin{aligned} &\mathrm{h}^0(E \otimes U) = 0 && \mathrm{h}^3(E \otimes U) = 0 \\ &\mathrm{h}^0(E \otimes Q(-1)) = 0 && \mathrm{h}^3(E \otimes Q(-1)) = 0 \end{aligned}$$

Then the cohomology table is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{h}^2(Q \otimes E(-1)) & \mathrm{h}^2(U \otimes E) & 0 \\ 0 & \mathrm{h}^1(Q \otimes E(-1)) & \mathrm{h}^1(U \otimes E) & 0 \\ 0 & 0 & 0 & \mathrm{h}^0(E) \end{pmatrix}$$

Then E is the cohomology of the complex

$$0 \longrightarrow \mathrm{H}^1(Q \otimes E(-1)) \otimes U \longrightarrow \begin{matrix} \mathrm{H}^2(Q \otimes E(-1)) \otimes U \\ \oplus \\ \mathrm{H}^1(U \otimes E) \otimes Q^* \\ \oplus \\ \mathrm{H}^0(U) \otimes \mathscr{O} \end{matrix} \longrightarrow \mathrm{H}^1(U \otimes E) \otimes Q^* \longrightarrow 0$$

But in the above complex there are no nonzero maps bundles with the same base i.e. there are no vertical arrows in the spectral sequence associated to complex $\mathscr{C}_\Delta$. This means that E splits into the direct sum of two bundles E_1 and E_2 defined by

$$0 \to \mathrm{H}^1(E \otimes Q(-1)) \otimes U \to \mathrm{H}^1(E \otimes U) \otimes Q^* \oplus \mathrm{H}^0(E) \otimes \mathscr{O} \to E_1 \to 0$$

$$0 \to E_2 \to \mathrm{H}^2(E \otimes Q(-1)) \otimes U \to \mathrm{H}^2(E \otimes U) \otimes Q^* \oplus \mathrm{H}^0(E) \otimes \mathscr{O} \to 0$$

If we look at the cohomology of E a little more carefully, we see that

$$\mathrm{h}^2(U \otimes E) = \mathrm{h}^1(U(-1) \otimes E^*) = \mathrm{h}^0(Q(-1) \otimes E^*)$$

thus semistability tells that

$$\mathrm{h}^2(U \otimes E) = 0 \qquad \text{if} \qquad c_1(E) > -\frac{2}{3}\,\mathrm{rk}(E)$$

Analogously we have $\mathrm{h}^2(Q(-1) \otimes E) = \mathrm{h}^1(Q^*(-1) \otimes E^*) = \mathrm{h}^0(U \otimes E^*)$ so that

$$\mathrm{h}^2(Q(-1) \otimes E) = 0 \qquad \text{if} \qquad c_1(E) > -\frac{1}{2}\,\mathrm{rk}(E)$$

and notice that if a normalized bundle E does not satisfy the above, then $E^*(-1)$ does, except the case $c_1(E) = -\frac{1}{2}\,\mathrm{rk}(E)$. On the other hand if $c_1(E) = -\frac{1}{2}\,\mathrm{rk}(E)$, the resolution of E provided above tells that E is a sum of copies of U plus the cokernel of a map $U^a \to (Q^*)^b$ for certain integers a and b. But $2c_1(E) = -\,\mathrm{rk}(E)$ implies $b = 0$, so that E is a sum of copies of U. Summing up we can state the following theorem

Theorem 6.3. *Up to dualizing and twisting, any aCM semistable bundle E of over V_5 is a sum of copies of U plus the cokernel E'*

$$0 \longrightarrow U^a \longrightarrow (Q^*)^b \oplus \mathscr{O}^c \longrightarrow E' \longrightarrow 0$$

and we have

i) E stable and $\mathrm{rk}(E) > 1$ imply $c = 0$;
ii) $c_1(E) \neq 0$ implies $c = 0$;
iii) $c_1(E) = -\frac{1}{2}\,\mathrm{rk}(E)$ implies $E \simeq \mathrm{H}^1(Q^* \otimes E) \otimes U$;

where the above integers are

$$a = \mathrm{h}^1(Q(-1) \otimes E) \qquad b = \mathrm{h}^1(U \otimes E) \qquad c = \mathrm{h}^0(E)$$

Corollary 6.4. *The moduli space of semistable aCM bundles over V_5 with fixed invariants is unirational. So is the versal deformation space of any aCM bundle over V_5.*

Remark 6.5. One would get the same result for a normalized bundle E with nonzero degree replacing the semistability hypothesis with the assumption that

$$\mathrm{h}^0(E) = 0 \qquad \mathrm{h}^3(E(-1)) = 0$$

Remark 6.6. Arrondo and Costa classified rank-2 aCM bundles over Fano threefolds of index 2 in [AC00] by Hartshorne–Serre correspondence. There are three families of such bundles on any threefold as above, related respectively to a line, a conic and a projectively normal elliptic curve in the threefold. On V_5 we have the following cases.

i) E_L, the strictly semistable bundle whose unique section vanishes along a line L. Then we clearly have

$$M(2; 0, 1) \simeq \mathbb{P}^2$$

ii) E_C the stable bundle given by a conic. It is exactly U and has no moduli.
iii) E_S the stable bundle corresponding to a degree-7 elliptic curve S. $c_1(E_S) = 0$ and $c_2(E_S) = 2$. The moduli space of E_S is five–dimensional.

The resolutions of these bundles are respectively

$$0 \longrightarrow U \longrightarrow Q^* \oplus \mathscr{O} \longrightarrow E_L \longrightarrow 0 \qquad 0 \longrightarrow U^{\oplus 2} \longrightarrow Q^{*\,\oplus 2} \longrightarrow E_S \longrightarrow 0$$

The following lemma allows to recover Arrondo and Costa's classification of aCM bundles in rank 2 quoted in Remark (6.6).

Lemma 6.7. *An indecomposable aCM rank-2 bundle E over V_5 is semistable. In particular, the only families of such bundles are the ones listed in Remark (6.6).*

Proof. Take the first twist t such that E has sections. Consider such section and its vanishing locus Z. The subvariety Z cannot be empty because $\mathrm{h}^1(\mathscr{O}(t)) = 0$ and E is indecomposable. Moreover Z cannot have divisorial components since we took the first twist with sections. Thus Z is a curve in V_5 with ideal sheaf I_Z

$$0 \to \mathscr{O} \to E(t) \to I_Z \otimes \mathscr{O}(c_1(E(t))) \to 0$$

Now since E is aCM we find $\mathrm{h}^1(I_Z) = 0$. This implies that Z is connected since $\mathrm{h}^0(\mathscr{O}_Z) = 1 + \mathrm{h}^1(I_Z)$. But if E is not semistable the Chern class $c = c_1(E(t))$ must be strictly negative so the dualizing sheaf of Z is

$$\omega_Z \simeq K_{V_5} \otimes \det(E(t))_{|Z} \simeq \mathscr{O}(-2+c)_{|Z}$$

Then the degree of the dualizing sheaf of the connected curve Z would be strictly less than two, which is impossible.

The last claim follows from Theorem (6.3). Indeed if $c_1(E) = -1$ we get $b = 0$, so that $E \simeq U$. If $c_1(E) = 0$ we have, besides $\mathscr{O} \oplus \mathscr{O}$, the two cases $c = 1$, $c = 0$, corresponding respectively to (i) and (iii) of Remark (6.6). □

If we denote $A_S = \mathrm{H}^1(U \otimes E_S)$ and $B_S = \mathrm{H}^1(Q \otimes E_S(-1))$ then one has the exact sequence

$$0 \longrightarrow B_S \longrightarrow B \otimes A_S \longrightarrow \mathrm{H}^1(Q^* \otimes E_S) \longrightarrow 0$$

which gives the dual presentation of E_S

$$0 \longrightarrow E_S \longrightarrow U^4 \longrightarrow Q(-1)^2 \longrightarrow 0$$

and we also recover the exact sequence

$$0 \to k \to B_S \otimes B_S \to A_S \otimes \mathrm{H}^1(Q^* \otimes E_S) \to \mathrm{H}^1(E_S \otimes E_S) \to 0$$

identifying the tangent to the moduli space. Notice that a degree-7 elliptic curve is given by an element of $\mathrm{H}^0(E(1)) \simeq k^{10}$. This agrees with the dimension (given by Riemann–Roch) of the Hilbert scheme $\dim(\mathrm{Hilb}_{7t}(V_5)) = 14$, a $\mathbb{P}^9$ fibration over the 5-dimensional moduli space. Since $\mathrm{Ext}^1(E_S, E_S) \simeq k^5$ there are nontrivial extensions

$$0 \longrightarrow E_S \longrightarrow E_4^S \longrightarrow E_S \longrightarrow 0$$

Then we have a rank-4 aCM bundle E_4^S and its resolution reads

$$0 \longrightarrow U^{\oplus 4} \longrightarrow Q^{*\,\oplus 4} \longrightarrow E_4^S \longrightarrow 0$$

The map $U^{\oplus 4} \to Q^{*\,\oplus 4}$ deforms the diagonal map (φ, φ) where $\varphi\colon U^{\oplus 2} \to Q^{*\,\oplus 2}$.

Finally, to show the effectiveness of the method, we classify aCM semistable bundles in rank 3.

Proposition 6.8. *Let E be an aCM semistable normalized rank-3 bundle over V_5. Then E is isomorphic to one of the following*

i) The dual universal Q^* or the twisted universal $Q(-1)$;
ii) $E^{(0,3)} \simeq \mathscr{O}^{\oplus 3}$;
iii) $E^{(1,2)} \simeq \mathscr{O} \oplus E_L$;
iv) $E^{(2,1)} \simeq \mathscr{O} \oplus E_S$;
v) $E^{(3,0)} \simeq \operatorname{Sym}^2 U(-1)$ or a deformation of $\operatorname{Sym}^2 U(-1)$. The deformation space of $\operatorname{Sym}^2 U(-1)$ has dimension 10.

Proof. The first case is $c_1(E) = -1$. Then by

$$0 \longrightarrow U^a \longrightarrow (Q^*)^b \longrightarrow E \longrightarrow 0$$

we get $3 = \operatorname{rk}(E) = 3b - 2a$, and $-1 = c_1(E) = -b + a$. Then $b = 1$ and $a = 0$, that is $E = Q^*$. Clearly if $c_1(E) = -2$, then $c_1(E^*(-1)) = -1$, therefore $E = Q(-1)$.

Now look at the case $c_1(E) = 0$. Then we have

$$0 \longrightarrow U^a \longrightarrow (Q^*)^b \oplus \mathscr{O}^c \longrightarrow E \longrightarrow 0$$

Here we have $a = b$ and the rank 3 says we have only the 4 cases

$$(a, c) \in \{(0,3), (1,2), (2,1), (3,0)\}$$

The first is the obvious $E \simeq \mathscr{O}^3$. Cases $(1,2)$ and $(2,1)$ are described by

$$\begin{array}{ccccc} & & \mathscr{O} & = & \mathscr{O} \\ & & \downarrow & & \downarrow \\ U & \longrightarrow & Q^* \oplus \mathscr{O}^2 & \longrightarrow & E^{(1,2)} \\ \| & & \downarrow & & \downarrow \\ U & \longrightarrow & Q^* \oplus \mathscr{O} & \longrightarrow & E_L \end{array} \qquad \begin{array}{ccccc} & & \mathscr{O} & = & \mathscr{O} \\ & & \downarrow & & \downarrow \\ U^2 & \longrightarrow & (Q^*)^2 \oplus \mathscr{O} & \longrightarrow & E^{(2,1)} \\ \| & & \downarrow & & \downarrow \\ U^2 & \longrightarrow & (Q^*)^2 & \longrightarrow & E_S \end{array}$$

So $E^{(1,2)} \simeq E_L \oplus \mathscr{O}$ and $E^{(2,1)} \simeq E_S \oplus \mathscr{O}$ since E_L and E_S have no H^1. The last possibility is

$$0 \longrightarrow U^3 \xrightarrow{\ \varphi\ } (Q^*)^3 \longrightarrow E \longrightarrow 0 \tag{16}$$

This corresponds to $\operatorname{Sym}^2 U$. First normalize to $\operatorname{Sym}^2 U(1)$, $c_1(\operatorname{Sym}^2 U(1)) = 0$. Then recall from Proposition (5.6) that $\operatorname{Sym}^2 U$ is indeed aCM. Then using the dual of sequences (7) and (2), plus the symmetric square of (2), one computes

$$0 \longrightarrow Y_4 \longrightarrow Y_4 \oplus Y_2 \longrightarrow \mathrm{H}^1(U \otimes \operatorname{Sym}^2 U(1)) \longrightarrow 0$$

All the maps here are invariant so $\mathrm{H}^1(U \otimes \operatorname{Sym}^2 U(1)) \simeq Y_2$. Furthermore, notice that

$$\begin{aligned} \mathrm{H}^1(Q(-1) \otimes \operatorname{Sym}^2 U(1)) &\simeq \mathrm{H}^2(U(-2) \otimes \operatorname{Sym}^2 U(2)) \simeq \\ &\simeq \mathrm{H}^1(U \otimes \operatorname{Sym}^2 U(1))^* \simeq Y_2 \end{aligned}$$

Clearly $\mathrm{Sym}^2 U$ is stable. Now if we put the invariant map $\varphi_0 \in Y_2 \otimes Y_2 \otimes Y_2$ we get the resolution

$$0 \longrightarrow U^3 \xrightarrow{\varphi_0} (Q^*)^3 \longrightarrow \mathrm{Sym}^2 U(1) \longrightarrow 0$$

Hence we generically get a stable aCM bundle as image of φ with the same invariants as $\mathrm{Sym}^2(U)(1)$. One may also notice that the copresentation of $E(-1)$ is

$$0 \longrightarrow E(-1) \longrightarrow U^6 \longrightarrow (Q^*)^3 \longrightarrow 0$$

In the case of $\mathrm{Sym}^2 U$ the above arrow is an invariant element in $Y_2 \otimes Y_2 \otimes (Y_4 \oplus k)$. By taking non invariant maps we choose a deformation of $\mathrm{Sym}^2 U(-1)$. The last statement follows, since tensorizing (16) by $\mathrm{Sym}^2 U(-1)$ and taking cohomology yields $\mathrm{h}^1(\mathrm{End}(\mathrm{Sym}^2 U)) = 10$. □

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Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67/a, I–50134, Florence, Italy

E-mail address: `faenzi@math.unifi.it`

URL: `http://www.math.unifi.it/~faenzi/`