SOME REMARKS ON VECTOR BUNDLES WITH NO INTERMEDIATE COHOMOLOGY

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Abstract. We prove that a general surface of degree d is the Pfaffian of a square matrix with (almost) quadratic entries if and only if $d \leq 15$. We classify rank 2 aCM bundles (i.e. with no intermediate cohomology) on a general sextic surface. A recursively defined matrix presenting the spinor bundle on a smooth quadric hypersurface is shown.

1. Introduction

Given a sheaf $\mathscr{E}$ on a projective variety Y polarized by $\mathscr{O}_Y(1)$, we consider the cohomology modules:

$$\mathrm{H}^p_*(Y, \mathscr{E}) = \bigoplus_{t \in \mathbb{Z}} \mathrm{H}^p(Y, \mathscr{E} \otimes \mathscr{O}_Y(t)).$$

Here we will focus on those sheaves $\mathscr{E}$ that satisfy $\mathrm{H}^p_*(Y, \mathscr{E}) = 0$ for all $0 < p < \dim(Y)$. These are called sheaves with no intermediate cohomology, or aCM sheaves, standing for arithmetically Cohen-Macaulay, indeed $\mathrm{H}^0_*(Y, \mathscr{E})$ is a Cohen-Macaulay module over the coordinate ring of Y iff $\mathscr{E}$ is an aCM sheaf.

It is possible to classify all aCM bundles on projective space, (Horrocks, [Hor64]), quadrics (Knörrer, [Knö87]) and few other varieties, see [BGS87] and [EH88]. On the other hand, a detailed study of the families aCM bundles of low rank has been carried out in some cases, for instance some Fano threefolds (see e.g. [Mad02], [AC00], [AF06]) and Grassmannians, [AG99]. An even richer literature is devoted to aCM bundles of rank 2 on hypersurfaces Y_d of degree d in $\mathbb{P}^n$. If $n \geq 4$, and Y_d is general, the classification is complete, as it results from the papers [Kle78], [CM00], [CM04], [CM05], [KRR05], [KRR06].

On the other hand, for $n = 3$, the classification has been completed only up to $d \leq 5$, see [Fae05], [CF06], while only partial results are available for higher d, see for instance [Bea00].

Here we will draw a few remarks on aCM bundles, mainly on hypersurfaces. In the next section we provide a slight generalization of a result of Beauville [Bea00], namely the structure of the minimal graded free resolution of an aCM sheaf on a projective variety.

In Section 3, we focus on surfaces Y_d of degree d in $\mathbb{P}^3$. We first note that the general Y_d supports an aCM bundle $\mathscr{E}$ of rank 2 with $\det(\mathscr{E}) \cong \mathscr{O}_{Y_d}(d-2)$ if and only if $d \leq 15$. This is proved with `Macaulay 2` and amounts to writing the

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equation of Y_d as the Pfaffian of a certain skew-symmetric matrix. Next we will extend the classification of aCM bundles of rank 2 to sextic surfaces.

Finally, in Section 4 we give a recursive formula for the matrix appearing in the minimal graded free resolution of a spinor bundle on a smooth quadric.

2. A general remark

Here we reformulate a result of Beauville, see [Bea00] in a slightly more general setup, with essentially the same proof. We refer to the minimal graded free resolution of a sheaf $\mathscr{E}$ on $\mathbb{P}^n$ as to the sheafification of the minimal graded free resolution of the module $\mathrm{H}^0_*(\mathbb{P}^n, \mathscr{E})$ over the polynomial ring in $n+1$ variables.

Remark 2.1. Let $\iota : X \hookrightarrow \mathbb{P} = \mathbb{P}^n$ be a subscheme of codimension r, and let $\mathscr{E}$ be an aCM sheaf of rank s on X. Then the minimal graded free resolution of $\iota_* \mathscr{E}$ takes the form:

$$0 \to \mathsf{P}_r \xrightarrow{f_{r-1}} \cdots \xrightarrow{f_0} \mathsf{P}_0 \xrightarrow{p} \iota_* \mathscr{E} \to 0, \qquad \text{with } \mathsf{P}_i = \bigoplus_{j=1}^{\ell_i} \mathscr{O}_{\mathbb{P}}(a_{ij}).$$

We have $\sum (-1)^i \ell_i = 0$, and the ideal of minors of order $s+1$ of f_0 is contained in the ideal I_X.

Proof. Given the inclusion $\iota : X \hookrightarrow \mathbb{P}$, we consider a minimal graded free resolution of the sheaf $\iota_* \mathscr{E}$ over $\mathbb{P}$. This takes the form:

$$\cdots \to \mathsf{P}_i \to \cdots \to \mathsf{P}_0 \to \iota_* \mathscr{E} \to 0. \tag{2.1}$$

Set $K_i = \ker(f_{i-1} : \mathsf{P}_i \to \mathsf{P}_{i-1})$ and $K_0 = \ker(p)$. For each $i \geq 0$, consider the induced exact sequence:

$$\mathrm{H}^0_*(\mathbb{P}, \mathsf{P}_{i+1}) \to \mathrm{H}^0_*(\mathbb{P}, K_i) \to \mathrm{H}^1_*(\mathbb{P}, K_{i+1}) \to \mathrm{H}^1_*(\mathbb{P}, \mathsf{P}_{i+1}).$$

In this sequence, the rightmost group vanishes, and the leftmost map is surjective by definition of a graded free resolution. Then we have $\mathrm{H}^1_*(\mathbb{P}, K_{i+1}) = 0$, and by the same reason $\mathrm{H}^1_*(\mathbb{P}, K_0) = 0$. Since $\mathscr{E}$ aCM on X, we obtain $\mathrm{H}^p_*(K_i) = 0$ for $0 < p < n - r + i + 1$. Therefore we get $\mathrm{H}^p_*(\mathbb{P}, K_{r-1}) = 0$ for $0 < p < n$ so K_r splits by Horrocks criterion. The ℓ_i's sum to zero for $\iota_* \mathscr{E}$ is a torsion sheaf. The condition on minors follows, since the rank of f_0 must decrease by s on the support of $\mathscr{E}$. □

Given a vector bundle $\mathscr{E}$ and an invertible sheaf $\mathscr{L}$ on X, if ε is 1 or -1, we say that an isomorphism $\kappa : \mathscr{E} \to \mathscr{E}^* \otimes \mathscr{L}$ is an ε-symmetric duality if $\kappa = \varepsilon\, \kappa^\top$.

Proposition 2.2 (Odd codimension)**.** Let $\iota : X \hookrightarrow \mathbb{P} = \mathbb{P}^n$ be a subvariety of pure codimension $r = 2\,s + 1$, and let $\mathscr{E}$ be an aCM vector bundle on X. Assume $\mathrm{char}(\mathbf{k}) \neq 2$, and suppose that there exists an ε-symmetric duality:

$$\kappa : \mathscr{E} \xrightarrow{\cong} \mathscr{E}^* \otimes \omega_X(t + n + 1).$$

Then the minimal resolution of $\iota_* \mathscr{E}$ takes the form:

$$0 \to \mathsf{P}_0^*(t) \to \cdots \xrightarrow{g_{s-1}^\top} \mathsf{P}_s^*(t) \xrightarrow{g_s^\top} \mathsf{P}_s \xrightarrow{g_{s-1}} \cdots \to \mathsf{P}_1 \xrightarrow{g_0} \mathsf{P}_0 \to \iota_* \mathscr{E} \to 0, \tag{2.2}$$

where the map g_s satisfies:

$$g_s^\top = \varepsilon\, g_s.$$

Proof. Consider the minimal graded free resolution provided by Remark 2.1. Apply the functor $\mathscr{H}om_{\mathbb{P}}(-,\mathscr{O}_{\mathbb{P}})$, and recall the Grothendieck duality isomorphism:

$$\mathscr{E}xt^r_{\mathbb{P}}(\mathscr{E},\mathscr{O}_X) \cong \iota_* \mathscr{E}', \quad \text{with:} \quad \mathscr{E}' = \mathscr{E}^* \otimes \omega_X(n+1).$$

We have thus a second minimal graded free resolution of $\iota_*\mathscr{E}$, of the form $f_i^\top : \mathsf{P}^*_{i-1}(t) \to \mathsf{P}^*_i(t)$. The duality $\kappa : \mathscr{E} \to \mathscr{E}'(t)$ lifts to an isomorphism between the two resolutions. This means that, for each $0 \le i \le r$, we have an isomorphism $\xi_i : \mathsf{P}_i \to \mathsf{P}^*_{2\,s+1-i}(t)$ satisfying:

$$\xi_i \circ f_i = f^\top_{2\,s-i} \circ \xi_{i+1}, \qquad f_i^\top \circ \xi_i^\top = \xi^\top_{i+1} \circ f_{2\,s-i}. \tag{2.3}$$

Now we may lift the map $\xi^\top_{2\,s+1-i} - \varepsilon\,\xi_i$ to $\mathsf{P}^*_{2\,s-i}(t)$, and thus getting a morphism $\varphi_i : \mathsf{P}_i \to \mathsf{P}^*_{2\,s-i}(t)$ which satisfies the relation:

$$f^\top_{2\,s-i} \circ \varphi_i = \xi^\top_{2\,s+1-i} - \varepsilon\,\xi_i. \tag{2.4}$$

We define the map $\psi_i : \mathsf{P}_i \to \mathsf{P}^*_{2\,s+1-i}(t)$ by:

$$\psi_i = \xi_i + \frac{\varepsilon}{2} f^\top_{2\,s-i} \circ \varphi_i. \tag{2.5}$$

Note that ψ_i is an isomorphism since $f^\top_{2\,s+1-i} \circ \psi_i = f^\top_{2\,s+1-i} \circ \xi_i$ and ξ_i is an isomorphism. The required maps g_i are defined by:

$$g_i = \begin{cases} \psi_i^{-1} \circ f^\top_{2\,s-i}, & \text{for } i \ge s+1, \\ f_i, & \text{otherwise.} \end{cases} \tag{2.6}$$

We obtain thus a resolution of the form (2.2), and one can easily check the relation $g_s^\top = \varepsilon\, g_s$. □

Proposition 2.3 (Even codimension)**.** Let $\mathbf{k}$, $\iota : X \hookrightarrow \mathbb{P}$, $\mathscr{E}$, κ be as in the hypothesis of Proposition 2.2, and assume $r = 2\,s+2$ for some integer s. Then the minimal resolution of $\iota_*\mathscr{E}$ takes the form:

$$0 \to \mathsf{P}^*_0(t) \to \cdots \to \mathsf{P}^*_s(t) \xrightarrow{J\circ g_s^\top} \mathsf{P}_{s+1} \xrightarrow{g_s} \mathsf{P}_s \to \cdots \to \mathsf{P}_1 \xrightarrow{g_0} \mathsf{P}_0 \to \iota_*\mathscr{E} \to 0,$$

where J is an isomorphism $\mathsf{P}^*_{s+1}(t) \to \mathsf{P}_{s+1}$ satisfying:

$$J^\top = \varepsilon\, J.$$

Proof. The proof is analogous to that of Proposition 2.2, where we replace the defining equation (2.4) of φ_i by the following:

$$\xi^\top_{2\,s+2-i} - \varepsilon\,\xi_i = f^\top_{2\,s+1-i} \circ \varphi_i, \tag{2.7}$$

and the definitions (2.5) and (2.6) of ψ_i and g_i by the following:

$$\psi_i = \xi_i + \frac{\varepsilon}{2} f^\top_{2\,s+1-i} \circ \varphi_i, \tag{2.8}$$

$$g_i = \begin{cases} \psi_i^{-1} \circ f^\top_{2\,s+1-i}, & \text{for } i \ge s+1, \\ f_i, & \text{otherwise.} \end{cases} \tag{2.9}$$

We obtain $2\,\psi_{s+1} = \xi_{s+1} + \varepsilon\,\xi^\top_{s+1}$, so setting $J = \psi_{s+1}$ we get $J^\top = \varepsilon\, J$. □

3. Pfaffian surfaces

From now on we will assume that the field $\mathbf{k}$ is algebraically closed of characteristic zero. Recall that a torsionfree sheaf $\mathscr{E}$ on a polarized variety Y is called *initialized* if $\mathrm{H}^0(Y, \mathscr{E}) \neq 0$, and $\mathrm{H}^0(Y, \mathscr{E}(-1)) = 0$.

Let us introduce some notation. Given a projective variety $Y \subset \mathbb{P}^n$, polarized by $\mathscr{O}_Y(1)$, we write h_Y for the Hilbert function of Y, and $R(Y)$ for the coordinate ring of Y, so that $h_Y(t) = \dim_{\mathbf{k}}(R(Y)_t)$. We define also the difference Hilbert function, $\Delta_Y(t) = h_Y(t) - h_Y(t-1)$.

Given a smooth projective surface Y, polarized by $H_Y = c_1(\mathscr{O}_Y(1))$, and given an integer r and the Chern classes (c_1, c_2), we denote by $\mathsf{M}_Y(r, c_1, c_2)$ the moduli space of Gieseker-semistable sheaves with respect to H_Y, of rank r with Chern classes c_1, c_2. We will often denote the Chern Classes by a pair integers: this stands for c_1 times H_Y and c_2 times the class of a point in Y.

Recall that the vanishing locus Z of a nonzero global section of a rank 2 initialized bundle $\mathscr{E}$ on a surface Y is arithmetically Gorenstein (i.e. R_Z is a Gorenstein ring) if and only if $\mathscr{E}$ is aCM. The *index* i_Z of a zero-dimensional aG subscheme Z is the largest integer c such that $h_Z(c) < \mathrm{len}(Z)$.

For basic material on aCM bundles and aG subschemes we refer to [IK99], [Die96]. In particular we recall the notation $\mathscr{G}_h(i, m, d)$, see [CF06, Section 3]. We will make use of the computer algebra package `Macaulay 2`, see [GS].

3.1. **Quadratic Pfaffians.** Here we will prove that a general surface Y_d of degree d is the Pfaffian of a skew-symmetric matrix with quadratic entries if and only if $d \leq 15$. This sentence makes sense only if d is an even number, so we will look for *almost quadratic* matrices when d is odd. This means a matrix of the form:

$$\mathscr{O}_{\mathbb{P}}(-2)^d \oplus \mathscr{O}_{\mathbb{P}}(-1) \to \mathscr{O}_{\mathbb{P}}^d \oplus \mathscr{O}_{\mathbb{P}}(-1). \tag{3.1}$$

A surface Y_d can be written as an (almost) quadratic Pfaffian if and only if there is an aCM initialized rank 2 bundle $\mathscr{E}$ on Y_d with $c_1(\mathscr{E}) = d-2$. Note by [CF06, Proposition 4.1] that in this case we have $c_2(\mathscr{E}) = d\,(d-1)\,(d-2)/3$, so our result amounts to the next proposition. Remark that we always have $c_1(\mathscr{E}) \leq d-1$, so $c_1(\mathscr{E}) = d-2$ is called the *submaximal* case.

Proposition 3.1 (Submaximal)**.** On the general surface $Y_d \subset \mathbb{P} = \mathbb{P}^3$, it is defined a rank 2 initialized aCM bundle $\mathscr{E}$ with:

$$c_1(\mathscr{E}) = d-2, \qquad c_2(\mathscr{E}) = \frac{d\,(d-1)\,(d-2)}{3},$$

if and only if $d \geq 15$.

Proof. Note that the aCM bundle $\mathscr{E}$ is defined on a surface Y_d if and only if Y_d contains an aG subscheme Z of length $m = d\,(d-1)\,(d-2)/3$, and index $i = 2\,d-6$. This means that the function h_Z must agree with $h_{\mathbb{P}}$ up to degree $d-3$ and symmetric around $d-2$. In particular h_Z is uniquely determined.

The dimension of the component $\mathscr{G}_{\mathrm{h}_Z}(i, m, d)$ of the scheme $\mathscr{G}(i, m, d)$ equals $4\,d^2 - 4\,d - 1$, see [IK99]. Then, given a surface Y_d in the image of $p_{m,i,d}$ we have:

$$\begin{aligned} \dim(\mathrm{Im}(p_{m,i,d})) &\leq 4\,d^2 - 4\,d - 1 - \dim(p_{m,i,d}^{-1}(Y_d)) \leq \\ &\leq 4\,d^2 - 4\,d - 1 - d + 1 - \dim(\mathsf{M}_{Y_d}(2, d-2, m)) \leq \\ &\leq 4\,d^2 - 5\,d + \frac{d^2 - 18\,d + 41}{6}. \end{aligned}$$

It is easy it see that this quantity is strictly less than $\mathrm{h}^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(d)) - 1$ for $d \geq 15$. So the map $p_{m,i,d}$ cannot be dominant for $d \geq 15$.

To prove the converse, we use the package `Macaulay 2`. We distinguish two cases according to the parity of d. If d is even, we take a matrix f of size d with random quadratic entries. If d is odd, we let f be a generic mapping of the form (3.1).

In both cases, we consider the map Pf which associates to a skew-symmetric matrix the square root of its determinant. We would like to prove that Pf is dominant at the point represented by the matrix f, for each $d \leq 15$. We consider the the ideal J of Pfaffians of order $d-2$, and we multiply it by the maximal ideal $\mathfrak{m}$. By Adler's method (see the appendix of [Bea00]), our claim takes place if we show the equality:

$$\dim_{\mathbf{k}}((J \cdot \mathfrak{m})_d) = 0.$$

For each $d \leq 15$, this is can be checked with the `Macaulay 2` script:

```
isPrime(32003)
kk=ZZ/32003
S = kk[x_0..x_3]
ranQuad = (e1,e2,S) -> (
     -- a random skew-symmetrix matrix on S of order e1+e2
     -- where e1 entries are linear
     -- and e2 entries are quadratic
     e:=e1+e2;
     N1:=binomial(e1,2);
     N2:=binomial(e2,2);
     N12:=e1*e2;
     N:=binomial(e,2);
     R:=kk[t_0..t_(N-1)];
     G:=genericSkewMatrix(R,t_0,e);
     substitute(G,random(S^{0},S^{N1:0,N12:-1,N2:-2}))
     )
isDom = (d)->(
     M := ranQuad((d-2*floor(d/2)),d,S);
     PF := ideal(x_0..x_3)*pfaffians(2*ceiling(d/2)-2,M);
     (0 == hilbertFunction(d,S/PF))
     )
```

This returns the value `True` for each $d \leq 15$, in the approximate time of two hours on a personal computer. □

3.2. **Pfaffian sextic surfaces.** Here we give a classify aCM bundles of rank 2 on a general sextic surface X. In particular we assume that X is smooth and $\mathrm{Pic}(X) \cong \mathbb{Z}$. We let $\mathscr{E}$ be a rank 2 initialized indecomposable aCM bundle on X, with Chern classes c_1, c_2, and Z be the vanishing locus of a nonzero global section of $\mathscr{E}$.

Lemma 3.2. Assume $c_1 = 1$. Then the following values for c_2 and Δ_Z take place:

(3.2)

c_2	0	1	2	3	4
5	1	1	1	1	1
8	1	2	2	2	1
11	1	3	3	3	1
12	1	3	4	3	1
13	1	3	5	3	1
14	1	3	5	3	1

Proof. The scheme Z with a Hilbert function h_Z as in the first (respectively, the second) row of (3.2) represents 5 collinear points (respectively, a planar complete intersection of type $2, 4$). Such a subscheme is indeed contained in X.

On the other hand, the subschemes represented by the remaining rows are defined on X by virtue of [CF06, Lemma 3.2 and Proposition 6.3]. This completes the proof. □

Lemma 3.3. Let $\mathscr{E}$ and X be as above, and suppose $c_1 = 2$. Then the following instances of c_2 and Δ_Z occur on X:

(3.3)

c_2	0	1	2	3	4	5
14	1	3	3	3	3	1
16	1	3	4	4	3	1
18	1	3	5	5	3	1
20	1	3	6	6	3	1

Proof. The case given by the last row of (3.3) is given by [CF06, Lemma 3.2 and Theorem 6.12]. To check the remaining cases, observe that a scheme Z with the required Hilbert function is contained in a quintic surface Y. We have the exact sequence:

$$0 \to \mathscr{O}_Y \to \mathscr{F} \to J_{Z,Y}(3) \to 0,$$

where $\mathscr{F}$ is an aCM indecomposable bundle on Y. Note that $\mathscr{F}$ is not initialized, while $\mathscr{F}(-1)$ is. We have $c_1(\mathscr{F}(-1)) = 1$ and $c_2(\mathscr{F}(-1)) = \mathrm{len}(Z) - 10 \in \{4, 6, 8\}$. Our conclusion follows, since an initialized aCM indecomposable bundle with these Chern classes is defined on the general quintic surface Y, see [CF06, Proposition 4.5 and Lemma 3.2]). □

Lemma 3.4. Let $\mathscr{E}$ and X be as above, and assume $c_1 = 3$. Then the following cases for c_2 and Δ_Z take place on X:

(3.4)

c_2	0	1	2	3	4	5	6
26	1	3	6	6	6	3	1
27	1	3	4	7	6	3	1
28	1	3	5	8	6	3	1
29	1	3	6	9	6	3	1
30	1	3	6	10	6	3	1

Proof. Consider first the first 4 cases. Working as in the previous lemma, we consider a general quintic surface Y and an exact sequence:

$$0 \to \mathscr{O}_Y \to \mathscr{F} \to J_{Z,Y}(4) \to 0,$$

where $\mathscr{F}$ is an aCM indecomposable bundle on Y (not initialized). We have $c_1(\mathscr{F}(-1)) = 2$, $c_2(\mathscr{F}(-1)) = \mathrm{len}(Z) - 15 \in \{11, 12, 13, 14\}$. Then we conclude by [CF06, Proposition 6.3 and Lemma 3.2]).

To work out the last case, we let f be a general skew-symmetric of the form:

$$f : \mathscr{O}_{\mathbb{P}}(-2)^9 \oplus \mathscr{O}_{\mathbb{P}}(-3) \to \mathscr{O}_{\mathbb{P}}(-1)^9 \oplus \mathscr{O}_{\mathbb{P}}.$$

Making use a `Macaulay2` script analogous to the one used in the proof of Proposition 3.1, one can easily prove that the general sextic surface is defined by the Pfaffian of a matrix of the this form. On the other hand, the sheaf $\mathrm{cok}(f)$ is a rank 2 aCM initialized bundle with the required invariants. □

We can now prove the following result.

Proposition 3.5. Let X be a general sextic surface. Then the following list describes all the rank 2 initialized indecomposable aCM bundles $\mathscr{E}$ on X.

$c_1(\mathscr{E})$	-3	-2	-1	0	1	2	3	4	5
$c_2(\mathscr{E})$	1	2	$3,4,5$	$4,6,8$	$5,8,11,12,13,14$	$14,16,18,20$	$26,27,28,29,30$	40	55

Proof. By the above lemmas, all the cases mentioned in the table take place on X.

On the other hand, by [CF06, Propositions 4.1 and 4.2], the Chern classes of $\mathscr{E}$ are bounded by the values of our table. This takes care of all the cases, except for $c_1 = 1$.

Considering the zero locus Z of a nonzero section of $\mathscr{E}$ and its ideal sheaf J_Z, assuming $\mathrm{H}^0(X, J_Z(1)) = 0$ we immediately get the possibilities $c_2 = \mathrm{len}(Z) \in \{11, \ldots, 14\}$. On the other hand if $\mathrm{H}^0(X, J_Z(1)) \neq 0$, Z must be a complete intersection subscheme of the projective plane. Since the aG subscheme Z has index 4, it corresponds either to 5 collinear points, or to the intersection of a plane quartic and a conic. This completes the proof. □

4. Quadric hypersurfaces

By the results of Knörrer, [Knö87] it is well-known that any aCM indecomposable initialized bundle on a smooth quadric Q_n is either trivial either isomorphic to a spinor bundle. Resolutions of the spinor bundle are obtained in [BEH87].

Here we provide a recursive explicit formula to define a matrix of linear forms whose cokernel is a spinor bundle $\mathscr{S}$. In other words, we write down an explicit minimal graded free resolution of $\mathscr{S}$. For basic material on spinor bundles we refer to [Ott88].

4.1. **Odd dimensional quadrics.** Fix a positive integer k and set $n = 2k - 1$. Let Q be the quadric hypersurface in $\mathbb{P}^{n+1}$ defined by the equation:

$$Q = \mathbb{V}(x_0^2 + x_1 x_2 + \ldots + x_n x_{n+1}). \tag{4.1}$$

Denote by $\iota : Q \hookrightarrow \mathbb{P}^{n+1}$ the natural inclusion. We denote by $\mathscr{S}$ the spinor bundle on Q, and we recall that it is uniquely determined up to isomorphism. The vector bundle $\mathscr{S}$ has rank 2^{k-1}. Moreover, the bundle $\mathscr{S}(1)$ is globally generated and we have $\mathrm{H}^0(Q, \mathscr{S}(1)) \cong \mathsf{S}$, where S is the spin representation of $\mathsf{Spin}(n+2)$. In particular we have $\mathrm{h}^0(Q, \mathscr{S}(1)) = 2^k$.

In order to show an explicit presentation matrix for the extension to zero $\iota_* \mathscr{S}(1)$, we first define recursively the following skew-diagonal matrices:

$$\Delta_j = \begin{pmatrix} 0 & \Delta_{j-1} \\ (-1)^{j-1}\Delta_{j-1} & 0 \end{pmatrix}, \qquad \Delta_0 = 1. \tag{4.2}$$

Remark 4.1. The matrices Δ_j satisfy the following duality:

$$\Delta_j^\top = (-1)^{\lfloor \frac{j}{2} \rfloor} \Delta_j.$$

Proposition 4.2 (Odd dimensional quadric)**.** Let $n = 2k - 1$, let Q be defined by (4.1), and let $\mathscr{S}$ be the spinor bundle on Q. The the bundle $\mathscr{S}(1)$ admits the following minimal graded free resolution:

$$0 \to \mathsf{S}^* \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(-1) \xrightarrow{f} \mathsf{S} \otimes \mathscr{O}_{\mathbb{P}^{n+1}} \to \iota_* \mathscr{S}(1) \to 0,$$

where f is defined by $f = f_k$, with f_j recursively constructed as:

$$f_j = \begin{pmatrix} x_{2j-1}\Delta_{j-1} & f_{j-1} \\ (-1)^{j-1} f_{j-1} & (-1)^j x_{2j}\Delta_{j-1} \end{pmatrix}, \qquad f_0 = x_0, \tag{4.3}$$

where Δ_j is given by (4.2). In particular, the duality $\mathscr{S} \cong \mathscr{S}^*(-1)$ is:

- skew symmetric if and only if $k \equiv 2,3 \mod 4$,
- symmetric if and only if $k \equiv 0,1 \mod 4$.

Proof. We work by induction on the integer k. Looking at the matrix f_k defined recursively by (4.3), we have to show that its rank decreases to 2^{k-1} over $Q = Q_n$. This is the case if the determinant of f equals $(x_0^2 + x_1 x_2 + \ldots + x_n x_{n+1})$, to the power 2^{k-1}.

Let us we start with $Q = Q_1 \subset \mathbb{P}^2$. In this case, the spinor bundle $\mathscr{S}$ is nothing but the line bundle $\mathscr{O}(-p)$, where p is a point in Q_1. In our basis we have $Q_1 = \mathbb{V}(x_0^2 + x_1 x_2)$. So the extension to zero $\iota_* \mathscr{S}(1)$ in this case is presented by the following matrix:

$$\begin{pmatrix} x_1 & x_0 \\ -x_0 & x_2 \end{pmatrix}. \tag{4.4}$$

This proves that the first step in the induction actually takes place. Now set q_k for the equation of $Q = Q_{2k-1}$ written as $x_0^2 + x_1 x_2 + \ldots + x_n x_{n+1}$, and set $N_k = 2^{k-1}$ for brevity. Now we develop the determinant of f_k in the following way. For any $j \leq k$ we choose j entries in the diagonal block $x_n \Delta_{k-1}$, and multiply x_n^j by the corresponding minor. Since Δ_{k-1} is a diagonal matrix, we will correspondingly obtain the factor x_{n+1}^j in the development.

By the induction hypothesis, all nonvanishing minors of f_{k-1} give a factor $q_{k-1}^{N_k - j}$. Since we have $\binom{N_k}{j}$ choices we get the equation:

$$\begin{aligned} \det(f_k) &= \sum_{j=0}^{N_k} \binom{N_k}{j} \; q_{k-1}^{N_k - j} \; (x_n x_{n+1})^j = \\ &= (q_{k-1} + x_n x_{n+1})^{N_k} = q_k^{N_k}. \end{aligned}$$

This proves the claim concerning the resolution of $\iota_* \mathscr{S}(1)$. The statements about duality follows from Remark 4.1. □

Note that, cutting the quadric Q_{2k-1} with two hyperplanes of the form $x_n = x_{n+1} = 0$, and substituting into (4.3), we obtain a block matrix presenting two copies of the spinor bundle on the quadric Q_{2k-3}.

4.2. **Even dimensional quadrics.** In an analogous way we consider even dimensional quadrics. Given a positive integer k, we set $n = 2\,k$ and we consider the quadric hypersurface of $\mathbb{P}^{n+1}$ given by:

$$Q = \mathbb{V}(x_1 x_2 + x_3 x_4 + \ldots + x_0 x_{n+1}). \tag{4.5}$$

The hypersurface Q is homogeneous under the action of the algebraic group $\mathsf{Spin}(n+2)$. Over Q, we have two non isomorphic spinor bundles, denoted by $\mathscr{S}^+$ and $\mathscr{S}^-$, both of rank 2^{k-1}. For $\epsilon \in \{+,-\}$, the space of global section $\mathrm{H}^0(Q, \mathscr{S}^\epsilon(1))$ is isomorphic to the spin representation S^ϵ of dimension 2^k of the group $\mathsf{Spin}(n+2)$.

This time we define recursively the matrices f_j and g_j by:

$$f_j = \begin{pmatrix} x_{2j-1}\Delta_{j-1} & f_{j-1} \\ (-1)^{j-1} g_{j-1} & (-1)^j x_{2j}\Delta_{j-1} \end{pmatrix}, \qquad f_0 = x_0, \tag{4.6}$$

$$g_j = \begin{pmatrix} x_{2j-1}\Delta_{j-1} & g_{j-1} \\ (-1)^{j-1} f_{j-1} & (-1)^j x_{2j}\Delta_{j-1} \end{pmatrix}, \qquad g_0 = x_{n+1}. \tag{4.7}$$

with Δ given by (4.2).

Proposition 4.3 (Even dimensional quadric)**.** Let $k \geq 1$, set $n = 2\,k$, and let $\mathscr{S}^+$, $\mathscr{S}^-$ be the spinor bundles on the smooth n-dimensional quadric Q defined by (4.5). Then $\iota_*\mathscr{S}^+(1)$ and $\iota_*\mathscr{S}^-(1)$ have the following minimal graded free resolution:

$$0 \to (\mathsf{S}^+)^* \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(-1) \xrightarrow{f} \mathsf{S}^+ \otimes \mathscr{O}_{\mathbb{P}^{n+1}} \to \iota_*\mathscr{S}^+(1) \to 0,$$

$$0 \to (\mathsf{S}^*)^* \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(-1) \xrightarrow{g} \mathsf{S}^- \otimes \mathscr{O}_{\mathbb{P}^{n+1}} \to \iota_*\S^-(1) \to 0,$$

with $f = f_k$, $g = g_k$. In particular for $n \equiv 0 \mod 4$, the dualities $\mathscr{S}^+(1) \cong (\mathscr{S}^+)^*$, $\mathscr{S}^-(1) \cong (\mathscr{S}^-)^*$ are:

- skew symmetric if and only if $k \equiv 2 \mod 4$,
- symmetric if and only if $k \equiv 0 \mod 4$.

Proof. The proof differs from that of Proposition 4.2 only the first step in the induction. So we consider the two dimensional quadric:

$$Q_2 = \mathbb{V}(x_0x_3 + x_1x_2).$$

The two non isomorphic spinor bundles in this case are the line bundles $\mathscr{O}(-1,0)$ and $\mathscr{O}(0,-1)$, corresponding to the two rulings of the quadric surface as $\mathbb{P}^1 \times \mathbb{P}^1$. Their resolutions are given by the following two matrices:

$$f_1 = \begin{pmatrix} x_2 & x_0 \\ -x_3 & x_1 \end{pmatrix}, \qquad g_1 = \begin{pmatrix} x_1 & x_0 \\ -x_3 & x_2 \end{pmatrix}.$$

And clearly the determinant of both matrices is the equation of the quadric Q_2. □

Note that, restricting to the hyperplane $x_{n+1} = x_0$, we obtain the matrix of the odd dimensional case.

Finally we formulate the following result, which is an analogue of Remark 2.1, only the ambient space is now a smooth quadric. We don't have an analogue of Proposition 2.2 or 2.3: we illustrate the reason with an example.

Remark 4.4. Let X be a subscheme of codimension r of a smooth quadric $Q \subset \mathbb{P}^{n+1}$ and let $\mathscr{E}$ be an aCM bundle on X. Then $\mathscr{E}$ admits the resolution:

$$0 \to \mathsf{P}_r \oplus \mathsf{L}^+ \oplus \mathsf{L}^- \to \mathsf{P}_{r-1} \to \cdots \to \mathsf{P}_0 \to \mathscr{E} \to 0, \quad \text{with}$$

$$\mathsf{P}_i = \bigoplus_{j=1}^{\ell_i} \mathscr{O}_{Q_n}(a_{ij}) \quad \mathsf{L}^+ = \bigoplus_{j=1}^{\ell^+} \mathscr{S}^+(a_j^+) \quad \mathsf{L}^- = \bigoplus_{j=1}^{\ell^-} \mathscr{S}^-(a_j^-). \tag{4.8}$$

Proof. The argument is analogous to that of Proposition 2.1. We consider a minimal graded free resolution of the form (2.1) of $\mathscr{E}$, with P_i free over Q, and the kernel $K_i = \ker(f_{i-1} : \mathsf{P}_i \to \mathsf{P}_{i-1})$.

This time, the bundle K_{r-1} is aCM on Q_n. Thus, by [Knö87], it must decompose in the form (4.8). □

Remark 4.5. Even if $\mathscr{E}$ admits a duality κ, the resolution might not be self dual. Indeed, we have:

$$\mathrm{Ext}^1_Q(\mathscr{S}^+(1) \oplus \mathscr{S}^-(1), \mathscr{S}^+ \oplus \mathscr{S}^-) \neq 0.$$

This provides a nontrivial obstruction to lifting κ to $\mathsf{P}_0 \to \mathsf{P}_r^*(t)$. A statement analogous to Proposition 2.2 and 2.3 holds if $\mathsf{P}_r \cong \oplus \mathscr{O}(a_{rj})$, see Example 4.6.

Example 4.6. Let X be the complete intersection of two general quadrics Q^1 and Q^2 in $\mathbb{P}^5$, and let $\mathscr{E}$ be an initialized indecomposable aCM bundle of rank 2 on X. Then by [AC00] there are precisely 3 deformation classes for $\mathscr{E}$. They are classified by the first and second Chern classes, which can be $(0,1)$, $(1,2)$ or $(2,6)$.

The resolution of $\iota_* \mathscr{E}$ over $\mathbb{P}^5$ takes the form:

$$\boxed{(c_1,c_2)=(0,1)} \qquad 0 \to \mathscr{O}(-4) \oplus \mathscr{O}(-3)^4 \xrightarrow{-f^\top} \mathscr{O}(-2)^{10} \xrightarrow{f} \mathscr{O} \oplus \mathscr{O}(-1)^4 \to \iota_* \mathscr{E} \to 0,$$

$$\boxed{(c_1,c_2)=(1,2)} \qquad 0 \to \mathscr{O}(-3)^4 \xrightarrow{-f^\top} \mathscr{O}(-2)^4 \oplus \mathscr{O}(-1)^4 \xrightarrow{f} \mathscr{O}^4 \to \iota_* \mathscr{E} \to 0,$$

$$\boxed{(c_1,c_2)=(2,6)} \qquad 0 \to \mathscr{O}(-2)^8 \xrightarrow{-f^\top} \mathscr{O}(-1)^{16} \xrightarrow{f} \mathscr{O}^8 \to \iota_* \mathscr{E} \to 0.$$

On the other hand, we let $\jmath : X \hookrightarrow Q = Q^1$ be the natural inclusion and we consider the resolutions of $\jmath_* \mathscr{E}$. This takes the form:

$$\boxed{(c_1,c_2)=(0,1)} \qquad 0 \to \mathscr{O}_Q(-2) \oplus \mathscr{S}^+ \oplus \mathscr{S}^- \to \mathscr{O}_Q \oplus \mathscr{O}_Q(-1)^4 \to \jmath_* \mathscr{E} \to 0, \tag{4.9}$$

$$\boxed{(c_1,c_2)=(2,6)} \qquad 0 \to \mathscr{O}_Q(-1)^4 \to \mathscr{O}_Q^4 \to \jmath_* \mathscr{E} \to 0, \tag{4.10}$$

$$\boxed{(c_1,c_2)=(1,2)} \qquad 0 \to (\mathscr{S}^+)^2 \oplus (\mathscr{S}^-)^2 \to \mathscr{O}_Q^8 \to \jmath_* \mathscr{E} \to 0. \tag{4.11}$$

Here the resolutions (4.9) and (4.11) are not self dual, while (4.10) is.

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