SURFACES OF MINIMAL DEGREE OF TAME REPRESENTATION TYPE AND MUTATIONS OF COHEN-MACAULAY MODULES

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ABSTRACT. We provide two examples of smooth projective surfaces of tame CM type, by showing that the parameter space of isomorphism classes of indecomposable ACM bundles with fixed rank and determinant on a rational quartic scroll in $\mathbb{P}^5$ is either a single point or a projective line. These turn out to be the only smooth projective varieties of tame CM type besides elliptic curves, [Ati57].

For surfaces of minimal degree and wild CM type, we classify rigid Ulrich bundles as Fibonacci extensions. For $\mathbb{F}_0$ and $\mathbb{F}_1$, embedded as quintic or sextic scrolls, a complete classification of rigid ACM bundles is given.

INTRODUCTION

Let $X \subset \mathbb{P}^n$ be a smooth positive-dimensional closed subvariety over an algebraically closed field $\Bbbk$, and assume that the graded coordinate ring $\Bbbk[X]$ of X is Cohen-Macaulay, i.e. X is ACM (arithmetically Cohen-Macaulay). Then X supports infinitely many indecomposable ACM sheaves $\mathscr{E}$ (i.e. whose $\Bbbk[X]$-module of global sections $\mathrm{H}^0_*(X,\mathscr{E})$ is Cohen-Macaulay), unless X is $\mathbb{P}^n$ itself, or a quadric hypersurface, or a rational normal curve, or one of the two sporadic cases: the Veronese surface in $\mathbb{P}^5$ and (cf. §1.2) the rational cubic scroll $S(1,2)$ in $\mathbb{P}^4$, see [EH88].

Actually, for most ACM varieties X, much more is true. Namely X supports families of arbitrarily large dimension of indecomposable ACM bundles, all non-isomorphic to one another (varieties like this are of "geometrically of wild CM type" or simply "CM-wild"). CM-wild varieties include curves of genus ≥ 2, hypersurfaces of degree $d \geq 4$ in $\mathbb{P}^n$ with $n \geq 2$, complete intersections in $\mathbb{P}^n$ of codimension ≥ 3, having one defining polynomial of degree ≥ 3 (cf. [CL11, DT14]), the third Veronese embedding of any variety of dimension ≥ 2 cf. [MR15]. In many cases, these families are provided by Ulrich bundles, i.e. those $\mathscr{E}$ such that $\mathrm{H}^0_*(X,\mathscr{E})$ achieves the maximum number of generators, namely $d_X \operatorname{rk}(\mathscr{E})$, where we write d_X for the degree of X. For instance, Segre embeddings are treated in [CMRPL12], smooth rational ACM surfaces in $\mathbb{P}^4$ in [MRPL13], cubic surfaces and threefolds in [CH11, CHGS12], del Pezzo surfaces in [PLT09, CKM13].

In spite of this, there is a special class of varieties X with intermediate behaviour, namely X supports continuous families of indecomposable ACM bundles, all non-isomorphic to one another, but, for each rank r, these bundles form finitely (or countably) many irreducible families of dimension at most one. Then X is called of tame CM type. It is the case of the elliptic curve, [Ati57].

In this note we provide the first examples of smooth positive-dimensional projective CM-tame varieties, besides elliptic curves. Part of this was announced in [FM13].

Theorem A. *Let X be a smooth surface of degree 4 in $\mathbb{P}^5$. Then, for any $r \geq 1$, there is a family of isomorphism classes of indecomposable Ulrich bundles of rank $2r$, parametrized by $\mathbb{P}^1$. Conversely, any indecomposable ACM bundle on X is rigid or belongs to one of these families (up to a twist). In particular, X is of tame CM type.*

Recalling the classification by del Pezzo and Bertini of smooth varieties of *minimal degree*, i.e. with $d_X = \operatorname{codim}(X)+1$, as the Veronese surface in $\mathbb{P}^5$ and rational normal scrolls (cf. [EH87]), we see two things. On one hand, both CM-finite varieties and our examples have minimal degree, actually a surface of degree 4 in $\mathbb{P}^5$ is a quartic scroll. Incidentally, these have the same graded Betti numbers as the Veronese surface in $\mathbb{P}^5$, which is CM-finite. On the other hand the remaining varieties of minimal degree are CM-wild by [MR13]; in fact in [MR13] also quartic scrolls are claimed to be of wild CM type: the gap in the argument only overlooks our two examples, cf. Remark 2.2. By the following result, these examples complete the list of non-CM-wild varieties in a broad sense.

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Theorem (cf. [FPL15]). *Let $X \subset \mathbb{P}^n$ be a closed normal ACM subvariety of positive dimension, which is not a cone. Then X is CM-wild if it is not one of the well-known CM-finite varieties, or an elliptic curve, or a quartic surface scroll.*

For CM-wild varieties, an interesting issue is to study rigid ACM bundles (by definition $\mathscr{E}$ is rigid if $\mathrm{Ext}^1_X(\mathscr{E},\mathscr{E}) = 0$). Important instances of classification of such bundles are given in [IY08, KMVdB11], cf. also [Fae15], for the third Veronese surface and second Veronese threefold, both embedded in $\mathbb{P}^9$.

Our first result in this direction deals with Ulrich bundles. Given a rational surface scroll X, we set H for the hyperplane class, F for the class of a fibre of the projection $X \to \mathbb{P}^1$, and $\mathscr{L} = \mathscr{O}_X((d_X-1)F-H)$. For $w \geq 2$ we define the Fibonacci numbers by the relations $\phi_{w,0} = 0$, $\phi_{w,1} = 1$ and $\phi_{w,k+1} = w\phi_{w,k} - \phi_{w,k-1}$. We set $\phi_{w,-1} = 0$.

Theorem B. *Let X be a smooth surface scroll of degree $d_X \geq 5$. Set $w = d_X - 2$. Then:*

(i) an indecomposable ACM bundle $\mathscr{E}$ on X is Ulrich if and only if, up to twist, $\mathscr{E}$ fits into:

$$0 \to \mathscr{O}_X(-F)^a \to \mathscr{E} \to \mathscr{L}^b \to 0, \qquad \text{for some } a, b \geq 0;$$

(ii) if moreover $\mathscr{E}$ is rigid then $a = \phi_{w,k}$ and $b = \phi_{w,k\pm 1}$, for some $k \geq 0$;
(iii) for any $k \in \mathbb{Z}$, there is a unique indecomposable rigid Ulrich bundle $\mathfrak{u}_k$ with:

$$\begin{aligned} a &= \phi_{w,k}, & b &= \phi_{w,k-1}, & &\text{for } k \geq 1,\\ a &= \phi_{w,-k}, & b &= \phi_{w,1-k}, & &\text{for } k \leq 0. \end{aligned}$$

Furthermore, each $\mathfrak{u}_k$ is a exceptional, and $\mathfrak{u}_k(-H) \simeq \mathfrak{u}^*_{1-k} \otimes \omega_X$.

Here, a bundle $\mathscr{E}$ is *exceptional* if $\mathrm{RHom}_X(\mathscr{E},\mathscr{E}) = \langle \mathrm{id}_{\mathscr{E}} \rangle$. "Up to twist" means "up to tensoring with $\mathscr{O}_X(tH)$ for some $t \in \mathbb{Z}$". The following result should be compared with [CH11].

Corollary. *There is no stable Ulrich bundle of rank greater than one on a surface of minimal degree. If this degree is 4, any non-Ulrich indecomposable ACM bundle is rigid.*

Our next result concerning the classification of rigid ACM sheaves deals with the CM-wild scrolls $S(\vartheta-1,\vartheta)$ and $S(\vartheta,\vartheta)$ for $\vartheta \geq 3$. To state it we anticipate from cf. §1.2.2 and §4.1. Consider the braid group B_3, whose standard generators σ_1 and σ_2 act by right mutation over 3-terms exceptional collections $(\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3)$ over X, starting with $\mathbf{B}_\emptyset = (\mathscr{L}[-1], \mathscr{O}_X(-F), \mathscr{O}_X)$. Given a vector $\vec{k} = (k_1,\ldots,k_s) \in \mathbb{Z}^s$, we let $\sigma^{\vec{k}} = \sigma_1^{k_1}\sigma_2^{k_2}\sigma_1^{k_3}\cdots$, and $\sigma^\emptyset = 1$. The exceptional collection $\mathbf{B}_{\vec{k}}$ obtained by $\sigma^{\vec{k}}$ can be extended to a full exceptional collection $\mathbf{C}_{\vec{k}} = (\mathscr{L}(-F)[-1], \mathbf{B}_{\vec{k}})$. Thinking of the Euler characteristic $\nu_{i-1} := \chi(\mathfrak{s}_i, \mathfrak{s}_{i+1})$ (with cyclic indexes) of pairs of bundles in the mutated 3-term collection $\mathbf{B}_{\vec{k}}$, we define an operation of B_3 on $\mathbb{Z}^3$, by:

$$\sigma_1 : (\nu_1,\nu_2,\nu_3) \mapsto (\nu_1\nu_3 - \nu_2, \nu_1, \nu_3), \qquad \sigma_2 : (\nu_1,\nu_2,\nu_3) \mapsto (\nu_1, \nu_1\nu_2 - \nu_3, \nu_2).$$

To $\mathbf{B}_\emptyset$ corresponds $\nu^\emptyset = (2, d_X-4, d_X-2)$. Set $\bar{t} \in \{0,1\}$ for the remainder of the division of an integer t by 2. Given $\vec{k} = (k_1,\ldots,k_s)$ and $t \leq s$, we write the truncation $\vec{k}(t) = (k_1,\ldots,k_{t-1})$. Having this set up, we finally define the set:

$$\mathfrak{K} = \{\vec{k} = (k_1,\ldots,k_s) \mid (-1)^t k_{t-1} (\sigma^{\vec{k}(t)}.\nu^\emptyset)_{2\bar{t}+1} \leq 0, \forall t \leq s\}.$$

Explicitly, a vector length-s vector $\vec{k} = (k_1,\ldots,k_s)$ belongs to $\mathfrak{K}$ if, for all subvectors $\vec{k}(t) = (k_1,\ldots,k_{t-1})$, applying $\sigma^{\vec{k}(t)}$ to ν we get a triple of integer whose

- third element has the same sign as k_{t-1} (for odd t);
- first element has opposite sign with respect to k_{t-1} (for even t).

Theorem C. *Let $\vec{k}$ be an element of $\mathfrak{K}$. Then, there is an exceptional ACM bundle $\mathfrak{f}_{\vec{k}}$ corresponding to $\vec{k}$, which is the middle element of the exceptional collection $\mathbf{B}_{\vec{k}}$. If $\vartheta = 3$, any indecomposable rigid ACM bundle is of the form $\mathfrak{f}_{\vec{k}}$ or $\mathfrak{f}^*_{\vec{k}} \otimes \omega_X$, for some $\vec{k} \in \mathfrak{K}$, up to twist.*

The structure of the paper is as follows. In §1 we write the basic form of a resolution of ACM bundles on scrolls relying on the structure of the derived category of a $\mathbb{P}^1$-bundle. We also provide here some cohomological splitting criteria for scrolls of low degree. In §2 we study Ulrich bundles on surfaces of minimal degree in terms of representations of Kronecker quivers and prove Theorem B using Kac's classification of Schur roots. In §3 we focus on quartic scrolls and prove their tameness according to Theorem A. In §4, we give the proof of Theorem C.

1. Resolutions for bundles on scrolls

In this section, after providing some essential terminology, we give our basic technique to classify ACM bundles on ruled surfaces. Indeed, it is shown in §1.4 that such bundle $\mathscr{E}$ has a functorial two-sided resolution computed upon certain cohomology groups of $\mathscr{E}$. We derive in §1.5 a splitting criterion over scrolls of low degree.

1.1. **Background.** Let $\Bbbk$ be an algebraically closed field. Given a vector space V over $\Bbbk$, we let $\mathbb{P}V$ be the projective space of 1-dimensional quotients of V. If $\dim(V) = n+1$, we write $\mathbb{P}^n = \mathbb{P}V$. It will be understood that a small letter denotes the dimension of a vector space in capital letter, for instance if $\mathscr{E}$ is a coherent sheaf on a variety X then $\mathrm{h}^i(X,\mathscr{E}) = \dim_{\Bbbk} \mathrm{H}^i(X,\mathscr{E})$. For a pair of coherent sheaves $\mathscr{E}_i$, $\mathscr{E}_2$ on X, the Euler characteristic is $\chi(\mathscr{E}_1,\mathscr{E}_2) = \sum(-1)^j \operatorname{ext}^j_X(\mathscr{E}_1,\mathscr{E}_2)$. We abbreviate $\chi(\mathscr{O}_X,\mathscr{E})$ to $\chi(\mathscr{E})$. A *(vector) bundle* is a coherent locally free sheaf.

1.1.1. *Derived categories.* We will use the derived category $\mathbf{D}^b(X)$ of bounded complexes of coherent sheaves on X. We refer to [Huy06] for a detailed account of it. An object $\mathscr{E}$ of $\mathbf{D}^b(X)$ is a bounded complex of coherent sheaves on X, we will denote by $\mathrm{H}^\bullet(X,\mathscr{E})$ the total complex associated with the hypercohomology of $\mathscr{E}$. If $\mathscr{E}$ is concentrated in degree i, then we write $|\mathscr{E}|$ for the coherent sheaf $\mathscr{H}^i(\mathscr{E})$, i.e. $|\mathscr{E}| = \mathscr{E}[i]$.

An object $\mathscr{E}$ of $\mathbf{D}^b(X)$ is *simple* if $\mathrm{Hom}_X(\mathscr{E},\mathscr{E}) \simeq \Bbbk$, and *exceptional* if $\mathrm{RHom}_X(\mathscr{E},\mathscr{E}) \simeq \Bbbk$. Given a set S of objects of $\mathbf{D}^b(X)$, we write $\langle S\rangle$ for the smallest full triangulated subcategory of $\mathbf{D}^b(X)$ containing all objects of S. The same notation is used for a collection S of subcategories of $\mathbf{D}^b(X)$. Given a pair of objects $\mathscr{E}$ and $\mathscr{F}$ of $\mathbf{D}^b(X)$ we have the left and right mutations $\mathrm{L}_{\mathscr{E}}(\mathscr{F})$ and $\mathrm{R}_{\mathscr{F}}(\mathscr{E})$ defined respectively by the distinguished triangles given by natural evaluations:

$$\mathrm{RHom}_X(\mathscr{E},\mathscr{F})\otimes\mathscr{E} \to \mathscr{F} \to \mathrm{L}_{\mathscr{E}}(\mathscr{F})[1], \qquad \mathrm{R}_{\mathscr{F}}(\mathscr{E})[-1] \to \mathscr{E} \to \mathrm{RHom}_X(\mathscr{E},\mathscr{F})^*\otimes\mathscr{F}.$$

1.1.2. *Arithmetically Cohen-Macaulay varieties and sheaves.* A *polarized* or *embedded variety* is a pair (X,H), where X is an integral m-dimensional projective variety and H is a *very ample* divisor class on X. The degree d_X is H^m. If H embeds X in $\mathbb{P}^n$, then the ideal I_X of X sits in $R = \Bbbk[x_0,\ldots,x_n]$ and the homogeneous coordinate $\Bbbk[X]$ is R/I_X. The variety $X \subset \mathbb{P}^n$ is *ACM* (for arithmetically Cohen-Macaulay) if $\Bbbk[X]$ is a graded Cohen-Macaulay ring, i.e. the R-projective dimension of $\Bbbk[X]$ is $n-m$.

Given a coherent sheaf $\mathscr{E}$ on a polarized variety (X,H), and $i,t \in \mathbb{N}$, we write $\mathscr{E}(t)$ for $\mathscr{E}(tH)$ and:

$$\mathrm{H}^i_*(X,\mathscr{E}) = \bigoplus_{t\in\mathbb{Z}} \mathrm{H}^i(X,\mathscr{E}(t)).$$

For each i, $\mathrm{H}^i_*(X,\mathscr{E})$ is a module over $\Bbbk[X]$. We say that $\mathscr{E}$ is *initialized* if $\mathrm{H}^0(X,\mathscr{E}) \neq 0$ and $\mathrm{H}^0(X,\mathscr{E}(-1)) = 0$, i.e. if $\mathrm{H}^0_*(X,\mathscr{E})$ is zero in negative degrees and non-zero in positive degrees. Any torsion-free sheaf $\mathscr{E}$ on a positive-dimensional variety has an *initialized twist*, i.e. there is a unique integer t_0 such that $\mathscr{E}(t_0)$ is initialized. The $\Bbbk[X]$-module $\mathrm{H}^0_*(X,\mathscr{E})$ is also finitely generated in this case.

Given $m \geq 1$, a vector bundle $\mathscr{E}$ on an smooth ACM m-dimensional variety X is *ACM* (for arithmetically Cohen-Macaulay) if $\mathscr{E}$ has no intermediate cohomology:

$$\mathrm{H}^i_*(X,\mathscr{E}) = 0, \qquad \text{for all } 1 \leq i \leq m-1.$$

Equivalently, $\mathscr{E}$ is ACM if $E = \mathrm{H}^0_*(X,\mathscr{E})$ is a maximal Cohen-Macaulay module over $\Bbbk[X]$, i.e., $\mathrm{depth}(E) = \dim(E) = m+1$.

The initialized twist $\mathscr{E}(t_0)$ of $\mathscr{E}$ satisfies $\mathrm{h}^0(X,\mathscr{E}(t_0)) \leq d_X \operatorname{rk}(\mathscr{E})$. We say that $\mathscr{E}$ is *Ulrich* if equality is attained in the previous inequality.

We will use Gieseker-Maruyama and slope-semi-stability of bundles with respect to a given polarization. We refer to [HL97].

1.2. **Reminder on Hirzebruch surfaces and their derived categories.** Let U be a 2-dimensional $\Bbbk$-vector space, and let $\mathbb{P}^1 = \mathbb{P}U$ so that $U = \mathrm{H}^0(\mathbb{P}^1,\mathscr{O}_{\mathbb{P}^1}(1))$. There is a identification $U \simeq U^*$, canonical up to the choice of a nonzero scalar. Let $\epsilon \geq 0$ be an integer and consider the Hirzebruch surface $\mathbb{F}_\epsilon = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(\epsilon))$. Write $\pi : \mathbb{F}_\epsilon \to \mathbb{P}^1$ for the projection on the base, let $F = c_1(\pi^*(\mathscr{O}_{\mathbb{P}^1}(1)))$ be the class of a fibre of π, and $\mathscr{O}_\pi(1)$ be the relatively ample tautological line bundle on $\mathbb{F}_\epsilon$. For any integer $\vartheta > 1$, setting $d_X = 2\vartheta + \epsilon$, the surface $\mathbb{F}_\epsilon$ is embedded in $\mathbb{P}^{d_X+1}$ by the line bundle $\mathscr{O}_\pi(1)\otimes\mathscr{O}_X(\vartheta F)$, as a ruled surface of degree d_X. We denote by $\mathscr{O}_X(H)$ this line bundle, so that the polarized variety $(\mathbb{F}_\epsilon,H)$ is a rational normal scroll $S(\vartheta,\vartheta+\epsilon)$. Of course, we have $F^2 = 0$ and $F\cdot H = 1$. The canonical bundle of $X = \mathbb{F}_\epsilon$ is $\omega_X \simeq \mathscr{O}_X((d_X-2)F - 2H)$.

For $\epsilon > 0$, denote by $\Delta \in |\mathscr{O}_X(H-(\vartheta+\epsilon)F)|$ the *negative section* of X, i.e. the section of π with self-intersection $-\epsilon$. We have for $a \ge 0$:

$$\mathrm{h}^k(X, \mathscr{O}_X(aH + bF)) = \textstyle\sum_{i=0,\dots,a} \mathrm{h}^k(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(a\vartheta + i\epsilon + b)).$$

On the other hand $\mathrm{H}^k(X, \mathscr{O}_X(bF - H)) = 0$ for all k, while $\mathrm{h}^k(X, \mathscr{O}_X(aH + bF))$ can be computed for $a \le -2$ by Serre duality. We fix the notation:

$$\mathscr{L} = \mathscr{O}_X((d_X - 1)F - H).$$

1.2.1. *Derived category of Hirzebruch surfaces.* By a result of Orlov [Orl92], we have the semiorthogonal decomposition (for a definition cf. for instance [Huy06, §1]):

$$\mathbf{D}^b(X) = \langle \pi^*\mathbf{D}^b(\mathbb{P}^1) \otimes \mathscr{O}_X(-H), \pi^*\mathbf{D}^b(\mathbb{P}^1)\rangle. \tag{1.1}$$

In turn, by Beilinson's theorem, see for instance [Huy06, §8], we have:

$$\mathbf{D}^b(\mathbb{P}^1) = \langle \mathscr{O}_{\mathbb{P}^1}(t-1), \mathscr{O}_{\mathbb{P}^1}(t)\rangle, \qquad \text{for any } t \in \mathbb{Z}. \tag{1.2}$$

The right adjoint of π^* is $\mathbf{R}\pi_*$. Let us denote by $\Theta : \mathbf{D}^b(\mathbb{P}^1) \to \mathbf{D}^b(X)$ the functor that sends $\mathscr{F}$ to $\pi^*(\mathscr{F}) \otimes \mathscr{O}_X(-H)$ and by Θ^* its left adjoint. By (1.1), any object $\mathscr{E}$ of $\mathbf{D}^b(X)$ fits into a functorial distinguished triangle:

$$\pi^*\mathbf{R}\pi_*\mathscr{E} \to \mathscr{E} \to \Theta\Theta^*\mathscr{E}. \tag{1.3}$$

To compute the expression of these functors, we first use (1.2) with $t = 0$ to check that $\pi^*\mathbf{R}\pi_*(\mathscr{E})$ fits into a functorial distinguished triangle:

$$\mathrm{H}^\bullet(X, \mathscr{E}(-F)) \otimes \mathscr{O}_X(-F) \xrightarrow{\alpha} \mathrm{H}^\bullet(X, \mathscr{E}) \otimes \mathscr{O}_X \to \pi^*\mathbf{R}\pi_*\mathscr{E}, \tag{1.4}$$

i.e., $\pi^*\mathbf{R}\pi_*\mathscr{E}$ is the cone of α.

Computing $\Theta^*(\mathscr{E})$, cf. [Huy06, §3] we get:

$$\Theta^*\mathscr{E} = \mathbf{R}\pi_*(\mathscr{E}(d_X F - H))[1].$$

Then, using (1.2) with $t = 1 - d_X$ we get the distinguished triangle:

$$\mathrm{H}^\bullet(X, \mathscr{E}(-H)) \otimes \mathscr{L}(-F) \to \mathrm{H}^\bullet(X, \mathscr{E}(F-H)) \otimes \mathscr{L} \to \Theta\Theta^*\mathscr{E}[-1]. \tag{1.5}$$

1.2.2. *Exceptional collections and braid group action.* Let $\epsilon \ge 0$ and set $X = \mathbb{F}_\epsilon$. A collection of objects $(\mathfrak{s}_0, \dots, \mathfrak{s}_r)$ in $\mathbf{D}^b(X)$ is *exceptional* if it consists of exceptional objects such that $\mathrm{RHom}_X(\mathfrak{s}_j, \mathfrak{s}_k) = 0$ if $j > k$. Such collection is *full* if it generates $\mathbf{D}^b(X)$. Any full exceptional collection on X has $r = 3$. As consequence of Orlov's theorem recalled above, one of them is:

$$\mathbf{C}_\emptyset = (\mathscr{L}(-F)[-1], \mathscr{L}[-1], \mathscr{O}_X(-F), \mathscr{O}_X). \tag{1.6}$$

A motivation for this notation will only be apparent in §4.1.

Let us describe the action of the braid group B_4 in 4 strands, which we number from 0 to 3, on the set of exceptional collections. Let σ_i be the generator of B_4 corresponding to the crossing of the i-th strand above the $(i+1)$-st one. With σ_i we associate $\mathrm{R}_{\mathfrak{s}_{i+1}} \mathfrak{s}_i$ so that σ_i sends an exceptional collection $\mathbf{C} = (\mathfrak{s}_0, \dots, \mathfrak{s}_3)$ to a new collection $\sigma_i\mathbf{C}$ where we replace $(\mathfrak{s}_i, \mathfrak{s}_{i+1})$ with $(\mathfrak{s}_{i+1}, \mathrm{R}_{\mathfrak{s}_{i+1}} \mathfrak{s}_i)$. Replacing $(\mathfrak{s}_i, \mathfrak{s}_{i+1})$ with $(\mathrm{L}_{\mathfrak{s}_i} \mathfrak{s}_{i+1}, \mathfrak{s}_i)$, gives σ_i^{-1}. This satisfies the braid group relations. The subgroups B_4 of braids not involving a given strand operate on partial exceptional collections.

Assume now $X = \mathbb{F}_\epsilon$ is a del Pezzo surface, i.e. if $\epsilon \in \{0, 1\}$. Then it turns out (although we will not need this) that this action is transitive on the set of all full exceptional collections, cf. [GK04, Theorem 6.1.1]. Also, in this case the objects $\mathfrak{s}_i$ are sheaves up to a shift, and actually (shifted) vector bundles if torsion-free. Finally, in this case for $j < k$ there is at most one i such that $\mathrm{Ext}^i_X(|\mathfrak{s}_j|, |\mathfrak{s}_k|) \ne 0$ and $i \in \{0, 1\}$, cf. [GK04, Proposition 5.3.5]. For any ϵ, an exceptional pair of shifted vector bundles $(\mathfrak{s}_j, \mathfrak{s}_k)$ on $\mathbb{F}_\epsilon$ is called *regular* if $i = 0$ and *irregular* if $i = 1$.

Given an irregular exceptional pair $(\mathscr{P}, \mathscr{N})$ on $X = \mathbb{F}_\epsilon$ (for any $\epsilon \ge 0$), we set $\mathfrak{g}_0 = |\mathscr{P}|[-1]$ and $\mathfrak{g}_1 = |\mathscr{N}|$ and define:

$$\mathfrak{g}_{k+1} = \mathrm{R}_{\mathfrak{g}_k} \mathfrak{g}_{k-1} \qquad \text{for } k \ge 1,$$
$$\mathfrak{g}_{k-1} = \mathrm{L}_{\mathfrak{g}_k} \mathfrak{g}_{k+1} \qquad \text{for } k \le 0.$$

Note that the two relations we have just written are both formally valid for any $k \in \mathbb{Z}$. It turns out that $\mathfrak{g}_k$ is concentrated in degree 1 for $k \le 0$, and in degree 0 for $k \ge 1$.

Explicitly, we set $w = \hom_X(\mathfrak{g}_0, \mathfrak{g}_1)$ and suppose $w \ge 2$. For $k \le 0$ we have the sequences:

$$0 \to \mathfrak{g}_{-2} \to \mathfrak{g}_{-1}^w \to \mathfrak{g}_0 \to 0,$$
$$0 \to \mathfrak{g}_{-3} \to \mathfrak{g}_{-2}^w \to \mathfrak{g}_{-1} \to 0,$$
$$0 \to \mathfrak{g}_{-4} \to \mathfrak{g}_{-3}^w \to \mathfrak{g}_{-2} \to 0,$$

etc., while for positive k the first mutation sequences read:

$$0 \to \mathfrak{g}_1 \to \mathfrak{g}_2^w \to \mathfrak{g}_3 \to 0,$$
$$0 \to \mathfrak{g}_2 \to \mathfrak{g}_3^w \to \mathfrak{g}_4 \to 0.$$

For the values around 0 the sequences take the special form:

$$0 \to \mathfrak{g}_1[-1] \to \mathfrak{g}_{-1} \to \mathfrak{g}_0^w \to 0,$$
$$0 \to \mathfrak{g}_1^w \to \mathfrak{g}_2 \to \mathfrak{g}_0[1] \to 0.$$

Let us recall one more feature of the exceptional objects $\mathfrak{g}_k$ generated by an irregular exceptional pair $(\mathscr{P},\mathscr{N})$, related to generalized Fibonacci numbers. Set $w = \mathrm{ext}^1_X(|\mathscr{P}|,|\mathscr{N}|)$ and define the integers $\phi_{w,k}$ recursively as:

$$\phi_{w,0} = 0, \quad \phi_{w,1} = 1, \quad \phi_{w,k+1} = w\phi_{w,k} - \phi_{w,k-1}, \quad \text{for } k \geq 1.$$

Then, for all k, there are exact sequences of sheaves:

$$(1.7) \qquad 0 \to |\mathscr{N}|^{\phi_{w,k}} \to \mathfrak{g}_k \to |\mathscr{P}|^{\phi_{w,k-1}} \to 0, \qquad \text{for } k \geq 1,$$

$$(1.8) \qquad 0 \to |\mathscr{N}|^{\phi_{w,-k}} \to \mathfrak{g}_k[1] \to |\mathscr{P}|^{\phi_{w,1-k}} \to 0, \qquad \text{for } k \leq 0.$$

The existence of these sequences (which we sometimes call *Fibonacci sequences*) is easily carried over to our case from [Bra08]. The first Fibonacci sequences look like:

$$0 \to \mathfrak{g}_1^{\phi_{w,2}} \to \mathfrak{g}_2 \to \mathfrak{g}_0[1]^{\phi_{w,1}} \to 0, \qquad 0 \to \mathfrak{g}_1^{\phi_{w,3}} \to \mathfrak{g}_3 \to \mathfrak{g}_0[1]^{\phi_{w,2}} \to 0,$$
$$0 \to \mathfrak{g}_1^{\phi_{w,1}} \to \mathfrak{g}_{-1}[1] \to \mathfrak{g}_0[1]^{\phi_{w,2}} \to 0, \qquad 0 \to \mathfrak{g}_1^{\phi_{w,2}} \to \mathfrak{g}_{-2}[1] \to \mathfrak{g}_0[1]^{\phi_{w,3}} \to 0.$$

1.3. **ACM line bundles on scrolls.** ACM line bundles on Hirzebruch surfaces, and more generally on rational normal scrolls are well-known, cf. [MR13].

Lemma 1.1. *Let $\mathscr{E}$ be an initialized ACM line bundle on $S(\vartheta,\vartheta+\epsilon)$. Then $\mathscr{E} \simeq \mathscr{O}_X(\ell F)$ for $0 \leq \ell \leq d_X - 1$, or $\mathscr{E} \simeq \mathscr{O}_X(H-F)$. Also, $\mathscr{E}$ is Ulrich iff $\mathscr{E} \simeq \mathscr{O}_X(H-F)$ or $\mathscr{E} \simeq \mathscr{O}_X((d_X-1)F) = \mathscr{L}(H)$.*

1.4. **Basic form of ACM bundles on surface scrolls.** Let now $\mathscr{E}$ be an ACM sheaf on X. We will compute in more detail the resolution (1.3). Set:

$$a_{i,j} = \mathrm{h}^i(X,\mathscr{E}(-jF)), \qquad b_{i,j} = \mathrm{h}^i(X,\mathscr{E}((1-j)F-H)).$$

We also set $a = a_{1,1}$ and $b = b_{1,0}$. Since $\mathscr{E}$ is ACM we have $a_{1,0} = b_{1,1} = 0$. Then, (1.4) can be broken up into two exact sequences:

$$(1.9) \qquad 0 \to \mathscr{O}_X(-F)^{a_{0,1}} \to \mathscr{O}_X^{a_{0,0}} \to \pi^*\pi_*\mathscr{E} \to \mathscr{O}_X(-F)^a \to 0,$$

$$(1.10) \qquad 0 \to \pi^*\mathbf{R}^1\pi_*\mathscr{E} \to \mathscr{O}_X(-F)^{a_{2,1}} \to \mathscr{O}_X^{a_{2,0}} \to 0.$$

Since $\Theta\Theta^*\mathscr{E}$ is the cone of (1.5), we read its cohomology in the exact sequences:

$$(1.11) \qquad 0 \to \mathscr{L}^b \to \mathscr{H}^0\Theta\Theta^*\mathscr{E} \to \mathscr{L}(-F)^{b_{2,1}} \to \mathscr{L}^{b_{2,0}} \to 0,$$

$$(1.12) \qquad 0 \to \mathscr{L}(-F)^{b_{0,1}} \to \mathscr{L}^{b_{0,0}} \to \mathscr{H}^{-1}\Theta\Theta^*\mathscr{E} \to 0.$$

Taking cohomology of (1.3) we get the long exact sequence:

$$(1.13) \qquad 0 \to \mathscr{H}^{-1}\Theta\Theta^*\mathscr{E} \xrightarrow{\varphi} \pi^*\pi_*\mathscr{E} \to \mathscr{E} \to \mathscr{H}^0\Theta\Theta^*\mathscr{E} \to \pi^*\mathbf{R}^1\pi_*\mathscr{E} \to 0.$$

Let I and J be the images of the middle maps of (1.9) and (1.11), so that:

$$(1.14) \qquad 0 \to \mathscr{O}_X(-F)^{a_{0,1}} \to \mathscr{O}_X^{a_{0,0}} \to I \to 0,$$

$$(1.15) \qquad 0 \to J \xrightarrow{g_1} \mathscr{L}(-F)^{b_{2,1}} \xrightarrow{g_2} \mathscr{L}^{b_{2,0}} \to 0.$$

We claim:

$$\pi^*\pi_*\mathscr{E} = I \oplus \mathscr{O}_X(-F)^a, \qquad \mathscr{H}^0\Theta\Theta^*\mathscr{E} = J \oplus \mathscr{L}^b.$$

Indeed, looking at (1.9), we have to check that $\pi^*\pi_*\mathscr{E} \to \mathscr{O}_X(-F)^a$ splits, and it suffices to prove that $\mathrm{Ext}^1_X(\mathscr{O}_X(-F), I) = 0$. This in turn is obtained twisting (1.14) by $\mathscr{O}_X(F)$ and computing cohomology. In a similar way one proves that $\mathscr{L}^b \to \mathscr{H}^0\Theta\Theta^*\mathscr{E}$ splits.

Now observe that the restriction of $\pi^*\pi_*\mathscr{E} \to \mathscr{E}$ to the summand $\mathscr{O}_X(-F)^a$ of $\pi^*\pi_*\mathscr{E}$ is injective. Indeed, its kernel is $\mathscr{H}^{-1}\Theta\Theta^*\mathscr{E}$, which is dominated by the bundle $\mathscr{L}^{b_{0,0}}$ in view of (1.12). But there are no non-trivial maps $\mathscr{L} \to \mathscr{O}_X(-F)$. Similarly, $\mathscr{E} \to \mathscr{H}^0\Theta\Theta^*\mathscr{E}$ is surjective onto $\mathscr{L}^b$. We define thus the sheaves P and Q by the sequences:

$$(1.16) \qquad 0 \to Q \to J \to \pi^*\mathbf{R}^1\pi_*\mathscr{E} \to 0,$$

$$(1.17) \qquad 0 \to \mathscr{H}^{-1}\Theta\Theta^*\mathscr{E} \to I \to P \to 0.$$

Summing up, we have shown the following basic result.

Lemma 1.2. *Let $\mathscr{E}$ be an ACM bundle on X. Then $\mathscr{E}$ fits into:*

$$0 \to P \oplus \mathscr{O}_X(-F)^a \to \mathscr{E} \to Q \oplus \mathscr{L}^b \to 0,$$

where P and Q fit into (1.17) *and* (1.16)*, and I and J fit into* (1.14) *and* (1.15).

1.5. **A splitting criterion for scrolls of low degree.** Let us now assume, for the rest of the section, $\epsilon + \vartheta \leq 3$. Namely, we focus on the following scrolls (we display their representation type, although tameness of quartic scrolls will be apparent from §3).

finite	tame	wild
$S(1,1)$	$S(2,2)$	$S(2,3)$
$S(1,2)$	$S(1,3)$	$S(3,3)$

Lemma 1.3. *Let $\mathscr{E}$ be ACM on X, and P and Q as in Lemma 1.2. Then* $\mathrm{Ext}^1_X(Q,P) = 0$.

Proof. To show this, we apply $\mathrm{Hom}_X(Q,-)$ to (1.17), obtaining:

$$\mathrm{Ext}^1_X(Q,I) \to \mathrm{Ext}^1_X(Q,P) \to \mathrm{Ext}^2_X(Q,\mathscr{H}^{-1}\Theta\Theta^*\mathscr{E}).$$

We want to show that the outer terms of this exact sequence vanish. For the leftmost term, using (1.14), we are reduced to show:

$$\mathrm{Ext}^1_X(Q,\mathscr{O}_X) \simeq \mathrm{H}^1(X, Q \otimes \mathscr{L}(-F-H))^* = 0, \tag{1.18}$$

$$\mathrm{Ext}^2_X(Q,\mathscr{O}_X(-F)) \simeq \mathrm{H}^0(X, Q \otimes \mathscr{L}(-H))^* = 0, \tag{1.19}$$

where the isomorphisms are given by Serre duality. For the rightmost term, by (1.12) we need:

$$\mathrm{Ext}^2_X(Q,\mathscr{L}) \simeq \mathrm{H}^0(X, Q(-H-F))^* = 0. \tag{1.20}$$

In turn, by (1.16), it suffices to show:

$$\mathrm{H}^0(X, J \otimes \mathscr{L}(-H)) = \mathrm{H}^0(X, J(-H-F)) = 0, \qquad \text{for (1.19), (1.20),}$$

$$\mathrm{H}^1(X, J \otimes \mathscr{L}(-F-H)) = \mathrm{H}^0(X, \pi^*\mathbf{R}^1\pi_*\mathscr{E} \otimes \mathscr{L}(-F-H)) = 0, \qquad \text{for (1.18).} \tag{1.21}$$

The first line follows by taking global sections of (1.15), twisted by $\mathscr{L}(-H)$, or by $\mathscr{O}_X(-H-F)$. Similarly, the first vanishing required for (1.21) follows from (1.15) and Serre duality, since $\mathrm{H}^2(X,\mathscr{L}^*) = 0$ and $\mathrm{H}^1(X,\mathscr{L}^*(F)) = 0$ for $\epsilon + \vartheta \leq 3$ (here is where the bound on the invariants appears). The last vanishing follows taking global sections of (1.10), twisted by $\mathscr{L}(-F-H)$. □

Proposition 1.4. *Any non-zero ACM bundle $\mathscr{E}$ satisfying:*

$$\mathrm{H}^1(X, \mathscr{E}(tH-F)) = 0, \quad \mathrm{H}^1(X, \mathscr{E}(tH+F)) = 0, \quad \textit{for all } t \in \mathbb{Z}, \tag{1.22}$$

splits as a direct sum of line bundles.

Proof. We borrow the notation from the above discussion, and we assume $a = b = 0$. First note that the sheaves I and $\mathscr{H}^{-1}\Theta\Theta^*\mathscr{E}$ are torsion-free and in fact locally free, since they are obtained as pull-back of direct images of $\mathscr{E}$ and $\mathscr{E}(-H)$ via π. More precisely, by (1.14), there are integers $r \geq 1$ and $p_i \geq 0$ such that:

$$I \simeq \bigoplus_{i=0,\ldots,r} \mathscr{O}_X(iF)^{p_i}. \tag{1.23}$$

We can assume that $\mathscr{E}$ is initialized, hence $\mathrm{H}^0(X,\mathscr{E}(-H)) = 0$. We get:

$$\mathscr{H}^{-1}\Theta\Theta^*\mathscr{E} \simeq \mathscr{L}^{b_{0,0}}. \tag{1.24}$$

In view of the previous lemma, we have that $\mathscr{E}$ is the direct sum of P and Q. Since $\mathrm{H}^0(X,Q) = 0$, to conclude, it remains to prove that P is a (possibly zero) direct sum of line bundles. Indeed, if $P \neq 0$, we may split off P from $\mathscr{E}$ and use induction on the rank to get out statement; on the other hand $P = 0$ leads to a contradiction since $\mathrm{H}^0(X,\mathscr{E} =\neq 0$ and $\mathrm{H}^0(X,Q) = 0$. Using (1.24), the exact sequence (1.17) becomes:

$$0 \to \mathscr{L}^{b_{0,0}} \to \bigoplus_{i=0,\ldots,r} \mathscr{O}_X(iF)^{p_i} \to P \to 0.$$

Twisting this sequence by $\mathscr{O}_X(-H-F)$, since $\mathrm{H}^k(X,\mathscr{O}_X((i-1)F-H)) = 0$ for any i and any k, we get $\mathrm{H}^1(X,P(-H-F)) \simeq \Bbbk^{b_{0,0}}$. But this space must be zero, since P is a direct summand of $\mathscr{E}$, and $\mathscr{E}$ satisfies (1.22). We deduce $P \simeq \bigoplus_{i=1,\ldots,r} \mathscr{O}_X(iF)^{p_i}$. □

This allows to give the following refinement of Lemma 1.2.

Lemma 1.5. *Let $\mathscr{E}$ be an indecomposable ACM bundle on X, assume $\vartheta \geq 2$ and let I, J, P, Q, $a_{i,j}$ and $b_{i,j}$ be as above. If $P \neq 0$ and $b \neq 0$, then $\vartheta + \epsilon = 3$ and $P \simeq I \simeq \mathscr{O}_X^{a_{0,0}}$.*

Proof. As in the proof of Proposition 1.4, I takes the form (1.23). Set $p = p_0$ and rewrite I as:

$$I = \mathscr{O}_X^p \oplus I', \qquad I' = \bigoplus_{i=1,\dots,r} \mathscr{O}_X(iF)^{p_i}.$$

This time, by (1.12) we get a splitting of $\mathscr{H}^{-1}\Theta\Theta^*\mathscr{E}$ of the form:

$$\mathscr{H}^{-1}\Theta\Theta^*\mathscr{E} \simeq \bigoplus_{i=1,\dots,s} \mathscr{L}(b_iF), \tag{1.25}$$

for some integers $s \geq 0$, $b_i \geq 0$.

We now observe that the map φ appearing in (1.13) is zero, when restricted to the summand $\mathscr{O}_X^p$ of I, because $\mathrm{H}^0(X, \mathscr{L}^*) = 0$ when $\vartheta \geq 2$. Therefore, $P = \mathscr{O}_X^p \oplus P'$, where P' fits into:

$$0 \to \mathscr{H}^{-1}\Theta\Theta^*\mathscr{E} \to I' \to P' \to 0. \tag{1.26}$$

Next, we use that $\mathscr{E}$ is indecomposable. We tensor the previous exact sequence with $\mathscr{L}^*$ and note that $i \geq 1$ implies $\mathrm{H}^1(X, I' \otimes \mathscr{L}^*)) = 0$, while (1.25) easily gives $\mathrm{H}^2(X, \mathscr{H}^{-1}\Theta\Theta^*\mathscr{E} \otimes \mathscr{L}^*) = 0$. Therefore $\mathrm{H}^1(X, P' \otimes \mathscr{L}^*) = 0$. In view of Lemma 1.3, since $P = P' \oplus \mathscr{O}_X^p$ and $\mathscr{E}$ is indecomposable, this implies $P' = 0$ and $\mathrm{H}^1(X, \mathscr{L}^*) \neq 0$. This says $\vartheta + \epsilon = 3$. Now (1.26) implies that $\mathscr{H}^{-1}\Theta\Theta^*\mathscr{E}$ and I' are both zero (indeed, they should be isomorphic, but they are direct sums of line bundles with different coefficients of H), so $P \simeq I \simeq \mathscr{O}_X^{a_{0,0}}$. □

An essentially identical argument shows that, if $\mathscr{E}$ is indecomposable ACM with $Q \neq 0$ and $a \neq 0$, then $Q \simeq J \simeq \mathscr{L}(-F)^{b_{2,1}}$ and $\vartheta + \epsilon = 3$. We have proved:

Proposition 1.6. *Let $\epsilon + \vartheta \leq 3$, $\vartheta \geq 2$ and $\mathscr{E}$ be an indecomposable ACM bundle on X. Set:*

$$a = a_{1,1}, \quad b = b_{1,0}, \quad c = a_{0,0}, \quad d = b_{2,1}.$$

Then $\mathscr{E}$ fits into:

$$0 \to \mathscr{O}_X^c \oplus \mathscr{O}_X(-F)^a \to \mathscr{E} \to \mathscr{L}^b \oplus \mathscr{L}(-F)^d \to 0.$$

1.6. **Monads.** Let $\epsilon+\vartheta \leq 3$ and $\vartheta \geq 2$. Given an indecomposable ACM bundle $\mathscr{E}$ on $X = S(\vartheta, \vartheta+\epsilon)$, by Proposition 1.6 we can consider the kernel $\mathscr{K}$ of the natural projection $\mathscr{E} \to \mathscr{L}(-F)^d$, and the cokernel $\mathscr{C}$ of the natural injection $\mathscr{O}_X^c \to \mathscr{E}$. This injection factors through $\mathscr{K}$ and we let $\mathscr{F}$ be cokernel of the resulting map. We will see in a minute that $\mathscr{F}$ is an Ulrich bundle. We have thus a complex whose cohomology is $\mathscr{F}$ (a *monad*) of the form:

$$0 \to \mathscr{O}_X^c \to \mathscr{E} \to \mathscr{L}(-F)^d \to 0.$$

The *display* of the monad is the following commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathscr{O}_X^c & = & \mathscr{O}_X^c & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathscr{K} & \to & \mathscr{E} & \to & \mathscr{L}(-F)^d & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathscr{F} & \to & \mathscr{C} & \to & \mathscr{L}(-F)^d & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{1.27}$$

Note that some monads can also be constructed when $\vartheta = 1$ cf. the following example.

Example 1.7. Let $X = S(1,3)$, and observe that $\mathrm{h}^1(X, \mathscr{L}^*) = 1$. Define the rank-2 bundle $\mathscr{V}$ as the non-trivial extension:

$$0 \to \mathscr{O}_X \to \mathscr{V} \to \mathscr{L} \to 0.$$

So in this case $c = b = 1$ and $a = d = 0$. Note that $\mathscr{V}$ is slope-unstable, and of course ACM. It is clear that $\mathrm{H}^1(X, \mathscr{V}^*) = 0$, since $1 \in \mathrm{H}^0(X, \mathscr{O}_X)$ is sent by the boundary map to the generator of $\mathrm{H}^1(X, \mathscr{L}^*)$. Also, $\mathscr{V} \otimes \mathscr{V}^*$ fits into:

$$0 \to \mathscr{V}^* \to \mathscr{V} \otimes \mathscr{V}^* \to \mathscr{V} \to 0,$$

and it easily follows that $\mathrm{Ext}_X^1(\mathscr{V}, \mathscr{V}) = 0$, so $\mathscr{V}$ is rigid (albeit not simple, since $\mathrm{hom}_X(\mathscr{V}, \mathscr{V}) = 2$). Taking $\mathscr{V}(-F)$, we get a monad with $a = d = 1$ and $b = c = 0$.

However, for $X = S(1,3)$, Proposition 1.6 does not always apply. Indeed, observe:

$$\mathrm{ext}_X^1(\mathscr{L}, \mathscr{O}_X(H-F)) = \mathrm{h}^1(X, \mathscr{O}_X(2H-4F)) = 1,$$

and define the bundle $\mathscr{W}$ as fitting in the associated non-trivial extension:

$$0 \to \mathscr{O}_X(H-F) \to \mathscr{W} \to \mathscr{L} \to 0.$$

Clearly $\mathscr{W}$ is an initialized ACM sheaf on X. Also, the displayed extension is Harder-Narasimhan filtration of $\mathscr{W}$, which shows that $\mathscr{W}$ is indecomposable. Just as for $\mathscr{V}$, one checks that $\mathscr{W}$ is rigid.

2. Parametrizing Ulrich bundles via the Kronecker quiver

Let again X be the scroll $S(\vartheta, \vartheta+\epsilon)$ of degree $d_X = 2\vartheta+\epsilon$. We will carry out here a description of Ulrich bundles on X as extensions, and relate them to representations of a Kronecker quiver.

2.1. **Ulrich bundles as extensions.** The first consequence of the computation we carried out in the previous section is the next result, which proves part (i) of Theorem B.

Proposition 2.1. *A vector bundle $\mathscr{E}$ on X is Ulrich iff, up to a twist, it fits into:*

$$0 \to \mathscr{O}_X(-F)^a \to \mathscr{E} \to \mathscr{L}^b \to 0, \tag{2.1}$$

with $a = \mathrm{h}^1(X, \mathscr{E}(-F))$ and $b = \mathrm{h}^1(X, \mathscr{E}(F-H))$.

Proof. Let us borrow notations from §1.4. By Lemma 1.1, if $\mathscr{E}$ fits into (2.1) for some a and b, then it is obviously Ulrich. Also, in this case the integers a and b equal $a_{1,1}$ and, respectively, $b_{1,0}$ as one sees by tensoring (2.1) with $\mathscr{O}_X(-F)$ and $\mathscr{O}_X(F-H)$ and taking cohomology.

Conversely, if $\mathscr{E}$ is Ulrich, then (up to a twist) we may assume $a_{0,0} = b_{2,1} = 0$ by [ESW03, Proposition 2.1]. Therefore, by (1.14) and (1.15) we get $I = J = 0$ hence $P = Q = 0$ by (1.17) and (1.16). We conclude by Lemma 1.2. □

Remark 2.2. We have $\mathrm{ext}^1_X(\mathscr{L}, \mathscr{O}_X(-F)) > 2$ for all $d_X > 4$. Then, extensions of the form (2.1) provide arbitrarily large families of Ulrich bundles, except for $d_X \leq 4$. This is the argument of [MR13], which overlooks the case $d_X = 4$ only.

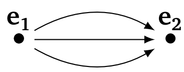
Recall that Ulrich bundles are semistable. Then, their moduli space is a subset of Maruyama's moduli space of S-classes of semistable sheaves (with respect to H) of fixed Hilbert polynomial.

Corollary 2.3. *For any integer $r \geq 1$, the moduli space of Ulrich bundles of rank r is supported at $r+1$ distinct points, characterized by $c_1(\mathscr{E})$.*

Proof. If $\mathscr{E}$ is an Ulrich bundle of rank r, then by Proposition 2.1 $\mathscr{E}$ fits into an exact sequence of the form (2.1), with $a_{1,1} + b_{1,0} = r$. Set $a = a_{1,1}$, so $b_{0,1} = r - a$. Since $\mathscr{O}_X(-F)$ and $\mathscr{L}$ have the same Hilbert polynomial, the graded object associated with $\mathscr{E}$ is $\mathscr{O}_X(-F)^a \oplus \mathscr{L}^{r-a}$. Hence the S-equivalence class of $\mathscr{E}$ only depends on a, and in turn a is clearly determined by $c_1(\mathscr{E})$. □

2.2. **Kronecker quivers.** Let $w \geq 2$ and let Υ_w be the Kronecker quiver with two vertexes $\mathbf{e_1}$, $\mathbf{e_2}$ and w arrows from $\mathbf{e_1}$ to $\mathbf{e_2}$.

Υ_3:



2.2.1. *Representations of Kronecker quivers.* Given integers a and b, a representation $\mathscr{R}$ of Υ_X of dimension vector (b, a) is given by a pair of vector spaces, B and A with $\dim(B) = b$ and $\dim(A) = a$ and w linear maps $B^* \to A$. We have the obvious notions such as direct sum, irreducible representation etc. If $\mathscr{R}$ has dimension vector (b, a), its deformation space has virtual dimension:

$$\psi(a, b) = wab - a^2 - b^2 + 1.$$

We think of a representation of Υ_w of dimension vector (b, a) as a matrix M of size $a \times b$ whose entries are linear forms in w variables, namely if our w linear maps have matrices $M_1, \ldots, M_w$ in a fixed basis, we write $M = x_1 M_1 + \cdots + x_w M_w$. If $w = 2$ we write $\boldsymbol{x} = x_1$ and $\boldsymbol{y} = x_2$.

Let W be a vector space of dimension w with a basis indexed by the arrows of Υ_w. The representation $\mathscr{R}$ then corresponds uniquely to an element ξ of $A \otimes B \otimes W^*$, so we write $\mathscr{R} = \mathscr{R}_\xi$. We write M_ξ the matrix corresponding to $\xi \in A \otimes B \otimes W^*$. It is customary to write $M_\xi = M_{\xi'} \boxplus M_{\xi''}$ when $\mathscr{R}_\xi = \mathscr{R}_{\xi'} \oplus \mathscr{R}_{\xi''}$. Geometrically, the space $\mathbb{P}W$ parametrizes isomorphism classes of non-zero representations of Υ_w with dimension vector $(1,1)$. The matrix M_ξ is naturally written as a morphism of sheaves over the dual space:

$$M : B^* \otimes \mathscr{O}_{\check{\mathbb{P}}W}(-1) \to A \otimes \mathscr{O}_{\check{\mathbb{P}}W}.$$

Via the natural isomorphism $H^0(\check{\mathbb{P}}W, \mathscr{O}_{\check{\mathbb{P}}W}(1)) \simeq W^* \simeq H^0(\mathbb{P}W, \mathscr{T}_{\mathbb{P}W}(-1))$, the matrix M_ξ is transformed into a matrix of twisted vector fields.

We write $\mathbf{D}^b(\Upsilon_w)$ for the derived category of $\Bbbk$-representations of Υ_w. It is naturally equivalent to the full subcategory $\langle \Omega_{\mathbb{P}W}(1), \mathscr{O}_{\mathbb{P}W}\rangle$ of $\mathbf{D}^b(\mathbb{P}W)$. Given a non-zero vector of W^* and the corresponding point $\boldsymbol{u} \in \mathbb{P}W$, there is a morphism $\Omega_{\mathbb{P}W}(1) \to \mathscr{O}_{\mathbb{P}W}$ vanishing at $\boldsymbol{u}$. The cone $S_{\boldsymbol{u}}$ of this morphism is mapped by this equivalence to the associated $(1,1)$-representation $\mathscr{R}_{\boldsymbol{u}}$ of Υ_w.

2.2.2. *Irregular exceptional pairs and the universal extension.* Let $(\mathscr{P},\mathscr{N})$ be an exceptional pair over a surface scroll X, with $\mathscr{P} = |\mathscr{P}|$ and $\mathscr{N} = |\mathscr{N}|$, and $\mathrm{Ext}^i_X(\mathscr{P},\mathscr{N}) = 0$ for $i \neq 1$. Define $W^* = \mathrm{Ext}^1_X(\mathscr{P},\mathscr{N})$ and $w = \dim(W)$. Over $X \times \mathbb{P}W$ we have the universal extension:

$$0 \to \mathscr{N} \boxtimes \mathscr{O}_{\mathbb{P}W} \to \mathscr{W} \to \mathscr{P} \boxtimes \mathscr{O}_{\mathbb{P}W}(-1) \to 0.$$

The functor $\Phi : \mathbf{D}^b(\mathbb{P}W) \to \mathbf{D}^b(X)$ defined by $\mathbf{R}q_*(p^*(-)\otimes \mathscr{W})$ gives an equivalence of $\mathbf{D}^b(\Upsilon_w)$ onto the subcategory $\langle \mathscr{P},\mathscr{N}\rangle \subset \mathbf{D}^b(X)$ sending $\mathscr{R}_\xi$ to the extension $\mathscr{F} = \mathscr{W}|_{X\times\{\xi\}}$ fitting into:

$$0 \to A \otimes \mathscr{N} \to \mathscr{F} \to B^* \otimes \mathscr{P} \to 0.$$

Write $\mathscr{F}_{\boldsymbol{u}}$ for the extension of $\mathscr{P}$ by $\mathscr{N}$ corresponding to $\boldsymbol{u} \in \mathbb{P}W$. We have:

$$\Phi(\mathscr{O}_{\mathbb{P}W}) \simeq \mathscr{N}, \qquad \Phi(\Omega_{\mathbb{P}W}(1)) \simeq \mathscr{P}[-1], \qquad \Phi(S_{\boldsymbol{u}}) \simeq \mathscr{F}_{\boldsymbol{u}}. \tag{2.2}$$

Lemma 2.4. *Let $w \geq 2$, $\xi \in A \otimes B \otimes W^*$ and set $\mathscr{R} = \mathscr{R}_\xi$ and $\mathscr{F} = \Phi(\mathscr{R})$.*

i) We have $\mathrm{Ext}^i_{\Upsilon_w}(\mathscr{R},\mathscr{R}) \simeq \mathrm{Ext}^i_X(\mathscr{F},\mathscr{F})$, and this space is zero for $i \neq 0,1$.
ii) The bundle $\mathscr{F}$ is indecomposable if and only if $\mathscr{R}$ is irreducible.
iii) Let (b,a) be the dimension vector of $\mathscr{R}$. Then we have the cases.
  (a) If $\psi(a,b) < 0$ then $\mathscr{R}$ is decomposable, hence so is $\mathscr{F}$.
  (b) If $\mathscr{F}$ is indecomposable and rigid then $\psi(a,b) = 0$, and this happens if and only if $\{a,b\} = \{\phi_{w,k}, \phi_{w,k+1}\}$ for some $k \geq 0$, cf. §1.2.2.
  (c) If $\psi(a,b) = 0$, then $\mathscr{F}$ is exceptional for general ξ. Also, $\mathscr{F}$ is exceptional if and only if there is a well-determined integer k such that $\mathscr{F} \simeq \mathfrak{g}_k$ and $k \geq 1$ or $\mathscr{F} \simeq \mathfrak{g}_k[1]$ and $k \leq 0$.
iv) The bundle $\mathscr{F}$ is rigid if and only if there are integers $k \neq 0$, a_k and a_{k+1} such that:

$$\mathscr{F} \simeq |\mathfrak{g}_k|^{a_k} \oplus |\mathfrak{g}_{k+1}|^{a_{k+1}}.$$

Proof. (i) is clear since Φ is fully faithful and representations of Υ_w form a hereditary category. For (ii), note that $\mathscr{F}$ is decomposable if and only if $\mathrm{Hom}_{\Upsilon_w}(\mathscr{R},\mathscr{R})$ contains a non-trivial idempotent. By (i), this happens if and only if $\mathrm{Hom}_{\Upsilon_w}(\mathscr{R},\mathscr{R})$ contains a non-trivial idempotent, i.e., if and only if $\mathscr{R}$ is irreducible.

Part (iii) follows from [Kac80, Theorem 4], cf. also [Kac80, Remarks a, b], as $\psi(a,b) < 0$ directly implies that $\mathscr{R}$ is decomposable, while $\psi(a,b) = 0$ precisely means that (b,a) is a Schur root in the sense of Kac. Since indecomposable Schurian representations are well-known to be uniquely determined by their dimension vector, we conclude that they correspond precisely to the Fibonacci bundles $|\mathfrak{g}_k|$ by [Bra08, Bra05]. Part (iv) follows from [Fae15, Lemma 5]. □

2.2.3. *Proof of parts* (ii) *and* (iii) *of Theorem B.* Now we apply this approach to Ulrich bundles. If $\mathscr{U}$ is an Ulrich bundle over X, then, up to a twist, $\mathscr{U}$ fits into an extension of the form (2.1) by Proposition 2.1. Set $W = \mathrm{Ext}^1_X(\mathscr{L}, \mathscr{O}_X(-F))^*$, so that $w = \dim(W) = d_X - 2$. Then, there is some $\xi \in A \otimes B \otimes W^*$ determined up to a non-zero scalar such that $\mathscr{U} \simeq \Phi(\mathscr{R}_\xi)$. We write $\mathscr{U} = \mathscr{U}_\xi$, and we call this the Ulrich bundle associated with ξ. Note that this is an instance of a bundle referred to as $\mathscr{F} = \Phi(\mathscr{R}_\xi)$ from §2.2.2, but we use the letter $\mathscr{U}$ when dealing with Ulrich bundles.

Having this in mind we see that, if $\mathscr{U}$ is indecomposable and rigid, then looking at the weight vector (b,a) of $\mathscr{R}_\xi$, the unordered pair $\{a,b\}$ must give a Schur root, i.e. a and b must be two Fibonacci numbers of the form $\phi_{w,k}$, $\phi_{w,k-1}$, for $k \geq 1$. Moreover for the weight vector $(b,a) = (\phi_{w,k-1}, \phi_{w,k})$ there is a unique indecomposable bundle, which we denote by $\mathfrak{u}_k$, and each $\mathfrak{u}_k$ is exceptional (we will use gothic letters only for rigid objects).

We notice that, since $\mathscr{O}_X(-F)$ and $\mathscr{L}$ are interchanged by dualizing and twisting by $\omega_X(H)$, the bundles $\mathfrak{u}_k(-H)$ and $\mathfrak{u}^*_{1-k} \otimes \omega_X$ are both indecomposable and rigid with the same weight vector, and must then be isomorphic by Lemma 2.4, part (iiic).

As an explicit example, over $X = S(2,3)$, the first Fibonacci sequences for $\mathfrak{u}_k$ are:

$$\mathfrak{u}_1 = \mathscr{O}_X(-F), \qquad 0 \to \mathscr{O}_X(-F)^3 \to \mathfrak{u}_2 \to \mathscr{L} \to 0,$$
$$0 \to \mathscr{O}_X(-F)^8 \to \mathfrak{u}_3 \to \mathscr{L}^3 \to 0,$$
$$\mathfrak{u}_0 = \mathscr{L}, \qquad 0 \to \mathscr{O}_X(-F) \to \mathfrak{u}_{-1} \to \mathscr{L}^3 \to 0,$$
$$0 \to \mathscr{O}_X(-F)^3 \to \mathfrak{u}_{-2} \to \mathscr{L}^8 \to 0,$$
$$0 \to \mathscr{O}_X(-F)^8 \to \mathfrak{u}_{-3} \to \mathscr{L}^{21} \to 0.$$

2.3. **Matrix pencils.** Representations of Kronecker quivers are completely classified only if $w = 2$. In this case we canonically have $W^* \simeq W$. We write $\mathbb{P}^1 = \mathbb{P}W$, so $\mathbf{D}^b(\Upsilon_2) \simeq \mathbf{D}^b(\mathbb{P}^1)$, and morphisms of the form $M = M_\xi$ are matrix pencils. These are classified by Kronecker-Weierstrass theory, which we recall for the reader's convenience. We refer to [BCS97, Chapter 19.1] for proofs. Fixing variables $\boldsymbol{x}, \boldsymbol{y}$ on $\mathbb{P}^1$, and given positive integers u, v, n and $\boldsymbol{u} \in \Bbbk$ one defines:

$$\mathfrak{c}_u = \begin{pmatrix} \boldsymbol{x} & & & \\ \boldsymbol{y} & \boldsymbol{x} & & \\ & \boldsymbol{y} & \ddots & \\ & & \ddots & \boldsymbol{x} \\ & & & \boldsymbol{y} \end{pmatrix}, \quad \mathfrak{b}_v = \begin{pmatrix} \boldsymbol{x} & \boldsymbol{y} & & & \\ & \boldsymbol{x} & \boldsymbol{y} & & \\ & & \ddots & \ddots & \\ & & & \boldsymbol{x} & \boldsymbol{y} \end{pmatrix}, \quad J_{\boldsymbol{u},n} = \begin{pmatrix} \boldsymbol{u} & 1 & & \\ & \ddots & \ddots & \\ & & \boldsymbol{u} & 1 \\ & & & \boldsymbol{u} \end{pmatrix},$$

and $\mathscr{J}_{\boldsymbol{u},n} = \boldsymbol{x} I_n + \boldsymbol{y} J_{\boldsymbol{u},n}$, where $\mathfrak{c}_u$ has size $(u+1) \times u$, $\mathfrak{b}_v$ has size $v \times (v+1)$, and $J_{\boldsymbol{u},n} \in \Bbbk^{n\times n}$. The next lemma is obtained combining [BCS97, Theorem 19.2 and 19.3], with the caveat that, up to changing basis in $\mathbb{P}^1$, we can assume that a matrix pencil M has no *infinite elementary* divisors, i.e., the morphism $M : B^* \otimes \mathscr{O}_{\mathbb{P}^1}(-1) \to A \otimes \mathscr{O}_{\mathbb{P}^1}$ has constant rank around $\infty = (0:1) \in \mathbb{P}^1$.

Lemma 2.5. *Up to possibly changing basis in $\mathbb{P}^1$, any matrix pencil M is equivalent to:*

$$\mathfrak{c}_{u_1} \boxplus \cdots \boxplus \mathfrak{c}_{u_r} \boxplus \mathfrak{b}_{v_1} \boxplus \cdots \boxplus \mathfrak{b}_{v_s} \boxplus \mathscr{J}_{n_1,\boldsymbol{u}_1} \boxplus \cdots \boxplus \mathscr{J}_{n_t,\boldsymbol{u}_t} \boxplus \mathfrak{z}_{a_0,b_0},$$

for some integers r, s, t, a_0, b_0 and u_i, v_j, n_k, and some $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_t \in \Bbbk$.

We have the following straightforward isomorphisms:

$$\operatorname{coker}(\mathfrak{c}_u) \cong \mathscr{O}_{\mathbb{P}^1}(u), \qquad \ker(\mathfrak{b}_v) \cong \mathscr{O}_{\mathbb{P}^1}(-v-1), \qquad \operatorname{coker}(\mathscr{J}_{n,\boldsymbol{u}}) \cong \mathscr{O}_{n\boldsymbol{u}},$$

where $\mathscr{O}_{n\boldsymbol{u}}$ is the skyscraper sheaf over the point $\boldsymbol{u}$ with multiplicity n. Allowing $\boldsymbol{u}$ to vary in $\mathbb{P}^1$ instead of $\Bbbk$ only amounts to authorizing infinite elementary divisors too.

Proposition 2.6. *Assume $w = 2$, let ξ be an element of $A \otimes B \otimes W$ and set $\mathscr{E} = \mathscr{U}_\xi$, $M = M_\xi$. If $\mathscr{E}$ is indecomposable, then $\operatorname{Ext}^2_X(\mathscr{E}, \mathscr{E}) = 0$, $|a - b| \le 1$ and:*
- *(i) if $a = b + 1$ then $M \simeq \mathfrak{b}_b$ and $\mathscr{E}$ is exceptional;*
- *(ii) if $a = b - 1$, then $M \simeq \mathfrak{c}_{b-1}$ and $\mathscr{E}$ is exceptional;*
- *(iii) if $a = b$ then $M \simeq \mathscr{J}_{a,\boldsymbol{u}}$ for some $\boldsymbol{u} \in \mathbb{P}^1$, and the indecomposable deformations of $\mathscr{E}$ vary in a 1-dimensional family, parametrized by a projective line.*

Proof. By Lemma 2.4 we can assume that M is irreducible, so that M is itself isomorphic to one of the summands appearing in Lemma 2.5. Consider the equivalence:

$$\langle \mathscr{P}, \mathscr{N} \rangle \simeq \mathbf{D}^b(\Upsilon_2) \simeq \mathbf{D}^b(\mathbb{P}^1). \tag{2.3}$$

If $M \simeq \mathfrak{c}_u$, then this equivalence maps $\mathscr{E}$ to $\operatorname{coker}(\mathfrak{c}_u) \simeq \mathscr{O}_{\mathbb{P}^1}(u)$, with $u \ge 1$. Since $\mathscr{O}_{\mathbb{P}^1}(u)$ is exceptional, Using part (i) of Lemma 2.4, we get that $\mathscr{E}$ is an also an exceptional bundle. For the case $\mathfrak{b}_v$, (2.3) sends $\mathscr{E}$ to $\ker(\mathfrak{b}_v)[1] \simeq \mathscr{O}_{\mathbb{P}^1}(-v-1)[1]$, with $v \ge 1$. Again $\mathscr{E}$ is then exceptional. The same argument works for $\mathfrak{z}_{1,0}$ in which case $\mathscr{E} \simeq \mathscr{L}$ is mapped to $\mathscr{O}_{\mathbb{P}^1}(-1)[1]$, and for $\mathfrak{z}_{0,1}$, whose counterpart on $\mathbb{P}^1$ is $\mathscr{O}_{\mathbb{P}^1}$, and of course $\mathscr{E} \simeq \mathscr{O}_X(-F)$.

It remains to look at the case $M \simeq \mathscr{J}_{n,\boldsymbol{u}}$, so that $\mathscr{E}$ is mapped under (2.3) to $\mathscr{F} = \mathscr{O}_{n\boldsymbol{u}}$. In this case, $\mathscr{F}$ is filtered by the sheaves $\mathscr{F}_m = \mathscr{O}_{m\boldsymbol{u}}$, for $1 \le m \le n$ and $\mathscr{F}_m / \mathscr{F}_{m-1} \simeq \mathscr{F}_1$. This induces a filtration of $\mathscr{E}$ be the sheaves $\mathscr{E}_m$, image of $\mathscr{F}_m$ via (2.3), having quotients $\mathscr{E}_m / \mathscr{E}_{m-1} \simeq \mathscr{E}_{\boldsymbol{u}}$, recall (2.2). Note that $\mathscr{E}_{\boldsymbol{u}}$ is a simple bundle with $\operatorname{Ext}^1_X(\mathscr{E}_{\boldsymbol{u}}, \mathscr{E}_{\boldsymbol{u}}) \simeq \Bbbk$ once more by Lemma 2.4, and $\operatorname{Ext}^2_X(\mathscr{E}_{\boldsymbol{u}}, \mathscr{E}_{\boldsymbol{u}}) = 0$. Bundles of the form $\mathscr{E}_{\boldsymbol{u}}$ are of course parametrized by $\mathbb{P}^1$. Deformations of $\mathscr{E}$ are thus provided by the motions in $\mathbb{P}^1$ of each of the factors $\mathscr{E}_{\boldsymbol{u}}$ of its filtration. But only bundles associated with sheaves of the form $\mathscr{O}_{n\boldsymbol{u}'}$ continue to be indecomposable, so deformations of $\mathscr{E}$ giving rise to indecomposable bundles correspond exactly to the motions of $n\boldsymbol{u}$ in $\mathbb{P}^1$ itself. □

3. Quartic scrolls are of tame representation type

Our goal here is to prove Theorem A. According to Bertini and del Pezzo's classification, cf. [EH87], a smooth non-degenerate surface X in $\mathbb{P}^5$ has degree 4 if and only if X is a quartic scroll or a Veronese surface, the last case being well understood and excluded of our study. So we actually show two stronger statements, one for each of the two non-isomorphic smooth quartic scrolls, namely $S(2,2)$ and $S(1,3)$, which we treat separately in §3.1 and §3.2. The outcome is a complete classification of ACM bundles on these scrolls. Here, continuous families of indecomposable ACM sheaves only exist for Ulrich bundles of even rank, which are parametrized by $\mathbb{P}^1$. In both cases, all other indecomposable ACM sheaves are rigid, but for $S(1,3)$ the classification of non-Ulrich ACM sheaves is a bit more involved.

3.1. **ACM bundles on** $S(2,2)$**.** We now show a stronger version of Theorem A for $X = S(2,2)$.

Theorem 3.1. *Let $X = S(2,2)$ and let $\mathscr{E}$ be an indecomposable ACM bundle on X. Then, up to a twist, either $\mathscr{E}$ is $\mathscr{O}_X$ or $\mathscr{L}_X(F)$ or $\mathscr{L}(-F)$, either $\mathscr{E}$ is Ulrich, and can be expressed as an extension:*

$$0 \to \mathscr{O}_X(-F)^a \to \mathscr{E} \to \mathscr{L}^b \to 0,$$

for some $a, b \geq 0$ with $|a-b| \leq 1$. In this case:

i) for any (a,b) with $a = b \pm 1$, there exists a unique indecomposable bundle of the form $\mathscr{E}$ as above, and moreover this bundle is exceptional;
ii) for $a = b \geq 1$, the isomorphism classes of indecomposable bundles of the form $\mathscr{E}$ are parametrized by $\mathbb{P}^1$.

Proof. We assume $\mathscr{E}$ is an indecomposable ACM bundle which is not Ulrich over $X = S(2,2)$, and prove that $\mathscr{E}$ is then a line bundle, all the remaining statements being clear by Proposition 2.6.

In view of Proposition 2.1, we know that replacing $\mathscr{E}$ with $\mathscr{E}(tH)$ for any $t \in \mathbb{Z}$, we always get $P \neq 0$ or $Q \neq 0$. But, if $P \neq 0$, then $b = 0$ by Lemma 1.5 since $\vartheta + \epsilon = \vartheta = 2$, and $Q = 0$ by Lemma 1.3. By Lemma 1.2 we now deduce $a = 0$. Likewise if we start with $Q \neq 0$ we conclude $a = b = 0$. By Proposition 1.4, we deduce that $\mathscr{E}$ is a line bundle. The conclusion follows. □

3.2. **ACM bundles on** $S(1,3)$**.** Here we take up the second type of quartic scroll, $X = S(1,3)$. In this case a bit more effort is required to classify ACM bundles, in order to take into account a larger set of sporadic rigid bundles, essentially $\mathscr{V}$ and $\mathscr{W}$ of Example 1.7. Our goal will be to prove the following result.

Theorem 3.2. *Let $X = S(1,3)$. Then any indecomposable ACM bundle $\mathscr{E}$ on X is either $\mathscr{E}$ an Ulrich bundle, or a line bundle, or a twist of $\mathscr{V}$, $\mathscr{V}(-F)$, $\mathscr{W}$. If $\mathscr{E}$ is Ulrich then it is an extension:*

$$0 \to \mathscr{O}_X(-F)^a \to \mathscr{E} \to \mathscr{L}^b \to 0,$$

for some $a, b \geq 0$ with $|a-b| \leq 1$. So:

i) for (a,b) with $a = b \pm 1$, there exists a unique indecomposable $\mathscr{E}$ as above and $\mathscr{E}$ is exceptional;
ii) for $a = b \geq 1$, the isomorphism classes of $\mathscr{E}$'s as above are parametrized by $\mathbb{P}^1$.

3.2.1. *The monad for $S(1,3)$.* Let $\mathscr{E}$ be an ACM bundle on X. The procedure of §1.6 in this case gives a monad which is tantamount to the following display:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & P & = & P & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & \mathscr{K} & \to & \mathscr{E} & \to & Q & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \mathscr{U} & \to & \mathscr{C} & \to & Q & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{3.1}$$

with P, Q defined in Lemma 1.2 and where $\mathscr{U}$ is an Ulrich bundle, called the *Ulrich part* of $\mathscr{E}$.

3.2.2. *Two vanishing results for Ulrich bundles and a lemma on rigid extensions.* We first need to prove that Ulrich bundles of rank greater than one do not mix with other elements of our monad.

Lemma 3.3. *Any Ulrich bundle $\mathscr{U}$ on X, not containing $\mathscr{L}$ as a direct summand, satisfies $\mathrm{H}^1(X, \mathscr{U}^*) = 0$. Likewise $\mathrm{H}^1(X, \mathscr{U} \otimes \mathscr{L}^*(F))) = 0$, unless $\mathscr{O}_X(-F)$ is a summand of $\mathscr{U}$.*

Proof. Recall the notation from §1.2 and 2.2.3. Observe that $U = \mathrm{H}^0(X, \mathscr{O}_X(F))$ is identified with $\mathrm{H}^1(X, \mathscr{L}^*(-F))$ and that W and U are also canonically identified. Let $\mathscr{U} = \mathscr{U}_\xi$, for some $\xi \in A \otimes B \otimes U$. We prove that $\mathrm{H}^1(X, \mathscr{U}^*) = 0$ if $\mathscr{U}$ has no copy of $\mathscr{L}$ as direct summand, the other statement is analogous. We dualize (2.1):

$$0 \to B \otimes \mathscr{L}^* \to \mathscr{U}^* \to A^* \otimes \mathscr{O}_X(F) \to 0. \tag{3.2}$$

Cup product gives a map:

$$\mathrm{H}^0(X, A^* \otimes \mathscr{O}_X(F)) \otimes \mathrm{Ext}^1_X(B \otimes \mathscr{L}, A \otimes \mathscr{O}_X(-F)) \to \mathrm{H}^1(X, B \otimes \mathscr{L}^*).$$

Restricting to $\langle \xi \rangle \subset \mathrm{Ext}^1_X(B \otimes \mathscr{L}, A \otimes \mathscr{O}_X(-F))$ we get a map $\hat{\xi} : A^* \otimes U \to B$ which is the boundary map associated with global sections of (3.2), and we have:

$$A^* \otimes U \xrightarrow{\hat{\xi}} B \to \mathrm{H}^1(X, \mathscr{U}^*) \to 0.$$

Now if $\mathrm{H}^1(X,\mathscr{U}^*)\neq 0$ then the map $\hat{\xi}$ is not surjective, say it factors through $A^*\otimes U \xrightarrow{\hat{\xi}'} B'$ with $B=B'\oplus B''$ and $B''\neq 0$. Then ξ can be written as $\xi'\in A\otimes B'\otimes U$, extended by zero to $A\otimes B\otimes U$. It follows that $B''\otimes\mathscr{L}$ is a direct summand of $\mathscr{U}^*$, which contradicts the assumption. The proof of the second statement follows the same pattern. □

Lemma 3.4. *Let $\mathscr{E}$ be an ACM bundle on X, P and Q as in Lemma 1.2, and $\mathscr{U}$ be an Ulrich bundle. If $\mathscr{U}$ does not contain $\mathscr{L}$ as a direct summand, then $\mathrm{Ext}^1_X(\mathscr{U},P)=0$. If $\mathscr{U}$ does not contain $\mathscr{O}_X(-F)$ as a direct summand, then $\mathrm{Ext}^1_X(Q,\mathscr{U})=0$.*

Proof. Again we prove only the first statement, the second one being similar. So assume $\mathscr{U}$ does not have $\mathscr{L}$ as a direct summand. We first show:

$$\mathrm{Ext}^2_X(\mathscr{U},\mathscr{H}^{-1}\Theta\Theta^*\mathscr{E})=0. \tag{3.3}$$

To check this, using (1.12) it suffices to show $\mathrm{Ext}^2_X(\mathscr{U},\mathscr{L})=0$. In turn, writing $\mathscr{U}$ in the form of Proposition 2.1, we are reduced to check $\mathrm{Ext}^2_X(\mathscr{L},\mathscr{L})=0$ and $\mathrm{Ext}^2_X(\mathscr{O}_X(-F),\mathscr{L})=0$, which are obvious. So (3.3) is proved.

We now look at the sheaf I of §1.4. In view of (3.3), in order to prove our statement it is enough to apply $\mathrm{Hom}_X(\mathscr{U},-)$ to (1.17) and verify $\mathrm{Ext}^1_X(\mathscr{U},I)=0$. To see this vanishing, we apply $\mathrm{Hom}_X(\mathscr{U},-)$ to the exact sequence (1.14) defining I. Then we have to prove:

$$\mathrm{Ext}^1_X(\mathscr{U},\mathscr{O}_X)=\mathrm{Ext}^2_X(\mathscr{U},\mathscr{O}_X(-F))=0.$$

The first vanishing is given by Lemma 3.3. The second follows again immediately by Proposition 2.1 by applying $\mathrm{Hom}_X(-,\mathscr{O}_X(-F))$. □

The lemma on rigid extensions that we will use is the following standard fact. We provide a proof for the reader's convenience.

Lemma 3.5. *Let $c,d\geq 0$ be integers, C and D be vector spaces of dimension c and d. Consider two sheaves $\mathscr{A}$ and $\mathscr{B}$ with $\mathrm{ext}^1_X(\mathscr{B},\mathscr{A})=1$ and corresponding extension sheaf $\mathscr{G}$. If $\mathscr{E}$ fitting into*

$$0\to C\otimes\mathscr{A}\to\mathscr{E}\to D\otimes\mathscr{B}\to 0,$$

is defined by $\eta\in\mathrm{Ext}^1_X(D\otimes\mathscr{B},C\otimes\mathscr{A})\simeq C\otimes D^*$ corresponding to a map $D\to C$ of rank r, then:

$$\mathscr{E}\simeq\mathscr{G}^r\oplus\mathscr{A}^{\max(c-r,0)}\oplus\mathscr{B}^{\max(d-r,0)}.$$

Proof. Apply $\mathrm{Hom}_X(\mathscr{B},-)$ to the exact sequence defining $\mathscr{E}$ and restrict to $D\otimes\langle\mathrm{id}_{\mathscr{B}}\rangle$. We obtain this way a map $D\to C\otimes\mathrm{Ext}^1_X(\mathscr{B},\mathscr{A})$ which is identified with the linear map $D\to C$ associated with η after choosing a generator of $\mathrm{Ext}^1_X(\mathscr{B},\mathscr{A})$. In a suitable basis, this map is just a diagonal matrix and each non-zero entry of this matrix corresponds to an instance of the generator $\mathrm{Ext}^1_X(\mathscr{B},\mathscr{A})$, hence provides a copy of $\mathscr{G}$. Completing to a basis of D and C gives the remaining copies of $\mathscr{B}$ and $\mathscr{A}$. □

3.2.3. *The reduced monad.* Let again $\mathscr{E}$ be ACM on X. We write its Ulrich part $\mathscr{U}$ as:

$$\mathscr{U}=\mathscr{U}_0\oplus\mathscr{L}^{b_0}\oplus\mathscr{O}_X(-F)^{a_0},$$

where $\mathscr{U}_0$ contains no copy of $\mathscr{L}$ or $\mathscr{O}_X(-F)$ as direct factor. We apply Lemma 3.4 to the summands of $\mathscr{U}$ and use the display of the monad (3.1), whereby getting subbundles $\mathscr{K}_0\subset\mathscr{K}$ and $\mathscr{C}_0\subset\mathscr{C}$ with decompositions:

$$\mathscr{K}=\mathscr{K}_0\oplus\mathscr{U}_0\oplus\mathscr{O}_X(-F)^{a_0},$$
$$\mathscr{C}=\mathscr{C}_0\oplus\mathscr{U}_0\oplus\mathscr{L}^{b_0}.$$

Here, the bundles $\mathscr{K}_0$ and $\mathscr{C}_0$ fit into:

$$0\to P\to\mathscr{K}_0\to\mathscr{L}^{b_0}\to 0, \tag{3.4}$$

$$0\to\mathscr{O}_X(-F)^{a_0}\to\mathscr{C}_0\to Q\to 0. \tag{3.5}$$

In turn, again by Lemma 3.4, we obtain a splitting:

$$\mathscr{E}=\mathscr{U}_0\oplus\mathscr{E}_0.$$

where $\mathscr{E}_0$ is an extension of $\mathscr{K}_0$ and Q, or equivalently of P and $\mathscr{C}_0$.

Therefore, if $\mathscr{E}$ is indecomposable ACM and not Ulrich, then $\mathscr{U}_0=0$ and $a=a_0$, $b=b_0$. In this case $\mathscr{U}\simeq\mathscr{O}_X(-F)^a\oplus\mathscr{L}^b$, and we speak of the *reduced monad*.

Lemma 3.6. *Let $\mathscr{E}$ be an indecomposable non-Ulrich ACM bundle on X.*

i) For any twist of $\mathscr{E}$ one has either $a=0$ and $Q=0$, or $b=0$ and $P=0$.
ii) If $\mathscr{E}$ is initialized, the first case of the two cases above occurs.

iii) If $\mathscr{E}$ is initialized with $b_{0,0} = 0$ then $\mathscr{E}$ is $\mathscr{V}$ or $\mathscr{O}_X(iF)$ with $0 \le i \le 3$.

Proof. We prove (i). We have just seen that the Ulrich part of $\mathscr{E}$ is $\mathscr{O}_X(-F)^a \oplus \mathscr{L}^b$. Let us show:

$$\mathrm{Ext}^1_X(Q, \mathscr{K}_0) = 0.$$

We know by Lemma 1.3 that $\mathrm{Ext}^1_X(Q,P) = 0$, so by (3.4) it suffices to check $\mathrm{Ext}^1_X(Q,\mathscr{L}) = 0$. But this follows from Lemma 3.4 since $\mathscr{L}$ is Ulrich.

Looking at the reduced monad, we have proved $\mathscr{E} \simeq \mathscr{K}_0 \oplus \mathscr{C}_0$. By indecomposability of $\mathscr{E}$, one of these two bundles must be zero, which is precisely what we need for (i).

Part (ii) follows, since $\mathrm{H}^0(X,\mathscr{C}_0) = \mathrm{H}^0(X,Q) = 0$ by (1.15), (1.16) and (3.5).

For (iii), in view of (1.12) and (1.17), $b_{0,0} = 0$ implies $P \simeq I \simeq \bigoplus_{i \ge 0} \mathscr{O}_X(iF)^{p_i}$, where the second isomorphism is (1.23). We would then have an exact sequence:

$$0 \to \bigoplus_{i\ge 0} \mathscr{O}_X(iF)^{p_i} \to \mathscr{E} \to \mathscr{L}^b \to 0,$$

Note that any $i \ge 4$ is actually forbidden. Indeed, $\mathrm{H}^1(X, \mathscr{O}_X(-2H + iF)) \neq 0$ for $i \ge 4$. In turn this would easily imply $\mathrm{H}^1(X, \mathscr{E}(-2H)) \neq 0$, which is absurd for $\mathscr{E}$ is ACM. Moreover $\mathrm{ext}^1_X(\mathscr{L}, \mathscr{O}_X(iF))$ is zero for $i \le 3$ and 1 for $i = 3$. This, together with Lemma 3.5, implies (iii). □

We need also another vanishing, this time for the bundle $\mathscr{V}$ of Example 1.7.

Lemma 3.7. *The bundle $\mathscr{V}$ satisfies $\mathrm{Ext}^1_X(\mathscr{L}, \mathscr{V}(H-F)) = 0$.*

Proof. Applying $\mathrm{Hom}_X(\mathscr{L}(F-H), -)$ to the sequence of Example 1.7 defining $\mathscr{V}$ we see that the space under consideration is the cokernel of the linear map:

$$\rho : \mathrm{H}^0(X, \mathscr{O}_X(H-F)) \to \mathrm{H}^1(X, \mathscr{L}^*(H-F))$$

coming from cup product with the generator δ of $\mathrm{H}^1(X,\mathscr{L}^*)$.

We have to prove that this map is surjective. To see it, consider its effect on a non-zero element corresponding to a curve $C \subset X$ of class $H-F$ corresponding to a global section s of $\mathscr{O}_X(H-F)$. The curve C gives:

$$0 \to \mathscr{L}^* \to \mathscr{L}^*(H-F) \to \mathscr{L}^*(H-F)|_C \to 0,$$

and taking cohomology we get a map $\sigma : \mathrm{H}^1(X,\mathscr{L}^*) \to \mathrm{H}^1(X,\mathscr{L}^*(H-F))$, which sends δ to the image of s via ρ. Also $\mathrm{H}^1(C, \mathscr{L}^*(H-F)|_C) = 0$ since C is rational and $C \cdot \mathscr{L}^*(H-F) > 0$.

But working over the exceptional curve Δ, i.e. on the generator of $\mathrm{H}^0(X,\mathscr{L}^*)$, we easily see that $\mathrm{h}^1(X,\mathscr{L}^*(H-F)) = 1$. Therefore σ is surjective (actually an isomorphism) so δ has non-zero image in $\mathrm{H}^1(X,\mathscr{L}^*(H-F))$. This says that ρ is surjective. □

3.2.4. *The second reduction of the monad.* In the hypothesis of Lemma 3.6, part (ii), we proved that $\mathscr{E} \simeq \mathscr{K}_0$ where $\mathscr{K}_0$ fits into (3.4) and in turn P fits into:

$$0 \to \mathscr{L}^{b_{0,0}} \to I \to P \to 0.$$

Recall also the form (1.23) of I. In the next lemma we carry out the final step in the study of P.

Lemma 3.8. *Let $\mathscr{E}$ be an indecomposable initialized ACM non-Ulrich bundle of rank $r > 1$ on X with $b_{0,0} \neq 0$. Then P is a direct sum of copies of $\mathscr{V}(H-F)$, $\mathscr{O}_X(H-F)$ and $\mathscr{O}_X$.*

Proof. We know by Lemma 3.6 that $a = 0$ and $Q = 0$. Define $\mathscr{E}' = P(-H)$. Observe that, since $\mathscr{E}$ is ACM and $\mathrm{H}^0(X,\mathscr{U}) = 0$, from the monad we get $\mathrm{H}^1(X,\mathscr{E}'(tH)) = 0$ for $t \le 0$. This vanishing allows to apply the construction of §1.4 to $\mathscr{E}'$. We can then define the corresponding integers $a'_{i,j}$, a', $b'_{i,j}$ and b', together with the associated sheaves Q' and J'. It is now easy to compute:

$$a'_{0,0} = a'_{0,1} = a'_{2,0} = a'_{2,1} = 0, \qquad a'_{1,1} = b_{0,0}.$$

We rely on exact sequences analogous to (1.10), (1.15) and (1.16) to deduce that $\mathscr{E}'$ fist into:

$$0 \to \mathscr{O}_X(-F)^{b_{0,0}} \to \mathscr{E}' \to Q' \to 0,$$

with Q' appearing as kernel:

$$0 \to Q' \to \mathscr{L}(-F)^{b'_{2,1}} \to \mathscr{L}^{b'_{2,0}} \to 0.$$

This implies that there are integers q_j such that Q' takes the form:

$$Q' \simeq \bigoplus_{j \ge 1} \mathscr{L}(-jF)^{q_j}.$$

Now $\mathrm{ext}^1_X(\mathscr{L}(-jF), \mathscr{O}_X(-F))$ equals 0 for $j \ge 2$, and 1 for $j = 1$. So by Lemma 3.5 $\mathscr{E}'$ is a direct sum of copies of $\mathscr{O}_X(-F)$, $\mathscr{V}(-F)$ and $\mathscr{L}(-jF)$ for $j \ge 1$. Actually $\mathrm{H}^1(X, \mathscr{L}(-H-jF)) \neq 0$ for $j \ge 5$ implies that only $j \le 4$ can occur. Note that $\mathscr{O}_X(-F)$ or $\mathscr{V}(-F)$ must occur because $a'_{1,1} \neq 0$.

We now add H and use indecomposability of $\mathscr{E}$, after observing that $\mathrm{Ext}^1_X(\mathscr{L},\mathscr{L}(H-jF)) = 0$ for $j \le 2$. Because $\mathscr{E}$ is not a line bundle, this implies that only $j = 3,4$ can occur in the decomposition of P. But $\mathscr{L}(H-4F) \simeq \mathscr{O}_X(-F)$ does not appear because Lemma 3.6 says that $a = 0$. Summing up, besides $\mathscr{V}(H-F)$ and $\mathscr{O}_X(H-F)$, only $j = 3$ is allowed, giving $\mathscr{O}_X$. □

3.2.5. *Proof of Theorem 3.2.* After the second reduction we are in position to prove that $S(1,3)$ is of tame CM type. So let $\mathscr{E}$ be an indecomposable ACM bundle on $X = S(1,3)$. In case $\mathscr{E}$ is Ulrich, Proposition 2.6 shows assertions (i) and (ii), so we only have to take case of the non-Ulrich case.

We can assume that $\mathscr{E}$ is initialized of rank $r > 1$, so that Lemma 3.8 applies. Then $\mathscr{E}$ fits into:

$$0 \to P \to \mathscr{E} \to \mathscr{L}^b \to 0,$$

where P is a direct sum of copies of $\mathscr{V}(H-F)$, $\mathscr{O}_X(H-F)$ and $\mathscr{O}_X$.

By Lemma 3.7, since $\mathscr{E}$ is indecomposable we get that either $\mathscr{E} \simeq \mathscr{V}(H-F)$, or $\mathscr{V}(H-F)$ does not occur in P. We study this second case again by means of a monad. Indeed, factoring out all copies of $\mathscr{O}_X(H-F)$ from P we write $\mathscr{E}$ as an extension by $\mathscr{L}^b$ of a quotient bundle Q'' of $\mathscr{E}$ which we express as extension of copies of $\mathscr{O}_X$ and $\mathscr{L}$. This quotient bundle is thus a direct sum of copies of $\mathscr{O}_X$, $\mathscr{L}$ and $\mathscr{V}$ by Lemma 3.5.

But again $\mathrm{Ext}^1_X(\mathscr{O}_X,\mathscr{O}_X(H-F)) = 0$ so in fact $\mathscr{O}_X$ does not occur as direct summand of Q''. Moreover, since $\mathscr{V}$ has rank 2 with $\wedge^2\mathscr{V} \simeq \mathscr{L}$ we have $\mathscr{V}\otimes\mathscr{L}^* \simeq \mathscr{V}$, so from Lemma 3.7 we deduce $\mathrm{Ext}^1_X(\mathscr{V},\mathscr{O}_X(H-F)) = 0$. Hence either $\mathscr{E}$ is $\mathscr{V}$, or $\mathscr{V}$ does not appear either as direct summand of Q''.

We have proved that $\mathscr{E}$ is then an extension of copies of $\mathscr{O}_X(H-F)$ and $\mathscr{L}$, which is hence necessarily isomorphic to $\mathscr{W}$, again by Lemma 3.5. The proof of Theorem 3.2 is now complete.

Remark 3.9. Theorem 3.1 (and hence Theorem A) hold, except for part (ii), over an arbitrary field $\Bbbk$. In turn, if $\Bbbk$ is not algebraically closed, part (ii) holds for geometrically indecomposable $\mathscr{E}$ (i.e., $\mathscr{E}$ is indecomposable over the algebraic closure of $\Bbbk$). Otherwise, indecomposable ACM bundles over $\Bbbk$ are obtained from companion block matrices associated with irreducible polynomials in one variable over $\Bbbk$, cf. again [BCS97].

4. Rigid bundles on scrolls of higher degree

In this section we set $X = \mathbb{F}_\epsilon$ with $\epsilon = 0,1$, so X is a del Pezzo surface, and we assume $d_X \ge 5$, hence X is of wild representation type. For the rest of the paper, we assume $\mathrm{char}(\Bbbk) = 0$.

4.1. **Construction of rigid ACM bundles.** Let $s \ge 0$ be an integer, and consider a vector of s integers $\vec{k} = (k_1,\dots,k_s)$. With $\vec{k}$ we associate the word $\sigma^{\vec{k}}$ of the braid group B_4 by:

$$\sigma^{\emptyset} = 1, \quad \sigma^{\vec{k}} = \sigma_1^{k_1}\sigma_2^{k_2}\sigma_1^{k_3}\sigma_2^{k_4}\cdots \in B_3 = \langle \sigma_1,\sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\rangle.$$

This means that $\sigma^{\vec{k}}$ belongs to the copy of B_3 in B_4 consisting of braids not involving the first strand. Clearly, up to adding k_{i-1} and k_{i+1}, we may assume $k_i \ne 0$, for $i > 1$.

This subgroup operates on 3-terms exceptional collections over X, in view of the B_4-action on full exceptional collections described in §1.2.2. Set $\mathbf{B}_\emptyset$ for the subcollection $(\mathscr{L}[-1],\mathscr{O}_X(-F),\mathscr{O}_X)$ of $\mathbf{C}_\emptyset$, cf. (1.6). Put $\mathbf{B}_{\vec{k}} = \sigma^{\vec{k}}\mathbf{B}_\emptyset$ for the subcollection of $\mathbf{C}_{\vec{k}} = \sigma^{\vec{k}}\mathbf{C}_\emptyset$ obtained by mutation via $\sigma^{\vec{k}}$. Given a 3-term exceptional collection $\mathbf{B} = (\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3)$ we consider:

$$v = (\chi(\mathfrak{s}_2,\mathfrak{s}_3),\chi(\mathfrak{s}_1,\mathfrak{s}_3),\chi(\mathfrak{s}_1,\mathfrak{s}_2)) \in \mathbb{Z}^3.$$

The group B_3 thus operate on $\mathbb{Z}^3$, by sending a vector $v = (v_1,v_2,v_3)$ to:

$$\sigma_1 : (v_1,v_2,v_3) \mapsto (v_1v_3 - v_2, v_1, v_3), \qquad \sigma_2 : (v_1,v_2,v_3) \mapsto (v_1, v_1v_2 - v_3, v_2).$$

This action factors through the center of B_3, so that it actually defines an operation of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathbb{Z}^3$.

The inverse of σ_1 and σ_2 operate by similar formulas. For the basic collection $\mathbf{B}_\emptyset$ we have:

$$v = v^{\emptyset} = (2, d_X - 4, d_X - 2).$$

We set $v^{\vec{k}} \in \mathbb{Z}^3$ for the vector corresponding to $\mathbf{B}_{\vec{k}}$, i.e. $v^{\vec{k}} = \sigma^{\vec{k}}.v^{\emptyset}$.

Next, we set $\bar{t} \in \{0,1\}$ for the remainder of the division of an integer t by 2. Given $\vec{k} = (k_1,\dots,k_s)$ and $t \le s$, we write the truncation $\vec{k}(t) = (k_1,\dots,k_{t-1})$. We define:

$$\mathfrak{K} = \{\vec{k} = (k_1,\dots,k_s) \mid (-1)^t k_{t-1}(v^{\vec{k}(t)})_{2\bar{t}+1} \le 0, \forall t \le s\}. \tag{4.1}$$

Note that, if $\vec{k} \in \mathfrak{K}$, then $\vec{k}(t) \in \mathfrak{K}$, $\forall t \le s$. Observe that belonging to $\mathfrak{K}$ imposes no restriction on the last coordinate k_s of $\vec{k}$.

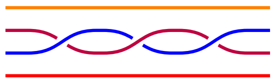
Next we provide an existence result for rigid ACM bundles indexed by $\vec{k} \in \mathfrak{K}$. There might be $\vec{k} \neq \vec{k}'$ such that $\sigma_{\vec{k}} = \sigma_{\vec{k}'}$ in view of the Braid group relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. The correspondence described by the next theorem has to be understood up to this ambiguity. The next result proves the existence part of Theorem C.

Theorem 4.1. *To any $\vec{k}$ in $\mathfrak{K}$ there corresponds an exceptional ACM bundle, denoted by $\mathfrak{f}_{\vec{k}}$, sitting as middle term of the exceptional collection $\mathbf{B}_{\vec{k}}$ obtained from $\mathbf{B}_\emptyset$ by mutation via $\sigma^{\vec{k}}$.*

Proof. We first give a step-by-step algorithm to construct $\mathfrak{f}_{\vec{k}}$.

Step 1. Take $s = 1$ and start with $\mathbf{B} = \mathbf{B}_\emptyset$, cf. (1.6). Observe that the exceptional pair $(\mathfrak{s}_1, \mathfrak{s}_2)$ is irregular, with $\mathfrak{s}_1 = |\mathfrak{s}_1|[-1]$, $\mathfrak{s}_2 = |\mathfrak{s}_2|$, $(\sigma^1.v)_3 = \chi(\mathfrak{s}_1, \mathfrak{s}_2) = d_X - 2 > 0$. Consider then the objects $\mathfrak{g}_k$ in the notation of §1.2.2, and set $\mathfrak{f}_{\vec{k}} = |\mathfrak{g}_{k_1+1}|$. In the notation of §2.2.3, we have $\mathfrak{f}_{k_1} \simeq \mathfrak{u}_{k_1+1}$, i.e. the case $s = 1$ corresponds to Ulrich bundles.

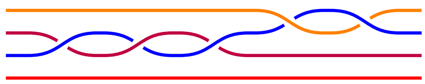
To prepare the next step, modify $\mathbf{B}$ by keeping $\mathfrak{s}_3$ unchanged, but replacing $\mathfrak{s}_1$ with $\mathfrak{g}_{k_1}$ and $\mathfrak{s}_2$ with $\mathfrak{g}_{k_1+1}$. We observed that (4.1) imposes no condition for $s = 1$ so k_1 is arbitrary in $\mathbb{Z}$ if $s = 1$. On the other hand, for $s \geq 2$, taking $t = 2$ leads to assume $k_1(\sigma^{\vec{k}(2)}.v)_1 \leq 0$. One can check right away that this simply means $k_1 \leq 0$.



For instance, take $\vec{k} = (-3)$, i.e. $\sigma = \sigma_1^{-3}$, so $\mathfrak{f}_{-3} = \mathfrak{u}_{-2}$. We draw the corresponding braid of B_4, and the associated mutations, starting from the basic exceptional collection $\mathbf{B}_\emptyset$:

$$\begin{aligned} \sigma_1^{-1}\mathbf{B}_\emptyset = \mathbf{B}_{-1} &= (\mathfrak{u}_{-1}[-1], \mathfrak{u}_0[-1], \mathcal{O}_X), && \text{for } k_1 = -1; \\ \mathbf{B}_{-2} &= (\mathfrak{u}_{-2}[-1], \mathfrak{u}_{-1}[-1], \mathcal{O}_X), && \text{for } k_1 = -2; \\ \mathbf{B}_{-3} &= (\mathfrak{u}_{-3}[-1], \mathfrak{u}_{-2}[-1], \mathcal{O}_X), && \text{for } k_1 = -3. \end{aligned}$$

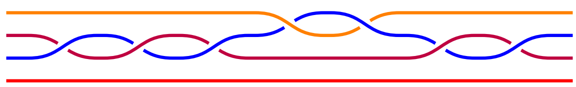
Step 2. Take $s \geq 2$ even. It turns out (cf. Lemma 4.2) that the condition $\vec{k} \in \mathfrak{K}$ implies that, in the collection $\mathbf{B}$ defined inductively by the truncation $\sigma^{\vec{k}(s)}$, the pair $(\mathfrak{s}_2, \mathfrak{s}_3)$ is irregular. Then, it can be used to construct the objects $\mathfrak{g}_k$ as in §1.2.2. Next, we define $\mathfrak{f}_{\vec{k}} = |\mathfrak{g}_{k_s}|$ and modify $\mathbf{B}$ by keeping $\mathfrak{s}_1$ unchanged and replacing the pair $(\mathfrak{s}_2, \mathfrak{s}_3)$ with $(\mathfrak{g}_{k_s}, \mathfrak{g}_{k_s+1})$.



Take e.g. $\sigma = \sigma_1^{-3}\sigma_2^2$, i.e. $\vec{k} = (-3, 2)$. The associated mutations start from $\mathbf{B}_{-3}$ and give:

$$\begin{aligned} \mathbf{B}_{-3} &= (\mathfrak{f}_{-4}[-1], \mathfrak{f}_{-3}[-1], \mathcal{O}_X), && \text{for } \vec{k} = (-3), && \mathcal{O}_X = \mathfrak{f}_{-3,1}; \\ \mathbf{B}_{-3,1} &= (\mathfrak{f}_{-4}[-1], \mathfrak{f}_{-3,1}, \mathfrak{f}_{-3,2}), && \text{for } \vec{k} = (-3, 1); \\ \mathbf{B}_{-3,2} &= (\mathfrak{f}_{-4}[-1], \mathfrak{f}_{-3,2}, \mathfrak{f}_{-3,3}), && \text{for } \vec{k} = (-3, 2). \end{aligned}$$

Step 3. Take $s \geq 3$ odd. Again apply Lemma 4.2 to the collection $\mathbf{C}_{\vec{k}(s)}$. This time the irregular pair is $(\mathfrak{s}_1, \mathfrak{s}_2)$, and we use it to construct the objects $\mathfrak{g}_k$, cf. §1.2.2. Put $\mathfrak{f}_{\vec{k}} = |\mathfrak{g}_{k_s+1}|$. As in Step 1, modify the collection $\mathbf{B}$ by keeping $\mathfrak{s}_3$ and replacing $\mathfrak{s}_1$ with $\mathfrak{g}_{k_s}$ and $\mathfrak{s}_2$ with $\mathfrak{g}_{k_s+1}$.



Take for example $\vec{k} = (-3, 2, -2)$. This lies in $\mathfrak{K}$ for $X = S(2, 3)$. The bundle $\mathfrak{f} = \mathfrak{f}_{-3,2,-1}$ has canonical slope $c_1(\mathfrak{f}) \cdot c_1(\omega_X) / \operatorname{rk}(\mathfrak{f}) = 37/216$. The associated mutations give:

$$\begin{aligned} \mathbf{B}_{-3,2} &= (\mathfrak{f}_{-4}[-1], \mathfrak{f}_{-3,2}, \mathfrak{f}_{-3,3}), && \text{for } \vec{k} = (-3, 2); \\ \mathbf{B}_{-3,2,-1} &= (\mathfrak{f}_{-3,2,-1}[-1], \mathfrak{f}_{-4}[-1], \mathfrak{f}_{-3,3}), && \text{for } \vec{k} = (-3, 2, -1); \\ \mathbf{B}_{-3,2,-2} &= (\mathfrak{f}_{-3,2,-2}[-1], \mathfrak{f}_{-3,2,-1}[-1], \mathfrak{f}_{-3,3}), && \text{for } \vec{k} = (-3, 2, -2). \end{aligned}$$

Having this in mind, it is clear that $\mathfrak{f}_{\vec{k}}$ is exceptional, by the basic properties of mutations. Also, by induction on s we see that $\mathfrak{f}_{\vec{k}}$ is an ACM bundle, as it is an extension of ACM bundles, in view of the Fibonacci sequences (1.7) and (1.8). To conclude we need the following lemma. ☐

Lemma 4.2. *Let $\vec{k} \in \mathfrak{K}$ and let $k_{s+1} \in \mathbb{Z} \setminus \{0\}$. Set $\mathbf{B} = \mathbf{B}_{\vec{k}} = (\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3)$. For s even, $(\mathfrak{s}_1, \mathfrak{s}_2)$ is an irregular pair if and only if:*

(1) $k_s < 0$, $(\sigma^{\vec{k}}.v)_3 < 0$, in which case $\mathfrak{s}_i = |\mathfrak{s}_i|[-1]$ for all i, or:
(2) $k_s > 0$, $(\sigma^{\vec{k}}.v)_3 > 0$, so $\mathfrak{s}_1 = |\mathfrak{s}_1|[-1]$, $\mathfrak{s}_i = |\mathfrak{s}_i|$ for $i = 2, 3$.

For s odd, $(\mathfrak{s}_2, \mathfrak{s}_3)$ is an irregular pair if and only if:

(3) $k_s < 0$, $(\sigma^{\vec{k}}.v)_1 > 0$, so $\mathfrak{s}_3 = |\mathfrak{s}_3|$, $\mathfrak{s}_i = |\mathfrak{s}_i|[-1]$ for $i = 1, 2$;
(4) $k_s > 0$ and $(\sigma^{\vec{k}}.v)_1 < 0$, in which case $\mathfrak{s}_i = |\mathfrak{s}_i|$ for all i.

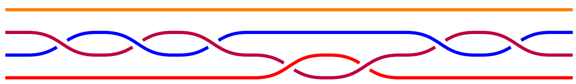
Finally, $(k_1, \ldots, k_{s+1})$ lies in $\mathfrak{K}$ if and only if one the cases (1) to (4) occurs.

The inequalities of the cases from (1) to (4) simply correspond to the conditions on $\vec{k}$ given by (4.1), so of course one (and only one) of them is verified if and only if $(k_1, \ldots, k_{s+1})$ lies in $\mathfrak{K}$.

Proof of Lemma 4.2. Let s be even. Note that, as soon as the objects $\mathfrak{s}_i$ are concentrated in the degree prescribed by the sign of k_s, we see that the sign of $(\sigma^{\vec{k}(s)}.v)_3$ corresponds precisely to $(\mathfrak{s}_1, \mathfrak{s}_2)$ being a regular or irregular exceptional pair. The same happens for odd s. Therefore it suffices to check that $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3$ are concentrated in the correct degrees, as listed in (1) to (4).

In turn, this last fact is easy to see by induction on s. Indeed, each time we pass from $t \leq s$ to $t+1$, we operate mutations based on an irregular exceptional pair of bundles $(\mathscr{P}, \mathscr{N}) = (\mathfrak{s}_{\bar{t}+1}, \mathfrak{s}_{\bar{t}+2})$. So, in the notation of §1.2.2, we set $w = \mathrm{ext}^1_X(|\mathscr{P}|, |\mathscr{N}|)$, and we see that the bundles $\mathfrak{g}_{k_{t+1}}$ and $\mathfrak{g}_{k_{t+1}+1}$ fit as middle term of an extension of the form (1.7) (for $k_{t+1} > 0$) or (1.8) (for $k_{t+1} < 0$). This implies that $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3$ are concentrated in the desired degrees. ☐

Remark 4.3. One can define similarly an action of $B_3 = \langle \sigma_0, \sigma_1 \mid \sigma_0\sigma_1\sigma_0 = \sigma_0\sigma_1\sigma_0 \rangle \subset B_4$ on the set of three-terms exceptional collections leaving $\mathscr{O}_X$ fixed. Write $\mathfrak{h}_{\vec{k}}$ for the bundle associated in this sense with the word $\sigma_0^{k_1}\sigma_1^{k_2}\sigma_0^{k_3}\cdots$. Then we have $\mathfrak{h}_{\vec{k}}(-H) \simeq \mathfrak{f}^*_{-\vec{k}} \otimes \omega_X$. Here is the braid for $\mathfrak{h}_{3,-2,2}$, which has canonical slope 253/216:



Also, if $d_X \geq 6$ and $k_1 < 0$, or if $k_1 < -2$, it is possible to prove, by induction on s, that any $\vec{k} = (k_1, \ldots, k_s)$ with $k_{t-1}k_t < 0$ for all $t < s$ lies in $\mathfrak{K}$.

4.2. **Classification of rigid ACM bundles on two rational normal scrolls.** Here we assume that X is $S(2,3)$ or $S(3,3)$. So X is a del Pezzo surface embedded as a CM-wild rational normal scroll of degree 5 or 6. The next results classifies indecomposable rigid ACM bundles on X, whereby concluding the proof of Theorem C.

Theorem 4.4. *Let $X = S(2,3)$ or $X = S(3,3)$, and let $\mathscr{E}$ be an indecomposable rigid ACM bundle on X. Then there exists $\vec{k} = (k_1, \ldots, k_s)$ in $\mathfrak{K}$ such that, up to a twist, $\mathscr{E} \simeq \mathfrak{f}_{\vec{k}}$ or $\mathscr{E}^* \otimes \omega_X \simeq \mathfrak{f}_{\vec{k}}$.*

Lemma 4.5. *Let $(\mathscr{P}, \mathscr{N})$ be an exceptional pair of bundles on X, and let p and q be integers. Let $\mathscr{E}$ be a bundle on X equipped with two maps f, g as in the diagram:*

$$|\mathscr{N}|^p \xrightarrow{f} \mathscr{E} \xrightarrow{g} |\mathscr{P}|^q, \qquad \text{with } g \circ f = 0,\ f \text{ injective, } g \text{ surjective.} \tag{4.2}$$

Write $\mathscr{K} = \ker(g)$, $\mathscr{C} = \mathrm{coker}(f)$, and $\mathscr{F} = \mathscr{K}/|\mathscr{N}|^p$. Finally, assume:

$$\mathrm{RHom}_X(\mathscr{N}, \mathscr{F}) = 0, \qquad \mathrm{RHom}_X(\mathscr{F}, \mathscr{P}) = 0, \tag{4.3}$$

$$\mathrm{Ext}^2_X(|\mathscr{P}|, \mathscr{F}) = 0, \qquad \mathrm{Ext}^2_X(\mathscr{F}, |\mathscr{N}|) = 0. \tag{4.4}$$

Then, whenever $\mathscr{E}$ is rigid, also $\mathscr{K}$, $\mathscr{C}$ and $\mathscr{F}$ are rigid.

Proof. Write the following exact commutative diagram as display of (4.2).

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & |\mathscr{N}|^p & = & |\mathscr{N}|^p & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathscr{K} & \to & \mathscr{E} & \to & |\mathscr{P}|^q & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathscr{F} & \to & \mathscr{C} & \to & |\mathscr{P}|^q & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{4.5}$$

Applying $\mathrm{Hom}_X(-,\mathscr{F})$ to the leftmost column of (4.5) and using (4.3) we get:

$$\mathrm{Ext}^i_X(\mathscr{F},\mathscr{F}) \simeq \mathrm{Ext}^i_X(\mathscr{K},\mathscr{F}), \qquad \text{for all } i.$$

By the same sequence, using $\mathrm{Hom}_X(-,|\mathscr{N}|)$ and (4.4)), we get $\mathrm{Ext}^2_X(\mathscr{K},|\mathscr{N}|)=0$ since $\mathscr{N}$ is exceptional. Then, applying $\mathrm{Hom}_X(\mathscr{K},-)$ to the same sequence we get:

$$\mathrm{Ext}^1_X(\mathscr{K},\mathscr{K}) \twoheadrightarrow \mathrm{Ext}^1_X(\mathscr{F},\mathscr{F}).$$

Then, to prove that $\mathscr{F}$ is rigid, it suffices to prove that $\mathscr{K}$ is. To do it, we apply $\mathrm{Hom}_X(-,\mathscr{P})$ to the left column of (4.5). Since $(\mathscr{P},\mathscr{N})$ is an exceptional pair, in view of (4.3) we get: $\mathrm{RHom}_X(\mathscr{K},\mathscr{P})=0$. Therefore, applying $\mathrm{Hom}_X(\mathscr{K},-)$ to the central row of (4.5) we obtain:

$$\mathrm{Ext}^i_X(\mathscr{K},\mathscr{K}) \simeq \mathrm{Ext}^i_X(\mathscr{K},\mathscr{E}), \qquad \text{for all } i. \tag{4.6}$$

Note that, since $\mathrm{Ext}^2_X(|\mathscr{P}|,|\mathscr{N}|)=0$ (because $(\mathscr{P},\mathscr{N})$ is an exceptional pair, regular or not), from (4.4) we deduce, applying $\mathrm{Hom}_X(|\mathscr{P}|,-)$ to (4.5), the vanishing $\mathrm{Ext}^2_X(|\mathscr{P}|,\mathscr{E})$. Therefore, applying $\mathrm{Hom}_X(-,\mathscr{E})$ to the central row of (4.5) we get a surjection:

$$\mathrm{Ext}^1_X(\mathscr{E},\mathscr{E}) \twoheadrightarrow \mathrm{Ext}^1_X(\mathscr{K},\mathscr{E}).$$

This, combined with (4.6), says that $\mathscr{K}$ (and therefore $\mathscr{F}$) is rigid if $\mathscr{E}$ is. A similar argument shows that the same happens to $\mathscr{C}$. □

Proof of Theorem 4.4. Recall Proposition 1.6, the notation a,b,c,d and the construction of the monad associated with $\mathscr{E}$, cf. §1.6. We have thus a kernel bundle $\mathscr{K}$, a cokernel bundle $\mathscr{C}$ and an Ulrich bundle $\mathscr{F}$ associated with $\mathscr{E}$.

We use now Lemma 4.5 with $\mathscr{P}=\mathscr{L}(-F)[-1]$, $p=c$, $\mathscr{N}=\mathscr{O}_X$ and $q=d$. Since $\mathscr{F}$ is an extension of $\mathscr{L}$ and $\mathscr{O}_X(-F)$ and $\mathbf{C}_\emptyset$ is an exceptional collection we have the vanishing (4.3). Also, for $\mathbf{C}_\emptyset$ we have $\mathrm{Ext}^2_X(|\mathfrak{s}_i|,|\mathfrak{s}_j|)=0$ so (4.4) also holds. Therefore by Lemma 4.5 we see that $\mathscr{K}$, $\mathscr{C}$ and $\mathscr{F}$ are all rigid. By Lemma 2.4, part (iv), there must be $k\neq 0$ and a_k, a_{k+1} such that:

$$\mathscr{F} \simeq \mathfrak{u}_k^{a_k} \oplus \mathfrak{u}_{k+1}^{a_{k+1}},$$

where $\mathfrak{u}_k$ are the exceptional Ulrich bundles, and $\mathfrak{f}_k=\mathfrak{u}_{k+1}$. Now, in Lemma 4.2 we computed:

$$\chi(\mathscr{L}(-F),\mathfrak{u}_k)\geq 0, \qquad \text{iff } k\leq 0,$$
$$\chi(\mathfrak{u}_k,\mathscr{O}_X)\geq 0, \qquad \text{iff } k\geq 1.$$

Up to possibly a shift by 1, the exceptional pair $(\mathfrak{u}_k,\mathfrak{u}_{k+1})$ is completed to the full exceptional collection $\mathbf{C}_k$ by adding $\mathscr{L}(-F)[-1]$ to the left and to the right $\mathscr{O}_X$. So, by [GK04, Proposition 5.3.5], the previous display implies the vanishing:

$$\mathrm{Ext}^1_X(\mathscr{L}(-F),\mathscr{F})=0, \qquad \text{if } k<0, \tag{4.7}$$
$$\mathrm{Ext}^1_X(\mathscr{F},\mathscr{O}_X)=0, \qquad \text{if } k>0.$$

Let us consider the case $k<0$. Before going further, note that analyzing this case is enough, indeed if $k>0$, then replacing $\mathscr{E}$ with $\mathscr{E}^*\otimes\omega_X$ we get another rigid ACM bundle, whose Ulrich part will be this time in the range $k<0$.

Assuming thus $k<0$, by (4.7), looking back at (1.27), we see that $\mathscr{C}\simeq\mathscr{F}\oplus\mathscr{L}(-F)^d$. Then we note that, over $S(2,3)$ and $S(3,3)$, we have $\mathrm{Ext}^1_X(\mathscr{L}(-F),\mathscr{O}_X)=0$, so that $\mathscr{L}(-F)^d$ is a direct summand of $\mathscr{E}$. So, since $\mathscr{E}$ is indecomposable, we get $d=0$, or $\mathscr{E}\simeq\mathscr{L}(-F)$. In the second case, $\mathscr{E}^*\otimes\omega_X(H)\simeq\mathscr{O}_X=\mathfrak{f}_{-1,1}$. Hence we assume $d=0$, so $\mathscr{E}$ lies in the subcategory generated by the subcollection $\mathbf{B}_k$ of $\mathbf{C}_k$. Our goal is to show that $\mathscr{E}$ can be constructed by the steps of Theorem 4.1, when we set $k_1=k$. This is clear for $\mathscr{F}=0$, in which case $\mathscr{E}\simeq\mathscr{O}_X$. Also, it is clear for $c=0$, as we may assume $a_{k_1}=0$ and $a_{k_1+1}=1$ by indecomposability of $\mathscr{E}$, so $\mathscr{E}\simeq\mathfrak{u}_{k_1+1}\simeq\mathfrak{f}_{k_1}$.

To study the case $c\neq 0$, $\mathscr{F}\neq 0$, first note that again indecomposability of $\mathscr{E}$ forces $(\mathfrak{s}_2,\mathfrak{s}_3)$ to be an irregular pair, which gives back $k_1<0$ and $\vec{k}=(k_1,k_2)\in\mathfrak{K}, \forall k_2\neq 0$. Then, we use again Lemma 4.5, this time with $\mathscr{N}=\mathscr{O}_X$, $p=c$, $\mathscr{P}=\mathfrak{u}_{k_1}[-1]$ and $q=a_{k_1}$, so that the new bundle $\mathscr{F}$ is $\mathfrak{u}_{k_1+1}^{a_{k_1+1}}$. We get a new bundle $\mathscr{K}$ fitting into:

$$0\to\mathscr{O}_X^c\to\mathscr{K}\to\mathfrak{u}_{k_1+1}^{a_{k_1+1}}\to 0.$$

Since $\mathbf{B}_{k_1}=(\mathfrak{u}_{k_1}[-1],\mathfrak{u}_{k_1+1}[-1],\mathscr{O}_X)$ is an exceptional sequence with vanishing Ext^2 groups, Lemma 4.5 applies and shows that $\mathscr{K}$ is rigid. So, again by By Lemma 2.4, part (iv), there are integers k_2, a_{k_1,k_2} and a_{k_1,k_2+1} such that, in the notation of Theorem 4.1:

$$\mathscr{K}\simeq\mathfrak{f}_{k_1,k_2}^{a_{k_1,k_2}}\oplus\mathfrak{f}_{k_1,k_2+1}^{a_{k_1,k_2+1}},$$

Next, note that the bundle $\mathscr{E}$ belongs to the subcategory generated by the new exceptional sequence $\mathbf{B}_{k_1,k_2}$. Up to shifts, $\mathbf{B}_{k_1,k_2}$ takes the form $(\mathfrak{f}_{k_1-1},\mathfrak{f}_{k_1,k_2},\mathfrak{f}_{k_1,k_2+1})$. If $a_{k_1}=0$ then $\mathscr{E}\simeq\mathscr{K}$ so since $\mathscr{E}$ is indecomposable we see that $\mathscr{E}$ the form $\mathfrak{f}_{\vec{k}}$, with $s = 2$ in the notation of Theorem 4.1. Otherwise, the new exceptional pair $(\mathfrak{s}_1, \mathfrak{s}_2)$ in $\mathbf{B}_{k_1,k_2}$ must be irregular, so Lemma 4.2 ensures that $\vec{k} = (k_1, k_2, k_3)$ lies in $\mathfrak{K}$ for any $k_3 \neq 0$. Further, Lemma 4.5 can be used again, this time with $\mathscr{N} = \mathfrak{f}_{k_1,k_2}$, $p = a_{k_1,k_2+1}$ and $\mathscr{P} = \mathfrak{f}_{k_1-1}$, $q = a_{k_1}$ to see that the new bundle $\mathscr{C}$ is rigid and hence a direct sum $\mathfrak{f}_{k_1,k_2,k_3}^{a_{k_1,k_2,k_3}} \oplus \mathfrak{f}_{k_1,k_2,k_3+1}^{a_{k_1,k_2,k_3+1}}$ for some $k_3 \neq 0$ and integers a_{k_1,k_2,k_3}, a_{k_1,k_2,k_3+1}. This process may be iterated, and eventually must stop because $\mathscr{E}$ has finite rank. We finally get that the bundle $\mathscr{E}$ is of the form $\mathfrak{f}_{\vec{k}}$ for some vector $\vec{k}$ lying in $\mathfrak{K}$. □

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