ULRICH BUNDLES ON K3 SURFACES

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Abstract. We show that any polarized K3 surface supports special Ulrich bundles of rank 2.

Given an n-dimensional closed subvariety $X \subset \mathbb{P}^N$, a coherent sheaf $\mathcal{F}$ on X is Ulrich if $\mathrm{H}^*(\mathcal{F}(-t)) = 0$ for $1 \le t \le n$. We refer to [Cos17, Bea18] for an introduction. We mention that Ulrich sheaves are related to Chow forms (this was the main motivation for they study in [ESW03]), to determinantal representations and generalized Clifford algebras, to Boij-Söderberg theory (cf. [SE10]) to the minimal resolution conjecture, and to the representation type of varieties (cf. [FP15]).

Conjecturally, Ulrich sheaves exist for any X, see [ESW03]. They are known to exist for several classes of varieties e.g. complete intersections, curves, Veronese, Segre, Grassmann varieties. Low-rank Ulrich bundles on surfaces have been studied intensively, and Ulrich bundles of rank 2 (or sometimes 1) are known in many cases. We refer to [Cas17, Bea18] for a survey and further references. Let us only review some of the cases that are most relevant for us, namely among surfaces with trivial canonical bundle.

In [Bea16], Ulrich bundles of rank 2 are proved to exist on abelian surfaces. In [AFO17], it is proved that K3 surfaces support Ulrich bundles of rank 2, provided that some Noether-Lefschetz open condition is satisfied. The case of quartic surfaces was previously analyzed in detail in [CKM12]. The main techniques used so far are the Serre construction starting from points on X and Lazarsfeld-Mukai bundles.

In this note, we show that any K3 surface supports an Ulrich bundle $\mathcal{E}$ of rank 2 with $c_1(\mathcal{E}) = 3H$, for any polarization H. So these bundles are *special*, cf. [ESW03]. We allow singular surfaces with trivial canonical bundle. The main tool is an enhancement of Serre's construction based on unobstructedness of simple sheaves on a $K3$ surface.

Let us state the result more precisely. We work over an algebraically closed field $\boldsymbol{k}$. Let X be an integral (i.e. reduced and irreducible) projective surface with $\omega_X \simeq \mathcal{O}_X$ and $\mathrm{H}^1(\mathcal{O}_X) = 0$. We denote by X_{sm} the smooth locus of X.

Fix a very ample divisor H on X. Under the closed embedding given by the complete linear series $|\mathcal{O}_X(H)|$ we may view X as a subvariety of some projective space $\mathbb{P}^g$. A hyperplane section C of X is a projective Gorenstein curve of arithmetic genus g with $\omega_C \simeq \mathcal{O}_C(H)$, where H also denotes the restriction of H to C. We may choose C to be integral too.

A locally Cohen-Macaulay sheaf $\mathcal{E}$ on X is arithmetically Cohen-Macaulay (ACM) if $\mathrm{H}^1(\mathcal{E}(tH)) = 0$ for all $t \in \mathbb{Z}$. A special class of ACM sheaves are Ulrich sheaves, which are characterized by the property $\mathrm{H}^*(\mathcal{E}(-tH)) = 0$ for $t = 1, 2$. Of course all these notions depend on the polarization H. We call simple a sheaf whose only endomorphisms are homotheties.

Theorem 1. *Let X and H be as above. Then there exists a simple Ulrich vector bundle of rank 2 on X whose determinant is $\mathcal{O}_X(3H)$.*

The strategy to prove the theorem is the following. First we build an ACM vector bundle $\mathcal{E}$ of rank 2 by Serre's construction applied to a projective coordinate system in X. Then we perform

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an elementary modification of $\mathcal{E}$ along a single generic point $p \in X$, producing a simple non-reflexive sheaf having the Chern character of an Ulrich bundle. Finally we flatly deform such sheaf and check that generically this yields the desired Ulrich bundle.

Prior to all this, we start by observing that the trivial bundle is a (trivial) example of ACM line bundle. Indeed, using that $\mathrm{H}^1(\mathcal{O}_X) = 0$ and that C is connected, one checks that $\mathrm{H}^1(\mathcal{O}_X(-H)) = 0$. In turn, this easily implies $\mathrm{H}^1(\mathcal{O}_X(-tH)) = 0$ for all $t \geq 2$. Also, Serre duality and triviality of ω_X give $\mathrm{H}^1(\mathcal{O}_X(tH)) = 0$ for all $t \geq 0$. This way, we see that $\mathcal{O}_X$ is an ACM line bundle on X. Combining this with Max Noether's theorem on the generation of the canonical ring of curves (cf. [Ros52] for a version for Gorenstein curves) one obtains, working as in [SD74, Theorem 6.1], that $X \subset \mathbb{P}^g$ is an ACM surface of degree $2g - 2$.

However this line bundle is never Ulrich, nor is any line bundle of the form $\mathcal{O}_X(dH)$. So generically (for instance when X has Picard number 1) the surface X will not support Ulrich line bundles. We thus move to rank two and start by constructing a simple ACM bundle.

Lemma 1. *Let $Z \subset X_{\mathrm{sm}}$ be a set of $g + 2$ points in general linear position. Then there is a unique coherent sheaf $\mathcal{E}$ of rank 2 fitting into a non-splitting exact sequence:*

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{I}_Z(H) \to 0. \tag{1}$$

The sheaf $\mathcal{E}$ is locally free, simple and ACM. It satisfies:

$$\mathcal{E} \simeq \mathcal{E}^*(H), \qquad \mathrm{h}^0(\mathcal{E}) = 1, \qquad \mathrm{h}^1(\mathcal{E}) = \mathrm{h}^2(\mathcal{E}) = 0 \qquad \mathrm{ext}^1_X(\mathcal{E}, \mathcal{E}) = 2g + 4.$$

Proof. Taking cohomology of the exact sequence

$$0 \to \mathcal{I}_Z(H) \to \mathcal{O}_X(H) \to \mathcal{O}_Z \to 0, \tag{2}$$

and using the fact that Z is in general linear position and hence contained in no hyperplane, we get $\mathrm{H}^0(\mathcal{I}_Z(H)) = 0$ and $\mathrm{h}^1(\mathcal{I}_Z(H)) = 1$.

By Serre duality we get $\mathrm{ext}^1_X(\mathcal{I}_Z(H), \mathcal{O}_X) = \mathrm{h}^1(\mathcal{I}_Z(H)) = 1$ so, up to proportionality, there is a unique non-splitting extension of the desired form. Correspondingly, there exists a unique coherent sheaf $\mathcal{E}$ of rank two fitting into a non-splitting exact sequence of the form (1). The sheaf $\mathcal{E}$ we obtain this way satisfies $\mathrm{h}^0(\mathcal{E}) = 1$ and $\mathrm{H}^1(\mathcal{E}) \simeq \mathrm{Ext}^1_X(\mathcal{E}, \mathcal{O}_X)^* = 0$ because applying $\mathrm{Hom}_X(-, \mathcal{O}_X)$ to (1) we obtain a non-zero map (and thus an isomorphism) $\mathrm{H}^0(\mathcal{O}_X) \to \mathrm{Ext}^1_X(\mathcal{I}_Z(H), \mathcal{O}_X)$.

This map is the dual of the homomorphism $\mathrm{H}^1(\mathcal{I}_Z(H)) \to \mathrm{H}^2(\mathcal{O}_X)$ obtained by taking global sections in (1). So $\mathrm{H}^1(\mathcal{E}) = \mathrm{H}^2(\mathcal{E}) = 0$.

If X is smooth we deduce that $\mathcal{E}$ is locally free from the Cayley-Bacharach property, cf. for instance [HL97, Theorem 5.1.1]. Indeed, since Z is in general linear position (i.e. Z is a projective frame in $\mathbb{P}^g$), no hyperplane passes through any subset of $g+1$ points of Z. Anyway the statement follows in general by a minor modification of the argument appearing in [FP15, Lemma 7.2]. Indeed by the local-to-global spectral sequence, using $\mathrm{H}^1(\mathcal{O}_X(-H)) = 0$ and $\mathcal{H}om_X(\mathcal{I}_Z(H), \mathcal{O}_X) \simeq \mathcal{O}_X(-H)$ we get the following exact sequence:

$$0 \to \mathrm{Ext}^1_X(\mathcal{I}_Z(H), \mathcal{O}_X) \to \mathrm{H}^0(\mathcal{E}xt^1_X(\mathcal{I}_Z(H), \mathcal{O}_X)) \to \mathrm{H}^2(X, \mathcal{O}_X(-H)) \to 0.$$

In turn, using $\mathcal{E}xt^1_X(\mathcal{I}_Z(H), \mathcal{O}_X) \simeq \omega_Z \simeq \mathcal{O}_Z$ and $\mathrm{H}^2(X, \mathcal{O}_X(-H)) \simeq \mathrm{H}^0(X, \mathcal{O}_X(H))^*$, if we choose Z to be a projective coordinate system of $\mathbb{P}^g$, we rewrite this exact sequence as:

$$0 \to \mathrm{Ext}^1_X(\mathcal{I}_Z(H), \mathcal{O}_X) \to \mathrm{H}^0(\mathcal{O}_Z) \xrightarrow{\begin{pmatrix} 1 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 1 \end{pmatrix}} \mathrm{H}^0(X, \mathcal{O}_X(H))^* \to 0.$$

So $\mathrm{Ext}^1_X(\mathcal{I}_Z(H), \mathcal{O}_X)$ is generated by the vector $(1, \dots, 1, -1)^t$ and since this vector corresponds to an extension in $\mathcal{E}xt^1_X(\mathcal{I}_Z(H), \mathcal{O}_X)$ which is non-zero at any point of Z we have that the sequence defining $\mathcal{E}$ is locally non-split around each point of Z, which in turn implies that $\mathcal{E}$ is locally free at each such point (and hence everywhere). From $c_1(\mathcal{E}) = H$, since $\mathcal{E}$ is locally free of rank 2, we get a canonical isomorphism $\mathcal{E} \simeq \mathcal{E}^*(H)$.

Let us prove that $\mathcal{E}$ is ACM. We already have $h^1(\mathcal{E}) = 0$ and thus by Serre duality $h^1(\mathcal{E}(-H)) = h^1(\mathcal{E}^*(H)) = h^1(\mathcal{E}) = 0$. Also $h^0(\mathcal{E}(-H)) = 0$ and $h^2(\mathcal{E}(-H)) = 1$. Note that, choosing an integral hyperplane section curve C that avoids Z, (1) becomes:

$$0 \to \mathcal{O}_C \to \mathcal{E}|_C \to \mathcal{O}_C(H) \to 0.$$

From $H^k(\mathcal{E}(-H)) = 0$ for $k = 0, 1$ we deduce $h^0(\mathcal{E}|_C) = 1$ so the previous exact sequence does not split. Then $h^0(\mathcal{E}|_C(-H)) = 0$. This easily implies $H^1(\mathcal{E}(-2H)) = 0$ and actually $H^1(\mathcal{E}(-tH)) = 0$ for all $t \geq 2$. Serre duality now gives $H^1(\mathcal{E}(tH)) = 0$ for all $t \geq 1$. In other words $\mathcal{E}$ is ACM.

It remains to check that $\mathcal{E}$ is simple. Applying $\mathrm{Hom}_X(\mathcal{E}, -)$ to the exact sequence (2) we get that the non-zero space $\mathrm{Hom}_X(\mathcal{E}, \mathcal{I}_Z(H))$ is contained in $\mathrm{Hom}_X(\mathcal{E}, \mathcal{O}_X(H)) \simeq H^0(\mathcal{E}) \simeq \boldsymbol{k}$, so $\hom_X(\mathcal{E}, \mathcal{I}_Z(H)) = 1$. As $\mathcal{H}om_X(\mathcal{E}, \mathcal{O}_Z)$ is a skyscraper sheaf of rank 2 at Z we have $\mathrm{ext}^k_X(\mathcal{E}, \mathcal{O}_Z) = (2g + 4)\delta_{0,k}$. We deduce $\mathrm{ext}^1_X(\mathcal{E}, \mathcal{I}_Z(H)) = 2g + 4$ and $\mathrm{ext}^0_X(\mathcal{E}, \mathcal{I}_Z(H)) = 0$.

Therefore, applying $\mathrm{Hom}_X(\mathcal{E}, -)$ to the (1), since $\mathrm{Hom}_X(\mathcal{E}, \mathcal{O}_X) \simeq h^2(\mathcal{E}) = 0$ we get that $\mathrm{End}_X(\mathcal{E})$ is contained in $\mathrm{Hom}_X(\mathcal{E}, \mathcal{I}_Z(H))$ and is therefore 1-dimensional. This says that $\mathcal{E}$ is simple. By Serre duality $\mathrm{ext}^2_X(\mathcal{E}, \mathcal{E}) = 1$. We deduce $\mathrm{ext}^1_X(\mathcal{E}, \mathcal{E}) = \mathrm{ext}^1_X(\mathcal{E}, \mathcal{I}_Z(H)) = 2g + 4$. □

Given a reduced subscheme $Z \in \mathrm{Hilb}_{g+2}(X_{\mathrm{sm}})$ consisting of points in general linear position, there is a unique rank-2 bundle associated with Z according to the previous lemma. We denote it by $\mathcal{E}_Z$. We write $\mathcal{O}_p$ for the skyscraper sheaf of a point $p \in X$.

Lemma 2. *Assume $\eta : \mathcal{E}_Z \to \mathcal{O}_p$ is surjective. Then $\mathcal{E}^\eta = \ker(\eta)$ is a simple sheaf with:*

$$c_1(\mathcal{E}^\eta) = H, \qquad c_2(\mathcal{E}^\eta) = g + 3, \qquad \mathrm{ext}^1_X(\mathcal{E}^\eta, \mathcal{E}^\eta) = 2g + 8.$$

Proof. Recall that $\mathcal{E} = \mathcal{E}_Z$ is simple and observe that this implies $\mathrm{Hom}_X(\mathcal{E}, \mathcal{E}^\eta) = 0$, as the composition of any non-zero map $\mathcal{E} \to \mathcal{E}^\eta$ with $\mathcal{E}^\eta \hookrightarrow \mathcal{E}$ would provide a self-map of $\mathcal{E}$ which is not a multiple of the identity. Also, since $\mathcal{E}$ is locally free we have $\hom_X(\mathcal{E}, \mathcal{O}_p) = 2$ and $\mathrm{Ext}^k_X(\mathcal{E}, \mathcal{O}_p) = 0$ for $k > 0$. Therefore, using Lemma 1 and applying $\mathrm{Hom}_X(\mathcal{E}, -)$ to the exact sequence:

$$0 \to \mathcal{E}^\eta \to \mathcal{E} \to \mathcal{O}_p \to 0. \tag{3}$$

we obtain $\mathrm{ext}^1_X(\mathcal{E}, \mathcal{E}^\eta) = 2g + 5$ and $\mathrm{ext}^2_X(\mathcal{E}, \mathcal{E}^\eta) = 1$.

Next, Serre duality gives $\mathrm{ext}^k_X(\mathcal{O}_p, \mathcal{E}) = 2\delta_{2,k}$, while $\mathrm{ext}^k_X(\mathcal{O}_p, \mathcal{O}_p)$ is the dimension of the k-th exterior power of the normal bundle of p in X and thus takes value $\binom{2}{k}$. Therefore, applying $\mathrm{Hom}_X(\mathcal{O}_p, -)$ to (3) we find $\mathrm{ext}^1_X(\mathcal{O}_p, \mathcal{E}^\eta) = 1$ and $\mathrm{ext}^2_X(\mathcal{O}_p, \mathcal{E}^\eta) = 3$. Putting these computations together and applying $\mathrm{Hom}_X(-, \mathcal{E}^\eta)$ again to (3) we get:

$$\hom_X(\mathcal{E}^\eta, \mathcal{E}^\eta) = \mathrm{ext}^2_X(\mathcal{E}^\eta, \mathcal{E}^\eta) = 1, \qquad \mathrm{ext}^1_X(\mathcal{E}^\eta, \mathcal{E}^\eta) = 2g + 8.$$

The computation of Chern classes is straightforward. □

Lemma 3. *Let $p \in X_{\mathrm{sm}} \setminus Z$. Then, for a generic map $\eta : \mathcal{E}_Z \to \mathcal{O}_p$, the induced map on global sections $H^0(\eta) : H^0(\mathcal{E}_Z) \to H^0(\mathcal{O}_p)$ is an isomorphism.*

Proof. Put $\mathcal{E} = \mathcal{E}_Z$. It suffices to check that there exists η such that the induced map $H^0(\eta) : \boldsymbol{k} \simeq H^0(\mathcal{E}) \to H^0(\mathcal{O}_p) \simeq \boldsymbol{k}$ is an isomorphism, for this is an open condition. To do it, we apply $\mathrm{Hom}_X(\mathcal{I}_Z(H), -)$ to the exact sequence:

$$0 \to \mathcal{I}_p \to \mathcal{O}_X \to \mathcal{O}_p \to 0.$$

This gives an exact sequence:

$$\mathrm{Ext}^1_X(\mathcal{I}_Z(H), \mathcal{I}_p) \to \mathrm{Ext}^1_X(\mathcal{I}_Z(H), \mathcal{O}_X) \to \mathrm{Ext}^1_X(\mathcal{I}_Z(H), \mathcal{O}_p).$$

Observe that $\mathcal{H}om_X(\mathcal{I}_Z(H), \mathcal{O}_p) \simeq \mathcal{O}_p$ and $\mathcal{E}xt^1_X(\mathcal{I}_Z(H), \mathcal{O}_p) = 0$ as these sheaves are computed locally on X and, since $p \cap Z = \emptyset$, we may choose an open cover of X consisting of subsets where $\mathcal{I}_Z$ is trivial or $\mathcal{O}_p$ vanishes. Then the local-to-global spectral sequence gives $\mathrm{Ext}^1_X(\mathcal{I}_Z(H), \mathcal{O}_p) = 0$ so the extension corresponding to (1) admits a lifting to $\mathcal{I}_p$. In other words, we get the commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{I}_p & \longrightarrow & \mathcal{E}^\eta & \longrightarrow & \mathcal{I}_Z(H) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{I}_Z(H) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \eta} & & & & \\
 & & \mathcal{O}_p & = & \mathcal{O}_p & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where η and $\mathcal{E}^\eta$ are defined by the diagram. For this choice of η we get, by the top row of the diagram, $\mathrm{H}^0(\mathcal{E}^\eta) = 0$, which implies that $\mathrm{H}^0(\eta)$ is an isomorphism. □

By the previous lemma, we may choose $\mathcal{E}_Z$ as in Lemma 1, a point $p \in X_{\mathrm{sm}} \setminus Z$, some $\eta : \mathcal{E}_Z \twoheadrightarrow \mathcal{O}_p$ and consider the sheaf $\mathcal{E}^\eta$. The goal is to deform $\mathcal{E}^\eta(H)$ to an Ulrich bundle. We use the notation $\mathcal{F}_s^*$ for $(\mathcal{F}_s)^*$ (which is a priori not the same as $(\mathcal{F}^*)_s$).

Lemma 4. *There exist a smooth connected variety S_0 of dimension $2g+8$ and a flat family of simple sheaves $\mathcal{F}$ on $X \times S_0$ such that $\mathcal{F}_s(H)$ is an Ulrich bundle for s generic in S_0 and $\mathcal{F}_{s_0} \simeq \mathcal{E}^\eta$ for some distinguished point s_0 of S_0.*

Proof. We proved in Lemma 2 that $\mathcal{E}^\eta$ is simple. Since the non-locally free locus of $\mathcal{E}^\eta$ is disjoint from the singular locus of X, we may apply the arguments of [Muk84, Theorem 0.1]. In particular (cf. [AK80]) the moduli functor of simple sheaves on X is pro-represented by a moduli space Spl_X which can be constructed in the étale topology and which is smooth of dimension $2g + 8$ at $\mathcal{E}^\eta$ (this is essentially [Muk84, Theorem 0.3]). Therefore there exists an open piece of Spl_X which is a quasi-projective variety S equipped with a flat family $\mathcal{F}$ of simple sheaves on X, such that the induced map $S \to \mathrm{Spl}_X$ is a local isomorphism around the point corresponding to $\mathcal{E}^\eta$. We denote by s_0 this point, so that $\mathcal{F}_{s_0} \simeq \mathcal{E}^\eta$.

We may assume that S is smooth and connected of dimension $2g + 8$. Since the reflexive hull $\mathcal{E}$ of $\mathcal{E}^\eta$ is locally free and satisfies the assumption of [Art90, Corollary 1.5], we get that $\mathcal{F}_s$ is locally free for all s in an open dense subset S_1 of S.

Now observe that $\mathrm{H}^*(\mathcal{F}_{s_0}) = 0$ by Lemmas 1 and 3. Then, semicontinuity ensures that $\mathrm{H}^*(\mathcal{F}_s) = 0$ for all s in an open dense subset S_0 of S_1. Therefore, the isomorphism $\mathcal{F}_s^* \simeq \mathcal{F}_s(-H)$ and Serre duality give $\mathrm{H}^i(\mathcal{F}_s(-H)) \simeq \mathrm{H}^{2-i}(\mathcal{F}_s^*(H))^* \simeq \mathrm{H}^{2-i}(\mathcal{F}_s)^* = 0$. This says that $\mathcal{F}_s(H)$ is a special Ulrich bundle, for all $s \in S_0$. □

For the reader's benefit we also provide a proof of Lemma 4 independent of [Art90]. The point is to check that $\mathcal{F}_s$ is locally free for all s in an open dense subset of S. To do this, first recall again that the non-locally free locus of $\mathcal{E}^\eta$ is disjoint from the singular locus of X, so up to shrinking S we may assume that this happens for $\mathcal{F}_s$ for all $s \in S$. Then $\mathcal{F}_s^{**}$ is locally free for $s \in S$.

Next, we may find an integer $t_0 \le -1$ such that $\mathrm{H}^0(\mathcal{F}_s^{**}(t_0 H)) = \mathrm{H}^1(\mathcal{F}_s^{**}(t_0 H)) = 0$ for all $s \in S$. This can be done for instance using Kollar's theory of husks (cf. [Kol08]), which gives a stratification $(S_i)_{i=1,\ldots,r}$ of S such that $\mathcal{F}_s^{**}$ defines a flat family of sheaves on X parametrized by S_i. Using base change over each each S_i one finds t_i satisfying the required vanishing together with $\mathrm{H}^0(\mathcal{F}_s^{**}(t_i H)|_C) = 0$, for a fixed curve $C \in |\mathcal{O}_X(H)|$. Then t_0 can be taken to be the minimum among $t_1, \ldots, t_r$.

Recall that $\mathrm{H}^*(\mathcal{F}_{s_0}) = 0$ and observe that (3) gives:

$$\mathrm{h}^1(\mathcal{F}_{s_0}(tH)) = \begin{cases} 1 & \text{if } t \le -1, \\ 0 & \text{if } t \ge 0. \end{cases}$$

By semi-continuity, we have thus $\mathrm{H}^*(\mathcal{F}_s) = 0$, $\mathrm{h}^1(\mathcal{F}_s(tH)) = 0$ for all $t \geq 0$ and $\mathrm{h}^1(\mathcal{F}_s(tH)) \leq 1$ for $t \leq -1$ for all s in an open dense subset of S. We still call S this subset.

Next, for all $s \in S$ we consider the double dual sequence:

$$0 \to \mathcal{F}_s \to \mathcal{F}_s^{**} \to \tau(\mathcal{F}_s) \to 0. \tag{4}$$

where the torsion sheaf $\tau(\mathcal{F}_P)$ is defined by the sequence. Put ℓ_s for the length of $\tau(\mathcal{F}_s)$.

Since $\mathrm{H}^0(\mathcal{F}_s^{**}(t_0H)) = \mathrm{H}^1(\mathcal{F}_s^{**}(t_0H)) = 0$, from the previous exact sequence we get $\ell_s = \mathrm{h}^0(\tau(\mathcal{F}_s)) = \mathrm{h}^1(\mathcal{F}_s(t_0H)) \leq 1$ (we neglect to indicate the twist on zero-dimensional sheaves).

Now we have two alternatives. Namely, either for s general enough in S one has $\ell_s = 0$, i.e. $\tau(\mathcal{F}_s) = 0$; or otherwise for all $s \in S$ we get $\ell_s = 1$, i.e. $\tau(\mathcal{F}_s) \simeq \mathcal{O}_{p_s}$, for some point $p_s \in X$ with $p_{s_0} = p$.

In the first case, we have $\mathcal{F}_s \simeq \mathcal{F}_s^{**}$ and $\mathcal{F}_s$ is locally free. So we would like to rule out the second alternative. By contradiction we assume that, for all $s \in S$, we have $\tau(\mathcal{F}_s) \simeq \mathcal{O}_{p_s}$. This gives a map $\gamma : S \to X$ associating p_s to s. This time $\mathcal{F}^{**}$ is flat over S and (4) is the restriction to $X \times \{s\}$ of a sequence on $X \times S$:

$$0 \to \mathcal{F} \to \mathcal{F}^{**} \to \tau(\mathcal{F}) \to 0,$$

with $(\mathcal{F}_s)^{**} \simeq (\mathcal{F}^{**})_s$ and where $\tau(\mathcal{F})$ is a line bundle supported on the graph of γ.

Also, again the previous exact sequence together with $\mathrm{H}^*(\mathcal{F}_s) = 0$ gives $\mathrm{h}^0(\mathcal{F}_s^{**}) = 1$ so that $\mathcal{F}_s^{**}$ has a unique non-zero global section up to a scalar. This section vanishes along a subscheme $Z_s \subset X$ and, up to shrinking again S we may assume that Z_s is zero-dimensional reduced and in general linear position, because these are open conditions, so that $\mathcal{F}_s^{**} \simeq \mathcal{E}_{Z_s}$.

For each sheaf $\mathcal{F}_s^{**}$ of this family, we denote by $\eta_s : \mathcal{F}_s^{**} \twoheadrightarrow \mathcal{O}_{p_s}$ the induced surjection of $\mathcal{F}_s^{**}$ onto $\tau(\mathcal{F}_s)$. We think of η_s as an element of $\mathbb{P}(\mathrm{H}^0(\mathcal{F}_s^{**}|_{p_s})) \simeq \mathbb{P}^1$ (we adopt the convention of writing $\mathbb{P}(V)$ for the projective space of hyperplanes of a vector space V). Plainly, we have $\mathcal{F}_{s_0}^{**} \simeq \mathcal{E}^\eta$, $\tau(\mathcal{F}_{s_0}) \simeq \mathcal{O}_p$ and η_{s_0} is identified with η. Note that $\mathcal{F}_s = \ker(\eta_s)$.

We assert that the family $\mathcal{F}$ is parametrized by an open subset T of the set of triples:

$$\{(W, q, \xi) \mid W \in \mathrm{Hilb}_{g+2}(X),\ q \in X,\ \xi \in \mathbb{P}(\mathrm{H}^0(\mathcal{E}_W|_q))\}.$$

The subset T consists of (W, q, ξ) with $W \subset X_{\mathrm{sm}}$ is reduced and in general linear position in X, $q \in X_{\mathrm{sm}} \setminus W$ and ξ is surjective. Given such a triple, we get the sheaf $\ker(\xi)$ which is simple by Lemma 2. Clearly this gives a flat deformation of $\mathcal{E}^\eta$ so, because $S \to \mathrm{Spl}_X$ is a local isomorphism at $\mathcal{E}^\eta$, there is a possibly smaller open subset T_0 such that all the resulting sheaves $\ker(\xi)$ are of the form $\mathcal{F}_s$, for some $s \in S$. By construction any sheaf $\mathcal{F}_s$ should be of this form by taking $q = p_s$, $W = Z_s$ and $\xi = \eta_s$.

But T_0 is an open dense subset of a $\mathbb{P}^1$-bundle over an open subset of $\mathrm{Hilb}_{g+2}(X) \times X$ and thus has dimension $1 + 2(g + 2) + 2 = 2g + 7$. Therefore T_0 cannot dominate S, as $\dim(S) = 2g + 8$. This says that the second alternative does not take place, so we have proved that $\mathcal{F}_s(H)$ is an Ulrich bundle for general s.

Recall the notation $\mathrm{M}_X(v)$ for the moduli space of H-semistable sheaves $\mathcal{F}$ on X whose Mukai vector $v = (v_0, v_1, v_2)$ satisfies $v_0 = \mathrm{rk}(\mathcal{F})$, $v_1 = c_1(\mathcal{F})$ and $v_2 = \chi(\mathcal{F}) - \mathrm{rk}(\mathcal{F})$. From [Qin93, Lemma 2.1] we obtain the following stronger version of Theorem 1.

Corollary 1. *If X is smooth, $\mathrm{M}_X(2, H, -2)$ is of dimension $2g+8$ and a general point of it corresponds to a sheaf $\mathcal{E}$ which is stable (with respect to all polarizations) and such that $\mathcal{E}(H)$ is a special Ulrich bundle.*

Again, we also offer a proof independent of [Qin93, Art90]. Consider the family of Ulrich sheaves $\mathcal{F}(H)$ with parameter space S_0 constructed in the previous lemma. Recall that, for generic $s \in S_0$, the sheaf $\mathcal{F}_s(H)$ is Ulrich, hence semistable with Ulrich sheaves as Jordan-Hölder factors (cf. [FP15, Lemma 7.1]). So we have to check that $\mathcal{F}_s$ is not strictly semistable. If it was, we would have an exact sequence:

$$0 \to \mathcal{L} \to \mathcal{F}_s \to \mathcal{L}^*(H) \to 0, \tag{5}$$

where $\mathcal{L}(H)$ is an Ulrich sheaf or rank 1 on X. Actually $\mathcal{L}(H)$ is an Ulrich line bundle since X is smooth. Since $\mathcal{L}$ and $\mathcal{L}^*(H)$ are rigid in view of $\mathrm{H}^1(\mathcal{O}_X) = 0$, they do not depend on s, which justifies the notation. Since $\mathcal{L}(H)$ is an Ulrich line bundle we have $\chi(\mathcal{L}) = \chi(\mathcal{L}(-H)) = 0$ which gives $L^2 = -4$ and $LH = g-1$, where $L = c_1(\mathcal{L})$. Similar constraints hold for $H - L$. In particular, L and $H - L$ have the same degree with respect to H, hence $\mathrm{h}^0(\mathcal{O}_X(2L - H)) \le 1$, with equality being attained if and only if $L \equiv H - L$. Likewise, $\mathrm{h}^2(\mathcal{O}_X(2L - H)) = \mathrm{h}^0(\mathcal{O}_X(H - 2L)) \le 1$. Now we observe the following bound:

$$\begin{aligned}\mathrm{ext}^1_X(\mathcal{L}^*(H), \mathcal{L}) = \mathrm{h}^1(\mathcal{O}_X(2L - H)) = \\ &= \mathrm{h}^0(\mathcal{O}_X(2L - H)) + \mathrm{h}^2(\mathcal{O}_X(2L - H)) - \chi(\mathcal{O}_X(2L - H)) \le \\ &\le 2 - \chi(\mathcal{O}_X(2L - H)) = g + 7,\end{aligned}$$

the last equation being obtained by Riemann-Roch after plugging $L^2 = -4$ and $HL = g-1$. In view of the rigidity of $H - L$ and L, the family of sheaves appearing as an extension (5) is parametrized by $\mathbb{P}(\mathrm{Ext}^1_X(\mathcal{L}^*(H), \mathcal{L}))$ and hence has dimension at most $g + 6$. So this family cannot dominate the $(2g + 8)$-dimensional family S_0, a contradiction.

It follows from Theorem 1 that X is strictly Ulrich wild in the sense of [FP15]. The next result refines this fact in terms of moduli spaces. It was proved when $\mathrm{Pic}(X)$ is generated by H in [AFO17, Theorem 2.7]. A modification of that argument allows to prove the result in general.

Theorem 2. *Let X be a K3 surface and H be a very ample line bundle on X. Then, for any positive integer r, the moduli space $\mathrm{M}_X(2r, rH, -2r)$ is of dimension $2(r^2(g + 3) + 1)$. Given a general sheaf $\mathcal{F}$ in this space, $\mathcal{F}(H)$ is a stable Ulrich bundle.*

Proof. Given a coherent sheaf $\mathcal{E}$ or rank $r > 0$ on X we write $\mathrm{P}(\mathcal{E}) \in \mathbb{Q}[t]$ for the Hilbert polynomial of $\mathcal{E}$ and $\mathrm{p}(\mathcal{E})$ for its reduced version, namely $\mathrm{P}(\mathcal{E}) = \chi(\mathcal{E}(tH))$ and $\mathrm{p}(\mathcal{E}) = \mathrm{P}(\mathcal{E})/r$. We put $\mathrm{p}_0 = (g-1)(t+1)t$ so that, if $\mathcal{E}$ is an Ulrich sheaf, then $\mathrm{p}(\mathcal{E}(-H)) = \mathrm{p}_0$. Note that, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are non-isomorphic stable sheaves with $\mathrm{p}(\mathcal{E}_1) = \mathrm{p}(\mathcal{E}_2)$, then $\mathrm{Ext}^k_X(\mathcal{E}_i, \mathcal{E}_j) = 0$ for $k = 0, 2$ and $i \ne j$.

The proof goes by induction on r, the case $r = 1$ being given by Corollary 1. For $r \ge 1$, we select a stable bundle $\mathcal{E}_2$ in $\mathrm{M}_X(2r, rH, -2r)$ given by the induction hypothesis and a stable bundle $\mathcal{E}_1$ in $\mathrm{M}_X(2, H, -2)$, with $\mathcal{E}_i(H)$ Ulrich for $i = 1, 2$, taking care that $\mathcal{E}_1$ is not isomorphic to $\mathcal{E}_2$ for $r = 1$. This is of course possible since $\dim(\mathrm{M}_X(2, H, -2)) > 0$. This way we have:

(6) $$\mathrm{Ext}^k_X(\mathcal{E}_i, \mathcal{E}_j) = 0, \qquad \text{for } k = 0, 2 \text{ and } i \ne j,$$

(7) $$\mathrm{ext}^1_X(\mathcal{E}_i, \mathcal{E}_j) = 2r(g + 3) \qquad \text{for } i \ne j.$$

Note that, for any choice of $\zeta \in \mathbb{P}(\mathrm{Ext}^1_X(\mathcal{E}_2, \mathcal{E}_1))$, the sheaf $\mathcal{E}^\zeta$ fitting as middle term of the associated extension is a locally free semistable sheaf, with $\mathcal{E}^\zeta(H)$ (as extension of sheaves having these properties). By direct computation, we see that it lies $\mathrm{M}_X(2(r + 1), (r + 1)H, -2(r + 1))$. Of course this sheaf is not stable, as $\mathcal{E}_1$ is a sub-sheaf of $\mathcal{E}^\zeta$ with quotient $\mathcal{E}_2$ and the reduced Hilbert polynomial of all these sheaves is p_0. However, it follows by [FP15, Theorem A, ii)] that $\mathcal{E}^\zeta$ is simple, as the representation of the associated Kronecker consists of a single non-zero map of one-dimensional vector spaces, and as such it is simple. Alternatively one may apply [PLT09, Proposition 5.3].

We record the defining sequence:

(8) $$0 \to \mathcal{E}_1 \to \mathcal{E}^\zeta \to \mathcal{E}_2 \to 0.$$

In the same spirit as in Lemma 4, we take a deformation of $\mathcal{E}^\zeta$ in the space of simple sheaves, which is unobstructed of dimension $2((r + 1)^2(g + 3) + 1)$ at $\mathcal{E}^\zeta$. We consider thus an integral quasi-projective variety S as base of an S-flat family of simple sheaves $\mathcal{F}_s$ with $\mathcal{F}_s(H)$ Ulrich for all s and $\mathcal{F}_{s_0} \simeq \mathcal{E}^\zeta$ for some $s_0 \in S$, the base S being locally isomorphic to the moduli space of simple sheaves around the point s_0. We may assume that $\mathcal{F}_s$ is locally free for all $s \in S$.

Claim 1. *There is an open dense subset S_0 of S such that, for any stable sheaf $\mathcal{K}$ with $\mathrm{rk}(\mathcal{K}) < 2(r+1)$, $\mathrm{rk}(\mathcal{K}) \ne 2$ and $\mathrm{p}(\mathcal{K}) = \mathrm{p}_0$, we have $\mathrm{Hom}_X(\mathcal{K}, \mathcal{F}_s) = 0$, for all $s \in S_0$.*

Proof of the claim. Clearly it suffices to find such open subset for a fixed rank u of $\mathcal{K}$ and take the intersection of the corresponding open subsets for all $u < 2(r+1)$, $u \neq 2$.

So let N be the moduli space of stable sheaves $\mathcal{E}$ on X with Hilbert polynomial $\mathrm{P}(\mathcal{E}) = u\mathrm{p}_0$. Let $\mathcal{U}$ be a quasi-universal family over $X \times \mathrm{N}$, cf. [HL97, Proposition 4.6.2] and denote by σ and π the projection maps $X \times \mathrm{N} \to \mathrm{N}$ and $X \times \mathrm{N} \to X$, respectively.

For $y \in \mathrm{N}$ let $\mathcal{U}_y$ be the corresponding sheaf over X. We observe that, applying $\mathrm{Hom}_X(\mathcal{U}_y, -)$ to (8), using the definition of N and ζ and the fact that the $\mathcal{E}_i$'s are stable with $\mathrm{p}(\mathcal{E}_i) = \mathrm{p}(\mathcal{U}_y)$ we get $\mathrm{Hom}_X(\mathcal{U}_y, \mathcal{E}^\zeta) = 0$. Indeed, the only case to check is for $u = 2r$ when y corresponds to the sheaf $\mathcal{E}_2$, but $\mathrm{Hom}_X(\mathcal{E}_2, \mathcal{E}^\zeta) = 0$, for otherwise by stability of $\mathcal{E}_2$ the exact sequence (8) would split, contradicting our assumption on ζ.

Then, Serre duality gives, for all $y \in \mathrm{N}$:

$$\mathrm{H}^2((\mathcal{E}^\zeta)^* \otimes \mathcal{U}_y) \simeq \mathrm{Ext}^2_X(\mathcal{E}^\zeta, \mathcal{U}_y) = 0. \tag{9}$$

Now consider $X \times \mathrm{N} \times S$, put τ for the projection $\mathrm{N} \times S \to S$ and denote by $\bar\sigma$, $\bar\pi$, $\bar\tau$ the projection maps from $X \times \mathrm{N} \times S$ onto $X \times S$, $\mathrm{N} \times S$ and $X \times \mathrm{N}$, respectively. Let $\mathcal{V} = \bar\pi^*(\mathcal{F}^*) \otimes \bar\tau^*(\mathcal{U})$. Since $\mathcal{V}$ is flat over the integral base $\mathrm{N} \times S$ and $\bar\sigma$ has relative dimension 2, base-change gives, for all $(y, s) \in \mathrm{N} \times S$:

$$\boldsymbol{R}^2\bar\sigma_*(\mathcal{V})_{(y,s)} \simeq \mathrm{H}^2(\mathcal{F}_s^* \otimes \mathcal{U}_y). \tag{10}$$

Let W be the support of $\boldsymbol{R}^2\sigma_*(\mathcal{V})$, i.e. the closed subset of points $(y, s) \in \mathrm{N} \times S$ such that $\boldsymbol{R}^2\sigma_*(\mathcal{V})_{(y,s)} \neq 0$. By (9) and (10), we have $W \cap \mathrm{N} \times \{s_0\} = \emptyset$, i.e. s_0 does not lie in $\tau(W)$. Then there is an open neighbourhood $S_0 \subset S$ of s_0 which is disjoint from $\tau(W)$. Again by (10), we get $\mathrm{H}^2(\mathcal{F}_s^* \otimes \mathcal{U}_y) = 0$ for all $(y, s) \in \mathrm{N} \times S_0$, which proves the claim. □

Let us now conclude the proof of the theorem. In view of the claim, we have two alternatives for s generic in S_0: either $\mathrm{Hom}(\mathcal{K}, \mathcal{F}_s) = 0$ for any stable sheaf $\mathcal{K}$ with $\mathrm{rk}(\mathcal{K}) < 2(r+1)$ and $\mathrm{p}(\mathcal{K}) = \mathrm{p}_0$, or otherwise this happens for all such $\mathcal{K}$ except for $\mathrm{rk}(\mathcal{K}) = 2$ and there actually exists a stable $\mathcal{K}$ in N such that $\mathrm{Hom}(\mathcal{K}, \mathcal{F}_s) \neq 0$.

In the first alternative $\mathcal{F}_s$ is stable, so we assume that the second one takes place and look for a contradiction. We go back to Claim 1 and carry out the same argument for $u = 2$, with y_0 being the point corresponding to $\mathcal{E}_1$. Observe that $\mathcal{K}$ must lie in $\mathrm{M}_X(2, H, -2)$ as the proof of Claim 1 applies verbatim on any other component of N.

We note that $W \cap \mathrm{N} \times \{s_0\} = \{(y_0, s_0)\}$, as clearly $\mathrm{Hom}_X(\mathcal{K}, \mathcal{E}^\zeta) = 0$ for all $\mathcal{K}$ in $\mathrm{N} \setminus \{y_0\}$. So W is properly contained in $\mathrm{N} \times S$. Moreover, we easily have $\hom_X(\mathcal{E}_1, \mathcal{E}^\zeta) = 1$. Recall by construction of the quasi-universal family that there is u_0 such that $\mathrm{rk}(\mathcal{U}) = 2u_0$ and that, for $y \in \mathrm{N}$, the sheaf $\mathcal{U}_y$ is a direct sum of u_0 copies of the stable sheaf of rank 2 in $\mathrm{M}_X(2, H, -2)$ corresponding to y. Therefore, the sheaf $\boldsymbol{R}^2\bar\sigma_*(\mathcal{V})_{(y,s)}$ has rank at least u_0 at any $(y, s) \in W$, and rank precisely u_0 at (y_0, s_0). So there is an open dense subset W_0 of W where $\boldsymbol{R}^2\bar\sigma_*(\mathcal{V})$ is free of rank u_0. For any $(y, s) \in W_0$, the stable sheaf $\mathcal{K}$ corresponding to y satisfies $\hom_X(\mathcal{K}, \mathcal{F}_s) = 1$; up to proportionality we have thus a unique non-zero map $\eta_{y,s} : \mathcal{K} \to \mathcal{F}_s$. Stability easily impies that $\eta_{y,s}$ is injective, so there is an exact sequence:

$$0 \to \mathcal{K} \to \mathcal{F}_s \to \mathcal{K}' \to 0,$$

for a well-defined sheaf $\mathcal{K}' = \mathrm{coker}(\eta_{y,s})$, for all $(y, s) \in W_0$.

For $s = s_0$ the sheaf $\mathcal{K}'$ is just $\mathcal{E}_2$ so, by openness of stability, up to shrinking W_0 we may assume that $\mathcal{K}'$ is stable for all $(y, s) \in W_0$. Note that $\mathcal{K}'$ lies in $\mathrm{M}(2r, rH, -2r)$.

Under our assumption, such sequence should exist for any s in an open neighbourhood of s_0. Then the family of sheaves $\mathcal{F}$ should be dominated by the family of extensions of $\mathcal{K}$ by $\mathcal{K}'$ as s varies around s_0. We see that the dimension of this family of extensions is:

$$\dim(\mathrm{M}_X(2, H, -2)) + \dim(\mathrm{M}_X(2r, rH, -2r)) + \dim(\mathbb{P}\,\mathrm{Ext}^1_X(\mathcal{K}', \mathcal{K})),$$

which equals $2(r(r+1)+1)(g+3)+3$, as it follows by formulas (6), (7) applied to $\mathcal{K}$ and $\mathcal{K}'$ instead of $\mathcal{E}_1$ and $\mathcal{E}_2$. On the other hand, the dimension of S is $2((r+1)^2(g+3)+1)$. The difference of these dimensions is $2r(g+3)-1$ and since this is always positive for $r \geq 1$, $g \geq 3$, we get that the family of simples sheaves appearing as extensions cannot be dense in S_0. This contradiction concludes the proof. □

The previous result is in some sense optimal as general K3 surfaces do not support Ulrich bundles of odd rank, cf. [AFO17, Corollary 2.2].

Remark 1. An argument similar to the one of Theorem 1 has been used to construct ACM and Ulrich bundles on Fano threefolds of index 1. Indeed, it follows from the main result of [BF11] that any smooth Fano threefold of Picard number 1 and index 1, containing a line L with normal bundle $\mathcal{O}_L \oplus \mathcal{O}_L(-1)$ (such a threefold was called "ordinary" in that paper) admits an Ulrich bundle of rank 2. Ulrich sheaves of rank 2 are precisely ACM sheaves $\mathcal{E}$ with $c_1(\mathcal{E}(-H)) = H$ and $c_2(\mathcal{E}(-H)) = (g+3)L$, where $L \subset X$ is a line. We do not know if the same result holds for non-ordinary threefolds.

Remark 2. Theorem 1 implies for instance that any integral quartic surface supports an Ulrich bundle of rank 2. If X is not integral, then X must the union of (possibly multiple) surfaces of degree ≥ 3. For each component it is possible to find a rank-2 Ulrich bundle, we refer to [FP15, Lemma 7.2] for the slightly delicate case of singular cubic surfaces. This yields existence of an Ulrich sheaf of rank 2 on an arbitrary quartic surface.

However the resulting sheaf will fail to be locally free over the intersection of the components. Finding locally free Ulrich sheaves of rank 2 seems more tricky when X is not irreducible and might be impossible when X is not reduced. To justify this let us mention that, for instance if X the union of two distinct double planes, the rank of any locally free Ulrich sheaf on X must be a multiple of 4 by [BHMP16, Proposition 4.14].

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