Instanton bundles and their moduli spaces

From ADHM quivers to Kuznetsov components

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Geometry of instantons and quivers

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In algebraic geometry, instantons are special classes of sheaves and vector bundles, rooted in the Yang–Mills equations through the Atiyah–Ward correspondence. They are typically characterized by cohomological vanishing conditions together with numerical or topological constraints. A recurring theme is that their moduli spaces admit descriptions in terms of auxiliary geometric, algebraic, and categorical structures, such as framings, monads and quivers, exceptional collections and Kuznetsov components.

Lecture 1. Framings, monads and quivers on the plane.

We introduce the moduli spaces of torsion-free sheaves on the projective plane, framed along a line at infinity, discussing Beilinson's spectral sequence and the quiver-theoretic, or ADHM, description of the moduli space in the spirit of Barth–Nakajima.

Lecture 2. Twistor spaces: $\mathbb{P}^3$ and the flag threefold.

After a brief review of Penrose's twistor construction, we study instanton bundles on the two compact Kähler twistor threefolds: the projective space $\mathbb{P}^3$, the twistor space of S^4, and $\mathrm{F}(1,2;3)$, the point-line flag variety of the projective plane, which is the twistor space of $\mathbb{P}^2$. We explain how to construct natural families of vector bundles, such as 't Hooft bundles, and how these constructions give information about the geometry of their moduli spaces.

Lecture 3. Fano threefolds: from monads to semiorthogonal decompositions.

We discuss instanton sheaves on Fano threefolds of Picard number one and explain their relation with Ulrich bundles. We give an overview of the relevant semiorthogonal decompositions of the derived categories of these threefolds, emphasizing how exceptional collections and Kuznetsov components can be used to construct and describe moduli spaces of sheaves.

Lecture 4. Higher-dimensional Fano varieties and Kuznetsov components.

We discuss possible definitions of instanton sheaves on higher-dimensional projective varieties, focusing on cubic fourfolds. In this setting, the K3 structure of the Kuznetsov component plays a central role. We explain how the perspective developed for Fano threefolds suggests new constructions and moduli-theoretic questions in higher dimension.

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1. Framings, monads and quivers on the plane

The goal of this lecture is to understand a result, explained in [Nak99, Chapter 2], connecting moduli spaces of sheaves on $\mathbb{P}^2$ with representations of a certain quiver.

Nakajima's description of framed instantons. We work over an algebraically closed field $\Bbbk$ and we consider the projective plane $\mathbb{P}^2 = \mathbb{P}^2_{\Bbbk}$. We choose a line of $\mathbb{P}^2$ and call it the line at infinity, denoted by ℓ_∞.

Definition 1.1. Fix $\Bbbk$-vector spaces W and I with

$$\dim W = r, \qquad \dim I = k.$$

Given a quadruple $(B_1, B_2, i, j) \in \mathrm{End}(I) \oplus \mathrm{End}(I) \oplus \mathrm{Hom}(W, I) \oplus \mathrm{Hom}(I, W)$, we call *ADHM equation* the following:

$$[B_1, B_2] + ij = 0. \tag{1}$$

We say that the quadruple (B_1, B_2, i, j) is *stable* if there is no proper subspace $S \subsetneq I$ such that

$$B_1(S) \subset S, \qquad B_2(S) \subset S, \qquad \mathrm{Im}(i) \subset S. \tag{2}$$

The group $\mathrm{GL}(I)$ acts on the set of quadruples by

$$g \cdot (B_1, B_2, i, j) = \big(gB_1g^{-1},\, gB_2g^{-1},\, gi,\, jg^{-1}\big).$$

We observe that the $\mathrm{GL}(I)$ action on quadruples preserves the stable locus. Also, $\mathrm{GL}(I)$ sends a quadruple satisfying (1) to a quadruple satisfying (1).

Definition 1.2. We write $\mathfrak{MF}(r, k)$ for the moduli space of *torsion-free sheaves of rank r and charge k, framed along the line ℓ_∞*, up to equivalence. These are torsion-free sheaves $\mathcal{E}$ on $\mathbb{P}^2$ such that

$$\mathrm{rk}(\mathcal{E}) = r, \qquad c_2(\mathcal{E}) = k,$$

together with a framing

$$\Phi \colon \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} W \otimes \mathcal{O}_{\ell_\infty}.$$

Here, two framed sheaves $(\mathcal{E}, \Phi)$ and $(\mathcal{E}', \Phi')$ are equivalent if there is an isomorphism $\varphi : \mathcal{E} \to \mathcal{E}'$ such that $\Phi' \circ \varphi|_{\ell_\infty} = \Phi$.

Theorem 1.3. *The group $\mathrm{GL}(I)$ acts freely on the set of stable quadruples satisfying (1), and the resulting geometric quotient is naturally isomorphic to the moduli space $\mathfrak{MF}(r, k)$.*

Equivalently, $\mathfrak{MF}(r, k)$ is the Nakajima quiver variety associated with the Jordan quiver, with dimension vector k and framing vector r.

Remark 1.4. It may be useful to make the following observations.

1. A framed sheaf $\mathcal{E}$ along ℓ_∞, which is locally trivial around ℓ_∞, necessarily has $c_1(\mathcal{E}) = 0$.
2. There are very natural sheaves that cannot be framed. For instance, $\mathcal{E} = S^2\Omega_{\mathbb{P}^2}(3)$ is a slope-stable vector bundle of rank 3, with $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = 3$. But for all lines $\ell \subset \mathbb{P}^2$, we have $\mathcal{E}|_\ell \simeq \mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell \oplus \mathcal{O}_\ell(1)$.

The connection with the Hilbert scheme. For $r = 1$, a rank-one torsion-free sheaf on $\mathbb{P}^2$ is of the form $\mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^2}$, for a zero-dimensional subscheme $Z \subset \mathbb{P}^2$ and some $t \in \mathbb{Z}$. If such a sheaf is framed along a line ℓ_∞, then $t = 0$ and $Z \subset \mathbb{A}^2 = \mathbb{P}^2 \setminus \ell_\infty$. Therefore the rank-one case recovers

$$\mathfrak{MF}(1, k) \simeq \mathrm{Hilb}^k(\mathbb{A}^2).$$

More generally, an example of a rank-r torsion-free sheaf on $\mathbb{P}^2$ is $\mathcal{I}_{Z_1} \oplus \cdots \oplus \mathcal{I}_{Z_r}$, for some zero-dimensional subschemes $Z_1, \ldots, Z_r \subset \mathbb{A}^2$.

For $r = 1$, Nakajima's description of the Hilbert scheme $\mathrm{Hilb}^k(\mathbb{A}^2)$ in terms of ADHM data is slightly different. It reads as follows. Let I be a vector space of dimension k and consider $B_1, B_2 \in \mathrm{End}(I)$, and $i \in \mathrm{Hom}_{\Bbbk}(\Bbbk, I)$. Consider the following quasi-affine variety:

$$\mathbf{H} = \left\{ (B_1, B_2, i) \;\middle|\; \begin{array}{ll} \text{(i)} & [B_1, B_2] = 0, \\ \text{(ii)} & \text{there exists no proper subspace } S \subsetneq I \\ & \text{such that } B_1(S) \subset S, B_2(S) \subset S, \text{ and } \mathrm{Im}\, i \subset S \end{array} \right\},$$

This is the locus of stable triples of ADHM data associated with the Hilbert scheme of subschemes of length k on $\mathbb{A}^2$. This Hilbert scheme is denoted by $(\mathbb{A}^2)^{[k]}$.

More generally, given a quasi-projective variety X, one writes $X^{[k]}$ for the Hilbert scheme parametrizing subschemes Z of dimension 0 and length k on X, where the length of Z is $\dim_{\Bbbk}(\mathrm{H}^0(\mathcal{O}_Z))$. This is a quasi-projective scheme that represents the Hilbert scheme functor Hilb_X^k, which assigns to a scheme S the set of subschemes of $X \times S$ which are flat over S, with constant Hilbert polynomial equal to k. If X is a smooth connected surface, $X^{[k]}$ is a smooth connected quasi-projective variety of dimension $2k$.

The manifold $X^{[k]}$ has a natural map, called the Hilbert-Chow map, toward the symmetric product $X^{(k)} = X^k/\mathfrak{S}_k$, where the symmetric group $\mathfrak{S}_k$ acts by permuting the coordinates of X^k. Of course, $X^{(k)}$ has dimension $2k$. A point of $X^{(k)}$ can be written as $z_1 + \cdots + z_k$, where $z_i \in X$ for $1 \leq i \leq k$. Given a subscheme Z of length k of X, there are s points $z_1, \ldots, z_s$ of X, such that Z has multiplicity m_i at z_i for $1 \leq i \leq s$, and $k = m_1 + \cdots + m_s$. Then the Hilbert-Chow map associates to $Z \in X^{[k]}$ the point $m_1 z_1 + \cdots + m_s z_s \in X^{(k)}$.

This map is an isomorphism at the open subset of $X^{[k]}$ consisting of smooth (i.e. reduced) subschemes. However, $X^{(k)}$ is singular along the whole subset of unordered k-tuples having at least one repeated entry, as can be seen by a simple computation in local coordinates.

On the other hand, $X^{[k]}$ is actually smooth (and connected). One way to get intuition for the smoothness of $X^{[k]}$ is by Riemann–Roch. Indeed, this space is of dimension $2k$ hence it is smooth if, for any point of X, the tangent space has dimension $2k$. But the dimension of the tangent space at a point $Z \in X^{[k]}$ is computed by

$$\mathrm{ext}^1_X(\mathcal{O}_Z, \mathcal{O}_Z) = \hom_X(\mathcal{O}_Z, \mathcal{O}_Z) + \mathrm{ext}^2_X(\mathcal{O}_Z, \mathcal{O}_Z) - \chi(\mathcal{O}_Z, \mathcal{O}_Z).$$

Now one observes $\hom_X(\mathcal{O}_Z, \mathcal{O}_Z) = \mathrm{h}^0(\mathcal{O}_Z) = k$ and

$$\mathrm{ext}^2_X(\mathcal{O}_Z, \mathcal{O}_Z) = \hom_X(\mathcal{O}_Z, \mathcal{O}_Z \otimes \omega_X) = \mathrm{h}^0(\mathcal{O}_Z \otimes \omega_X) = k,$$

since on a finite-dimensional subscheme, the operation of tensoring with a line bundle has no effect. Then, since the Chern character of $\mathcal{O}_Z$ is $k[z]$, where $[z]$ is the cohomology class of a point $z \in X$, Riemann–Roch gives, by dimension reasons:

$$\chi(\mathcal{O}_Z, \mathcal{O}_Z) = \int_X \mathrm{ch}(\mathcal{O}_Z)^\vee \mathrm{ch}(\mathcal{O}_Z)\mathrm{td}(X) = 0.$$

Therefore $\mathrm{ext}^1_X(\mathcal{O}_Z, \mathcal{O}_Z) = 2k$, which implies smoothness of $X^{[k]}$ at Z. This argument is nicely explained in [Leh04].

Going back to $\mathbb{A}^2$, we define an action of $\mathrm{GL}_k(\Bbbk)$ on $\mathbf{H}$ by

$$g \cdot (B_1, B_2, i) = (gB_1g^{-1}, gB_2g^{-1}, gi),$$

for $g \in \mathrm{GL}_k(\Bbbk)$. Then, we consider the quotient space

$$\mathbf{H}/\mathrm{GL}_k(\Bbbk).$$

Theorem 1.5 (Nakajima)**.** *The quotient* $\mathbf{H}/\mathrm{GL}_k(\Bbbk)$ *is isomorphic to* $(\mathbb{A}^2)^{[k]}$.

Comparing this to Theorem 1.3, we see that only B_1, B_2 and i show up. Indeed, it turns out that the datum of j is superfluous for $r = 1$. The intuition behind this result is that, if Z is a subscheme of $\mathbb{A}^2$ of length k, then we have a surjection of $\Bbbk$-algebras

$$\Bbbk[z_1, z_2] \to \mathrm{H}^0(\mathcal{O}_Z).$$

We said that $\mathrm{H}^0(\mathcal{O}_Z)$ and I are k-dimensional vector spaces. So let us fix an isomorphism $\Psi : \mathrm{H}^0(\mathcal{O}_Z) \to I$. Then we have a natural map $\Bbbk \to I$ by sending $\lambda \in \Bbbk$ to the image of $\lambda 1_{\Bbbk[z_1,z_2]}$ in I. This map is our i. Also, we have natural $\Bbbk$-linear maps $B_1, B_2 : I \to I$ which are given by sending $f \in I \simeq \mathrm{H}^0(\mathcal{O}_Z)$ to $z_1 f$ and $z_2 f$, respectively (here, we tacitly write z_1 and z_2 for the images of z_1 and z_2 in I). The maps B_1 and B_2 commute, since the variables z_1 and z_2 themselves commute. Also, the $\Bbbk$-algebra $\mathrm{H}^0(\mathcal{O}_Z)$ is generated by the images of 1, z_1 and z_2, so there is no subspace $S \subsetneq I$ which contains the image of 1 and which is stable for the multiplication by z_1 and z_2. This gives our triple $(B_1, B_2, i) \in \mathbf{H}$. We note that such a triple is determined only up to the choice of Ψ, which is why, in order to get back the Hilbert scheme, we should take the quotient by the action of $\mathrm{GL}(I)$.

Minimal setting of notions to read the result

We work out the basic framework in the setting of complex projective manifolds. In this case, the category of coherent sheaves over the ring of regular functions is equivalent to the category of analytic coherent sheaves over the ring of holomorphic functions, by the GAGA correspondence.

If X is defined over a field $\mathbb{k}$ different from $\mathbb{C}$, then in principle we cannot use the GAGA correspondence, nor singular cohomology. However, all the notions presented here, notably Chern classes, still make sense provided that we replace the cohomology ring of X with the Chow ring of X. Thus, over an arbitrary field, the natural home of Chern classes is $\mathrm{CH}^*(X)$. When $\mathbb{k} = \mathbb{C}$, we may pass from Chow groups to singular cohomology by the cycle class map.

We use Grothendieck's convention for projective spaces and projective bundles. Thus, if $\mathcal{E}$ is a vector bundle on X, then $\mathbb{P}(\mathcal{E})$ parametrizes one-dimensional quotients of $\mathcal{E}$. In particular, if $\pi : \mathbb{P}(\mathcal{E}) \to X$ is the structure morphism, there is a tautological quotient

$$\pi^*\mathcal{E} \twoheadrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

The first Chern class. Given an n-dimensional complex projective manifold X, the isomorphism classes of algebraic line bundles on X form an abelian group, the Picard group of X, denoted by $\mathrm{Pic}(X)$. This group is isomorphic to $\mathrm{H}^1(\mathcal{O}_X^*)$, as one can see by identifying a line bundle with its transition functions, which satisfy the cocycle condition required to define an element of the Čech cohomology group $\mathrm{H}^1(\mathcal{O}_X^*)$. The Picard group is equipped with the first Chern class map

$$c_1 : \mathrm{Pic}(X) \simeq \mathrm{H}^1(\mathcal{O}_X^*) \longrightarrow \mathrm{H}^2(X, \mathbb{Z}).$$

This map is obtained from the long exact cohomology sequence associated with the exponential sequence in the complex analytic topology:

$$0 \to \mathbb{Z} \xrightarrow{2\pi\sqrt{-1}} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0.$$

Given a line bundle $\mathcal{L}$, we write $c_1(\mathcal{L})$ for its first Chern class.

Chern classes of vector bundles. Let $\mathcal{E}$ be a vector bundle of rank r over X. The projective bundle $\mathbb{P}(\mathcal{E})$ is equipped with the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. We denote by

$$\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$$

the corresponding class, either in $\mathrm{H}^2(\mathbb{P}(\mathcal{E}), \mathbb{Z})$ if $\mathbb{k} = \mathbb{C}$, or in $\mathrm{CH}^1(\mathbb{P}(\mathcal{E}))$ in the algebraic setting. The relative Euler sequence is

$$0 \to \Omega_\pi(1) \to \pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \to 0.$$

This specializes to the classical Euler sequence

$$0 \to \Omega_{\mathbb{P}^n}(1) \to \mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0. \tag{3}$$

The projective bundle formula gives

$$\mathrm{H}^*(\mathbb{P}(\mathcal{E}), \mathbb{Z}) \simeq \bigoplus_{0 \le p \le r-1} \xi^p \cdot \mathrm{H}^*(X, \mathbb{Z}),$$

and similarly for Chow groups. Hence there are uniquely determined classes

$$c_p(\mathcal{E}) \in \mathrm{H}^{2p}(X, \mathbb{Z}), \qquad 0 \le p \le r,$$

with $c_0(\mathcal{E}) = 1$, such that

$$\xi^r - c_1(\mathcal{E})\xi^{r-1} + c_2(\mathcal{E})\xi^{r-2} - \cdots + (-1)^r c_r(\mathcal{E}) = 0.$$

Equivalently,

$$\sum_{p=0}^{r} (-1)^p c_p(\mathcal{E})\xi^{r-p} = 0.$$

In the algebraic category, over an arbitrary field, we write

$$c_p(\mathcal{E}) \in \mathrm{CH}^p(X).$$

Basic properties of Chern classes. The total Chern class of $\mathcal{E}$ is

$$c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + \cdots + c_r(\mathcal{E}).$$

It is functorial under pull-back: if $f : Y \to X$ is a morphism, then

$$c_p(f^*\mathcal{E}) = f^*c_p(\mathcal{E}).$$

It is also multiplicative in short exact sequences. Namely, if

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

is an exact sequence of vector bundles, then

$$c(\mathcal{E}) = c(\mathcal{E}')c(\mathcal{E}'').$$

In particular, $c_1(\mathcal{E}) = c_1(\mathcal{E}') + c_1(\mathcal{E}'')$, and $c_2(\mathcal{E}) = c_2(\mathcal{E}') + c_1(\mathcal{E}')c_1(\mathcal{E}'') + c_2(\mathcal{E}'')$. For a line bundle $\mathcal{L}$, the total Chern class is simply

$$c(\mathcal{L}) = 1 + c_1(\mathcal{L}).$$

The Chern classes are therefore determined by line bundles, the projective bundle formula, and the splitting principle.

A nice property of Chern classes is their connection with degeneracy loci. Namely, if $\mathcal{E}$ is a vector bundle of rank r on X and $\varphi : \mathcal{O}_X^s \to \mathcal{E}$ is a morphism of coherent sheaves given by the choice of $s \leq r$ global sections of $\mathcal{E}$, then the degeneracy locus of φ is the closed subscheme of X defined locally by the maximal minors of φ. This subscheme is expected either to be empty or to have codimension $r - s + 1$ in X. If this is the case, then the cohomology class of this locus is $c_{r-s+1}(\mathcal{E})$. Also, this is indeed the case if $\mathcal{E}$ is globally generated and φ is sufficiently general.

Chern classes of coherent sheaves. If X is smooth, the Chern classes may be extended from vector bundles to coherent sheaves. Indeed, every coherent sheaf defines a class in the Grothendieck group $K_0(X)$, the free abelian group generated by coherent sheaves on X, modulo the relation $\sim$ given by $\mathcal{E} \sim \mathcal{E}' + \mathcal{E}''$ if there is an exact sequence of coherent sheaves

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

On a smooth variety, this group is generated by vector bundles. Thus, if a coherent sheaf $\mathcal{F}$ is represented in $K_0(X)$ by a finite locally free resolution

$$0 \to \mathcal{E}_m \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0,$$

then its total Chern class is defined by

$$c(\mathcal{F}) = \prod_{a=0}^{m} c(\mathcal{E}_a)^{(-1)^a}.$$

This expression is independent of the chosen resolution. Equivalently, one may first define the Chern character (see below), which is additive in exact sequences:

$$\mathrm{ch}(\mathcal{F}) = \sum_{a=0}^{m} (-1)^a \mathrm{ch}(\mathcal{E}_a).$$

The Chern classes and the Chern character determine each other after tensoring with $\mathbb{Q}$. For instance,

$$\mathrm{ch}(\mathcal{F}) = \mathrm{rk}(\mathcal{F}) + c_1(\mathcal{F}) + \frac{1}{2}\big(c_1(\mathcal{F})^2 - 2c_2(\mathcal{F})\big) + \cdots.$$

This convention is very useful for torsion-free sheaves: if $\mathcal{F}$ is torsion-free of rank r, then $c_1(\mathcal{F})$ is the first Chern class of its determinant line bundle

$$\det(\mathcal{F}) = (\wedge^r \mathcal{F})^{\vee\vee}.$$

Example 1.6. Let $\mathcal{I}_Z$ be the ideal sheaf of a subscheme $Z \subset \mathbb{P}^2$ obtained as complete intersection of two curves of degree a and b. Then we have

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-a-b) \to \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b) \to \mathcal{I}_Z \to 0.$$

Therefore, $c_1(\mathcal{I}_Z) = 0$ and $(1 + c_2(\mathcal{I}_Z))(1 - (a+b)H) = (1 - aH)(1 - bH)$, so $c_2(\mathcal{I}_Z) = abH^2$. If Z is disjoint from ℓ_∞, then $\mathcal{I}_Z$ belongs to $\mathfrak{MF}(1, ab)$.

Chern roots. The splitting principle says that, after pulling back to a suitable auxiliary space, a vector bundle $\mathcal{E}$ of rank r behaves as if it were a direct sum of line bundles

$$\mathcal{E} \simeq \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r.$$

If we write $\alpha_i = c_1(\mathcal{L}_i)$, then the classes $\alpha_1, \ldots, \alpha_r$ are called the Chern roots of $\mathcal{E}$. Formally,

$$c(\mathcal{E}) = \prod_{i=1}^{r} (1 + \alpha_i),$$

so that $c_p(\mathcal{E})$ is the p-th elementary symmetric polynomial in the α_i's. In these terms,

$$\mathrm{ch}(\mathcal{E}) = \sum_{i=1}^{r} e^{\alpha_i}.$$

This is often the quickest way to remember formulas involving Chern classes. For example,

$$c_1(\mathcal{E}) = \sum_i \alpha_i, \qquad c_2(\mathcal{E}) = \sum_{i<j} \alpha_i \alpha_j.$$

Riemann–Roch. Let X be a proper scheme over a field $\Bbbk$. Let $n = \dim(X)$. The Euler characteristic of a coherent sheaf $\mathcal{F}$ over X is defined as

$$\chi(\mathcal{F}) = \sum_{0 \leq p \leq n} (-1)^p \mathrm{h}^p(X, \mathcal{F}).$$

Assume X is smooth. Then the Hirzebruch-Riemann-Roch theorem expresses $\chi(\mathcal{F})$ in terms of characteristic classes:

$$\chi(\mathcal{F}) = \int_X \mathrm{ch}(\mathcal{F}) \, \mathrm{td}(X),$$

where $\mathrm{td}(X) = \mathrm{td}(\mathrm{T}_X)$ is the Todd class of the tangent bundle and $\int_X$ means taking the degree of the component of codimension n. More generally, for a line bundle $\mathcal{O}_X(tH)$,

$$\mathbf{P}_{\mathcal{F}}(t) = \chi(\mathcal{F}(tH)) = \int_X e^{tH} \mathrm{ch}(\mathcal{F}) \, \mathrm{td}(X).$$

Thus the Hilbert polynomial of $\mathcal{F}$ is determined by the Chern character of $\mathcal{F}$, the polarization H, and the Todd class of X. On the projective space, this gives the familiar fact that the Hilbert polynomial of $\mathcal{F}$ is encoded by its Chern classes.

Similarly, if X is a smooth proper n-dimensional scheme, over $\mathbb{C}$, and $\mathcal{E}$, $\mathcal{F}$ are coherent sheaves on X, one defines

$$\chi(\mathcal{E}, \mathcal{F}) = \sum_{0 \leq p \leq n} (-1)^p \mathrm{ext}^p_X(\mathcal{E}, \mathcal{F}).$$

Then, Riemann–Roch gives

$$\chi(\mathcal{E}, \mathcal{F}) = \int_X \mathrm{ch}(\mathcal{E})^\vee \, \mathrm{ch}(\mathcal{F}) \, \mathrm{td}(X),$$

Here, given a vector $v \in \mathrm{H}^*(X, \mathbb{Z})$, the dual $v^\vee$ is just defined as $v^\vee = \sum_{0 \leq i \leq n} (-1)^i v_i$, where v_i is the component of v lying in $\mathrm{H}^{2i}(X, \mathbb{Z})$.

Cohomology of the projective space. For the projective space $\mathbb{P}^n = \mathbb{P}^n_{\Bbbk}$, we have the tautological very ample line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. We usually denote $H = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Let us work with $\Bbbk = \mathbb{C}$, so that H lies naturally in $\mathrm{H}^2(\mathbb{P}^n, \mathbb{Z})$. We have, for $1 \leq p \leq n$:

$$\mathrm{H}^{2p}(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z} H^p.$$

Here, H^p represents a linear subspace of $\mathbb{P}^n$ of codimension p. Very often, for a coherent sheaf $\mathcal{E}$ on $\mathbb{P}^n$, we write $c_p(\mathcal{E}) = c_p \in \mathbb{Z}$, meaning that actually $c_p(\mathcal{E}) = c_p H^p \in \mathrm{H}^{2p}(\mathbb{P}^n, \mathbb{Z})$. If $\Bbbk$ is a different field, this can be conveniently reformulated in terms of the intersection ring, or the Chow ring of X, consisting of algebraic cycles modulo numerical, or rational equivalence.

Stability of sheaves. Let X be a smooth projective variety endowed with an ample line bundle $\mathcal{O}_X(1)$. If $\mathcal{E}$ is a torsion-free coherent sheaf on X, its slope is

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot H^{\dim X - 1}}{\mathrm{rk}(\mathcal{E})}, \qquad H = c_1(\mathcal{O}_X(1)).$$

The sheaf $\mathcal{E}$ is called slope-stable, respectively slope-semistable, if for every non-zero proper subsheaf

$$0 \neq \mathcal{F} \subsetneq \mathcal{E}$$

with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$, one has

$$\mu(\mathcal{F}) < \mu(\mathcal{E}), \qquad \text{respectively} \qquad \mu(\mathcal{F}) \leq \mu(\mathcal{E}).$$

There is also a slightly different notion, called Gieseker stability. For a coherent sheaf $\mathcal{E}$, let

$$p_{\mathcal{E}}(m) = \frac{\chi(\mathcal{E}(m))}{\alpha_d(\mathcal{E})}$$

be its reduced Hilbert polynomial, where $d = \dim \mathrm{Supp}(\mathcal{E})$ and $\alpha_d(\mathcal{E})$ is the leading coefficient of $\chi(\mathcal{E}(m))$. A pure sheaf $\mathcal{E}$ is Gieseker-stable, respectively Gieseker-semistable, if for every non-zero proper subsheaf $\mathcal{F} \subsetneq \mathcal{E}$ one has

$$p_{\mathcal{F}}(m) < p_{\mathcal{E}}(m), \qquad \text{respectively} \qquad p_{\mathcal{F}}(m) \leq p_{\mathcal{E}}(m), \qquad \text{for } m \gg 0.$$

For a torsion-free sheaf $\mathcal{E}$ of rank $r > 0$, (semi)stability amounts to asking that any non-zero subsheaf $\mathcal{F} \subsetneq \mathcal{E}$ has (strictly) smaller *reduced Hilbert polynomial*, $\chi(\mathcal{E}(m))/r$. One has the implications

$$\text{slope-stable} \Longrightarrow \text{Gieseker-stable} \Longrightarrow \text{Gieseker-semistable} \Longrightarrow \text{slope-semistable}.$$

Gieseker stability is the stability notion naturally used to construct projective moduli spaces of coherent sheaves, while slope stability is often more flexible for computations.

Gieseker's moduli space. Let us introduce the moduli space of semistable sheaves. Let X be a smooth projective variety and let H be an ample divisor class on X. We write

$$\mathfrak{M}_{X,H}(\mathbf{v})$$

for the moduli space of S-equivalence classes of H-semistable torsion-free sheaves on the projective scheme X, with Chern character $\mathbf{v}$. We write

$$\mathfrak{M}^s_{X,H}(\mathbf{v}) \subset \mathfrak{M}_{X,H}(\mathbf{v})$$

for the stable locus. This moduli space is a projective scheme. It is a closed subscheme of the moduli space of $\mathfrak{M}_{X,H}(P)$ of S-equivalence classes of H-semistable torsion-free sheaves on the projective scheme X, with Hilbert polynomial $\mathbf{P} = \mathbf{P}_{\mathcal{E}}$, where

$$\mathbf{P}_{\mathcal{E}}(t) = \chi(\mathcal{E}(tH)) = \int_X \mathrm{ch}(\mathcal{E})\mathrm{ch}(\mathcal{O}_X(tH))\mathrm{td}(X).$$

We refer to [HL10] for a comprehensive treatment.

However we mention here that $\mathfrak{M}_{X,H}(\mathbf{P})$ corepresents a moduli functor which we describe below. Let $\mathcal{M}' : (\mathrm{Sch}/\Bbbk)^\circ \to (\mathrm{Sets})$ the functor that assigns to a $\Bbbk$-scheme S the set of isomorphism classes of S-flat families of semistable sheaves on X with Hilbert polynomial $\mathbf{P}$.

Given a morphism $f : S' \to S$ of $\Bbbk$-schemes, we let $\mathcal{M}'(f)$ be the map obtained by taking pull-back of sheaves. If $F \in \mathcal{M}'(S)$ is an S-flat family of semistable sheaves, and if $L \in \mathrm{Pic}(S)$, then $F \otimes p^*(L)$ is also an S-flat family, and the fibres F_s and $(F \otimes p^*(L))_s$ are isomorphic for all $s \in S$.

This leads to consider the quotient functor $\mathcal{M} = \mathcal{M}'/\sim$, where $\sim$ is the equivalence relation that gives $F \sim F'$ for $F, F' \in \mathcal{M}'(S)$ if and only if $F = F' \otimes p^*(L)$ for some $L \in \mathrm{Pic}(S)$. If we take families of (geometrically) stable sheaves, we get open subfunctors $(\mathcal{M}')^s \subset \mathcal{M}'$ and $\mathcal{M}^s \subset \mathcal{M}$.

The moduli functor $\mathcal{M}$ is not representable in general, in the sense that there is no scheme M such that $\mathcal{M} = \mathrm{Hom}_{\mathrm{Sch}}(-, M)$. However, this functor is corepresentable. This means that there is a scheme M and a morphism of functors $\alpha : \mathcal{M} \to \mathrm{Hom}_{\mathrm{Sch}}(-, M)$ such that any other morphism of functors $\alpha' : \mathcal{M} \to \mathrm{Hom}_{\mathrm{Sch}}(-, M')$ factors through α, i.e., there is $\beta : M \to M'$ such that $\alpha' = \beta \circ \alpha$. The scheme M universally corepresents $\mathcal{M}$ if for any $T \to M$, the fibre product $\mathcal{M} \times_{\mathrm{Hom}_{\mathrm{Sch}}(-,M)} \mathrm{Hom}_{\mathrm{Sch}}(-, T)$ is corepresented by T.

If M represents $\mathcal{M}$, then for any $\Bbbk$-scheme S and an S-flat family $\mathcal{E}$ of H-semistable torsion-free sheaves on X with Hilbert polynomial $\mathbf{P}$, so that $\mathcal{E} \in \mathcal{M}(S)$, we get $\alpha(S)(\mathcal{E})$, which is a morphism $f : S \to M$, called the classifying map of $\mathcal{E}$. Also, for any scheme M' giving rise to a classifying map $f' : S \to M'$ we have $f' = \beta \circ f$ for some $\beta : M \to M'$.

Tangent space, obstruction, dimension estimates. We shall also use the standard deformation-theoretic dimension estimates for moduli spaces of stable sheaves. If $\mathcal{E}$ is an H-stable sheaf, with Chern character $\mathbf{v}$, then $\mathrm{Ext}^1(\mathcal{E},\mathcal{E})$ is the Zariski tangent space to the moduli space at $[\mathcal{E}]$, while obstructions lie in $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})$. Consequently, at least locally,

$$\mathrm{ext}^1(\mathcal{E},\mathcal{E}) - \mathrm{ext}^2(\mathcal{E},\mathcal{E}) \leq \dim_{[\mathcal{E}]} \mathfrak{M}_{X,H}(\mathbf{v}) \leq \mathrm{ext}^1(\mathcal{E},\mathcal{E}).$$

Note that, if $\mathcal{E}$ is stable, then $\mathcal{E}$ is simple, namely

$$\mathrm{Hom}(\mathcal{E},\mathcal{E}) = \mathbb{k}.$$

There is a theory of moduli spaces of simple sheaves, which can be developed using algebraic spaces, where the descriptions of tangent spaces and obstructions are similar to those for stable sheaves.

If X has dimension 3 and $\mathrm{Ext}^3(\mathcal{E},\mathcal{E}) = 0$, the expected dimension is

$$1 - \chi(\mathcal{E},\mathcal{E}).$$

For an instanton bundle of rank r and charge k on $\mathbb{P}^3$, since $c_1 = c_3 = 0$, Riemann–Roch gives

$$\chi(\mathcal{E},\mathcal{E}) = r^2 - 4rk,$$

hence the expected dimension of the unframed moduli space is

$$4rk - r^2 + 1.$$

In rank 2, this becomes the familiar number $8k - 3$. See [HL10, Chapter 4].

The following basic restriction result is the Grauert-Mülich theorem. Let $\mathcal{E}$ be a μ-semistable vector bundle of rank r on $\mathbb{P}^n$, over a field of characteristic zero. If $L \subset \mathbb{P}^n$ is a general line and

$$\mathcal{E}_{|L} \simeq \mathcal{O}_L(b_1) \oplus \cdots \oplus \mathcal{O}_L(b_r), \qquad b_1 \geq \cdots \geq b_r,$$

then

$$0 \leq b_i - b_{i+1} \leq 1 \qquad \text{for all } i.$$

Thus the restriction of a semistable bundle to a general line is as balanced as possible. In particular, for a rank 2 semistable bundle with $c_1 = 0$, the restriction to a general line is trivial. We shall call a line with this generic splitting type a non-jumping line. We refer to [HL10, §4.5, Corollary 4.5.2] and the references therein.

1.1. Derived categories of coherent sheaves

We work on a smooth projective variety X over a field $\Bbbk$. We let $\mathrm{D^b}(X)$ be the bounded derived category of complexes of coherent sheaves on X.

1.1.A. Complexes of coherent sheaves, quasi-isomorphisms, triangles

Let us introduce very briefly the derived category of coherent sheaves on a manifold X.

Quasi-isomorphisms. An object of $\mathrm{D^b}(X)$ is a cochain complex $\mathcal{E}^\bullet$ of quasi-coherent sheaves on X, whose cohomology has only finitely many non-zero terms, and moreover these terms are coherent sheaves on X.

A map of complexes $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ is a quasi-isomorphism if the induced map $\mathcal{H}(\mathcal{E}^\bullet) \to \mathcal{H}(\mathcal{F}^\bullet)$ is an isomorphism. A morphism $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ in $\mathrm{D^b}(X)$ is a morphism of complexes after possibly inverting a quasi-isomorphism. It is represented by a *roof* $\mathcal{E}^\bullet \leftarrow \mathcal{G}^\bullet \to \mathcal{F}^\bullet$, where $\mathcal{E}^\bullet \leftarrow \mathcal{G}^\bullet$ is a quasi-isomorphism. Two roofs represent the same morphism in $\mathrm{D^b}(X)$ if there is a common refinement.

Shifts and cones. Given a complex $\mathcal{E}^\bullet$, the shifted complex $\mathcal{E}^\bullet[1]$ has terms $\mathcal{E}^p[1] = \mathcal{E}^{p+1}$ for all $p \in \mathbb{Z}$ and is equipped with the differential $-d_{\mathcal{E}}^{p+1}$ in degree p. Given a morphism of complexes $f : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$, the cone $\mathrm{Con}(f)^\bullet$ is a complex defined by

$$\mathrm{Cone}(f)^n = \mathcal{F}^n \oplus \mathcal{E}^{n+1}$$

with differential

$$d^n_{\mathrm{Con}(f)} = \begin{pmatrix} d^n_{\mathcal{F}} & f^{n+1} \\ 0 & -d^{n+1}_{\mathcal{E}} \end{pmatrix}.$$

Given a morphism $f : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ in $\mathrm{D^b}(X)$, after choosing a representative of f by an actual morphism of complexes, we may define the cone $\mathrm{Con}(f)^\bullet$. This is unique up to a non-unique isomorphism. We have a sequence of morphisms in $\mathrm{D^b}(X)$:

$$\mathcal{E}^\bullet \xrightarrow{f} \mathcal{F}^\bullet \to \mathrm{Con}(f) \to \mathcal{E}^\bullet[1] \tag{4}$$

Triangulated structure. The category $\mathrm{D^b}(X)$ is triangulated. There is a class of triangles

$$(\mathcal{E}^\bullet)' \to \mathcal{E}^\bullet \to (\mathcal{E}^\bullet)'' \to (\mathcal{E}^\bullet)'[1]$$

which are quasi-isomorphic to (4). These triangles are said to be *distinguished*. Distinguished triangles have to satisfy a number of axioms and play a role similar to exact sequences in abelian categories. The first one is that a distinguished triangle gives rise to an exact cohomology sequence

$$\cdots \to \mathcal{H}^p((\mathcal{E}^\bullet)') \to \mathcal{H}^p(\mathcal{E}^\bullet) \to \mathcal{H}^p((\mathcal{E}^\bullet)'') \to \mathcal{H}^{p+1}((\mathcal{E}^\bullet)') \to \cdots$$

Basic functors. Let $f : X \to Y$ be a morphism of smooth projective varieties. Then we have functors

$$\mathrm{R}f_* : \mathrm{D^b}(X) \to \mathrm{D^b}(Y), \qquad \mathrm{L}f^* : \mathrm{D^b}(Y) \to \mathrm{D^b}(X).$$

These functors send distinguished triangles to distinguished triangles, i.e. they are *exact*. They are defined using injective resolutions. Namely, given a complex $\mathcal{E}^\bullet$ of $\mathrm{D^b}(X)$, we replace $\mathcal{E}^\bullet$ by a complex $\mathcal{I}^\bullet$ of quasi-coherent injective sheaves on X which is quasi-isomorphic to $\mathcal{E}^\bullet$. This is possible in the category of quasi-coherent sheaves, which has enough injectives. Then $f_*(\mathcal{I}^\bullet)$ is a complex of quasi-coherent sheaves on Y. Since X and Y are projective, the map f is proper. Therefore, a basic result about direct images of coherent sheaves shows that $f_*(\mathcal{I}^\bullet)$ lies in $\mathrm{D^b}(Y)$. This is what we call $\mathrm{L}f_*(\mathcal{E}^\bullet)$. A similar construction applies for $\mathrm{L}f^*(\mathcal{F}^\bullet)$ for any $\mathcal{F}^\bullet$ in $\mathrm{D^b}(Y)$.

1.1.B. Semiorthogonal decompositions

We will suppress the "bullet" from the notation $\mathcal{E}^\bullet$ and just denote by $\mathcal{E}$ an object of $\mathrm{D^b}(X)$.

Exceptional object and collections. An object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ is called exceptional if

$$\mathrm{Ext}^{\bullet}(\mathcal{E}, \mathcal{E}) = \mathbb{k},$$

that is, $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{E}, \mathcal{E}) = \mathbb{k}$ and $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{E}, \mathcal{E}[k]) = 0$. A sequence of exceptional objects

$$\mathcal{E}_1, \ldots, \mathcal{E}_m \in \mathrm{D}^{\mathrm{b}}(X)$$

is called an exceptional collection if

$$\mathrm{Ext}^{\bullet}(\mathcal{E}_i, \mathcal{E}_j) = 0 \qquad \text{for all } i > j.$$

Equivalently, there are no morphisms, in any degree, from an object appearing later in the collection to an object appearing earlier. The exceptional collection is called full if the smallest triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(X)$ containing $\mathcal{E}_1, \ldots, \mathcal{E}_m$ is the whole derived category:

$$\langle \mathcal{E}_1, \ldots, \mathcal{E}_m \rangle = \mathrm{D}^{\mathrm{b}}(X).$$

Semiorthogonal decomposition. More generally, a semiorthogonal decomposition of $\mathrm{D}^{\mathrm{b}}(X)$ is a sequence of full triangulated admissible subcategories

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$$

such that

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(A_i, A_j) = 0 \qquad \text{for all } A_i \in \mathcal{A}_i,\ A_j \in \mathcal{A}_j,\ i > j,$$

and such that the subcategories $\mathcal{A}_1, \ldots, \mathcal{A}_m$ generate $\mathrm{D}^{\mathrm{b}}(X)$ as a triangulated category. The word *admissible* here means that the embedding of the subcategory should admit a left adjoint and right adjoint.

1.1.C. Beilinson resolutions

A fundamental example is Beilinson's exceptional collection on projective space. On $\mathbb{P}^n$, the collection of line bundles

$$B = \big(\mathcal{O}_{\mathbb{P}^n}(-n), \mathcal{O}_{\mathbb{P}^n}(-n+1), \ldots, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n}\big) \tag{5}$$

forms a full exceptional collection in $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^n)$. Equivalently, one has a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}(-n), \mathcal{O}_{\mathbb{P}^n}(-n+1), \ldots, \mathcal{O}_{\mathbb{P}^n} \rangle\,.$$

This means that every object of $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^n)$ can be reconstructed, in a derived sense, from these line bundles. Beilinson's theorem gives a more precise and effective form of this statement: any coherent sheaf, or more generally any bounded complex of coherent sheaves, admits a resolution or spectral sequence whose terms are built from this collection and its dual collection.

Cohomology tables. Let us consider Beilinson's standard collection (5). Its dual collection is

$$B^{\vee} = \big(\Omega^n_{\mathbb{P}^n}(n), \Omega^{n-1}_{\mathbb{P}^n}(n-1), \ldots, \Omega^1_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}\big).$$

This duality is encoded in Beilinson's resolution of the diagonal. Let

$$p_1, p_2 \colon \mathbb{P}^n \times \mathbb{P}^n \longrightarrow \mathbb{P}^n$$

be the two projections, and denote by $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ the diagonal. Then $\mathcal{O}_\Delta$ admits the resolution

$$0 \longrightarrow \Omega^n_{\mathbb{P}^n}(n) \boxtimes \mathcal{O}_{\mathbb{P}^n}(-n) \longrightarrow \cdots \longrightarrow \Omega^1_{\mathbb{P}^n}(1) \boxtimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \boxtimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0.$$

Here, use the standard notation

$$a \boxtimes b = p_1^*(a) \otimes p_2^*(b).$$

Equivalently, $\mathcal{O}_\Delta$ is represented by the complex

$$\big(\Omega^n_{\mathbb{P}^n}(n) \boxtimes \mathcal{O}_{\mathbb{P}^n}(-n) \to \cdots \to \Omega^1_{\mathbb{P}^n}(1) \boxtimes \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \boxtimes \mathcal{O}_{\mathbb{P}^n}\big)\,,$$

where the last term is placed in degree 0.

Now let $\mathcal{E}$ be an object of $\mathrm{D^b}(\mathbb{P}^n)$. Since $\mathcal{E} \simeq \mathbf{R}p_{2*}(p_1^*\mathcal{E} \otimes \mathcal{O}_\Delta)$, the resolution of the diagonal gives a spectral sequence whose first page is

$$E_1^{-q,p} = \mathrm{H}^p\big(\mathbb{P}^n, \mathcal{E} \otimes \Omega^q_{\mathbb{P}^n}(q)\big) \otimes \mathcal{O}_{\mathbb{P}^n}(-q), \qquad 0 \le q \le n.$$

It converges to $\mathcal{E}$, viewed as an object of $\mathrm{D^b}(\mathbb{P}^n)$. Equivalently, one arranges these cohomology groups in the Beilinson table, with the columns ordered from $q = n$ to $q = 0$:

p	$q = n$	$q = n-1$	$\cdots$	$q = 1$	$q = 0$
n	$\mathrm{H}^n(\mathcal{E} \otimes \Omega^n_{\mathbb{P}^n}(n))$	$\mathrm{H}^n(\mathcal{E} \otimes \Omega^{n-1}_{\mathbb{P}^n}(n-1))$	$\cdots$	$\mathrm{H}^n(\mathcal{E} \otimes \Omega^1_{\mathbb{P}^n}(1))$	$\mathrm{H}^n(\mathcal{E})$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
1	$\mathrm{H}^1(\mathcal{E} \otimes \Omega^n_{\mathbb{P}^n}(n))$	$\mathrm{H}^1(\mathcal{E} \otimes \Omega^{n-1}_{\mathbb{P}^n}(n-1))$	$\cdots$	$\mathrm{H}^1(\mathcal{E} \otimes \Omega^1_{\mathbb{P}^n}(1))$	$\mathrm{H}^1(\mathcal{E})$
0	$\mathrm{H}^0(\mathcal{E} \otimes \Omega^n_{\mathbb{P}^n}(n))$	$\mathrm{H}^0(\mathcal{E} \otimes \Omega^{n-1}_{\mathbb{P}^n}(n-1))$	$\cdots$	$\mathrm{H}^0(\mathcal{E} \otimes \Omega^1_{\mathbb{P}^n}(1))$	$\mathrm{H}^0(\mathcal{E})$

(6)

The entry in column q and row p contributes the summand

$$\mathrm{H}^p\big(\mathbb{P}^n, \mathcal{E} \otimes \Omega^q_{\mathbb{P}^n}(q)\big) \otimes \mathcal{O}_{\mathbb{P}^n}(-q).$$

Thus the column labelled q contributes multiples of $\mathcal{O}_{\mathbb{P}^n}(-q)$.

The terms of the Beilinson complex are obtained by summing along the diagonals with fixed value of $p - q$. More precisely, the term in cohomological degree k is

$$\bigoplus_{q=0}^{n} \mathrm{H}^{q+k}\big(\mathbb{P}^n, \mathcal{E} \otimes \Omega^q_{\mathbb{P}^n}(q)\big) \otimes \mathcal{O}_{\mathbb{P}^n}(-q),$$

with the convention that the summand is zero if $q + k \notin \{0, \dots, n\}$.

Dual cohomology tables. There is a dual version of the same construction, obtained by using the other Beilinson resolution of the diagonal:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-n) \boxtimes \Omega^n_{\mathbb{P}^n}(n) \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \Omega^1_{\mathbb{P}^n}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \boxtimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0.$$

Applying again $\mathbf{R}p_{2*}(p_1^*\mathcal{E} \otimes -)$, one obtains a spectral sequence whose first page is

$$E'^{-q,p}_1 = \mathrm{H}^p\big(\mathbb{P}^n, \mathcal{E}(-q)\big) \otimes \Omega^q_{\mathbb{P}^n}(q), \qquad 0 \le q \le n.$$

It also converges to $\mathcal{E}$. Thus one may alternatively arrange the cohomology groups in the dual Beilinson table

p	$q = n$	$q = n-1$	$\cdots$	$q = 1$	$q = 0$
n	$\mathrm{H}^n(\mathcal{E}(-n))$	$\mathrm{H}^n(\mathcal{E}(-n+1))$	$\cdots$	$\mathrm{H}^n(\mathcal{E}(-1))$	$\mathrm{H}^n(\mathcal{E})$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
1	$\mathrm{H}^1(\mathcal{E}(-n))$	$\mathrm{H}^1(\mathcal{E}(-n+1))$	$\cdots$	$\mathrm{H}^1(\mathcal{E}(-1))$	$\mathrm{H}^1(\mathcal{E})$
0	$\mathrm{H}^0(\mathcal{E}(-n))$	$\mathrm{H}^0(\mathcal{E}(-n+1))$	$\cdots$	$\mathrm{H}^0(\mathcal{E}(-1))$	$\mathrm{H}^0(\mathcal{E})$

where the entry in column q and row p contributes the summand

$$\mathrm{H}^p\big(\mathbb{P}^n, \mathcal{E}(-q)\big) \otimes \Omega^q_{\mathbb{P}^n}(q).$$

Thus the column labelled q contributes multiples of $\Omega^q_{\mathbb{P}^n}(q)$. The terms of the associated dual Beilinson complex are again obtained by summing along the diagonals with fixed value of $p - q$: the term in cohomological degree k is

$$\bigoplus_{q=0}^{n} \mathrm{H}^{q+k}\big(\mathbb{P}^n, \mathcal{E}(-q)\big) \otimes \Omega^q_{\mathbb{P}^n}(q),$$

with the same convention that the summand is zero if $q + k \notin \{0, \dots, n\}$. This form is often useful when the twists $\mathcal{E}(-q)$ have simple cohomology, while the previous form is useful when the groups $\mathrm{H}^p(\mathcal{E} \otimes \Omega^q_{\mathbb{P}^n}(q))$ are easier to control.

1.1.D. Horrocks splitting

Consider the projective space $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{k}}$ and a rank-r torsion-free sheaf $\mathcal{E}$ on $\mathbb{P}^n$, with $r \geq 1$. The splitting theorem of Horrocks says:

Theorem 1.7. *We have* $\mathrm{H}^p(\mathcal{E}(t)) = 0$ *for all* $t \in \mathbb{Z}$ *and* $1 \leq p \leq n-1$ *if and only if there are* $a_1, \ldots, a_r \in \mathbb{Z}$ *such that*

$$\mathcal{E} \simeq \bigoplus_{1 \leq i \leq r} \mathcal{O}_{\mathbb{P}^n}(a_i).$$

Proof. The proof we give here appears in [AO91]. One direction is clear in view of the usual computation of cohomology of line bundles on $\mathbb{P}^n$. We can prove the converse by using Beilinson resolutions. By induction on the rank, it suffices to see that a torsion-free sheaf $\mathcal{E}$ satisfying our cohomological vanishing condition admits a line bundle $\mathcal{O}_{\mathbb{P}^n}(t)$ as a direct summand.

To find this summand, given a torsion-free sheaf $\mathcal{E}$, we note that there is a unique integer $t \in \mathbb{Z}$ such that $\mathrm{H}^0(\mathcal{E}(t)) \neq 0$, while $\mathrm{H}^0(\mathcal{E}(t-1)) = 0$. This is because, on the one hand, $\mathcal{E}(t_1)$ is non-zero and generated by global sections for some $t_1 \gg 0$, since $\mathcal{O}_{\mathbb{P}^n}(1)$ is very ample. On the other hand, the torsion-free sheaf $\mathcal{E}$ is embedded into some $\mathcal{O}_{\mathbb{P}^n}(t_0)^{\oplus N}$ for some $t_0 \gg 0$ and some $N > 0$, because $\mathcal{E} \hookrightarrow \mathcal{E}^{\vee\vee}$ and $\mathcal{E}^{\vee}(t_0)$ is globally generated for some $t_0 \gg 0$, so $\mathrm{H}^0(\mathcal{E}(-t_0-1)) = 0$.

Write $\mathcal{E}_0 = \mathcal{E}(t)$. Looking at the cohomology table in Beilinson's theorem for $\mathcal{E}_0$, we find

p	$q = n$	$q = n-1$	$\cdots$	$q = 1$	$q = 0$
n	$\mathrm{H}^n(\mathcal{E}_0(-n))$	$\mathrm{H}^n(\mathcal{E}_0(1-n))$	$\cdots$	$\mathrm{H}^n(\mathcal{E}_0(-1))$	$\mathrm{H}^n(\mathcal{E}_0)$
$\vdots$	0	0			0
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
1	0	0	$\cdots$	0	0
0	0	0	$\cdots$	0	$\mathrm{H}^0(\mathcal{E}_0)$

Hence the Beilinson complex looks like

$$\mathrm{H}^0(\mathcal{E}_0) \otimes \mathcal{O}_{\mathbb{P}^n} \oplus \mathrm{H}^n(\mathcal{E}_0(-n)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathrm{H}^n(\mathcal{E}_0(1-n)) \otimes \Omega^{n-1}_{\mathbb{P}^n}(n-1) \to \cdots$$

The first term above sits in homological degree 0 and the cohomology of this complex is $\mathcal{E}_0$. But the differential of this complex, restricted to $\mathrm{H}^0(\mathcal{E}_0) \otimes \mathcal{O}_{\mathbb{P}^n}$, is zero, because there are no non-trivial morphisms from $\mathcal{O}_{\mathbb{P}^n}$ to $\Omega^{n-1}_{\mathbb{P}^n}(n-1)$. Hence $\mathrm{H}^0(\mathcal{E}_0) \otimes \mathcal{O}_{\mathbb{P}^n}$ is a direct summand of the cohomology of this complex, i.e., a direct summand of $\mathcal{E}_0$. This concludes the proof. □

The following consequence is immediate.

Corollary 1.8. *Let* $\mathcal{E}$ *be a vector bundle on* $\mathbb{P}^1$. *Then there is* $a_1, \ldots, a_r \in \mathbb{Z}$ *such that*

$$\mathcal{E} \simeq \bigoplus_{1 \leq i \leq r} \mathcal{O}_{\mathbb{P}^1}(a_i).$$

1.2. Framed instantons on $\mathbb{P}^2$

Let us recall that the object we want to study is $\mathfrak{M}\mathfrak{F}(r,k)$. This is the moduli space of *torsion-free sheaves of rank* r *and charge* k*, framed along the line* ℓ_∞. It consists of torsion-free sheaves $\mathcal{E}$ on $\mathbb{P}^2$ such that

$$\mathrm{rk}(\mathcal{E}) = r, \qquad c_2(\mathcal{E}) = k,$$

together with a framing

$$\Phi \colon \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} W \otimes \mathcal{O}_{\ell_\infty}.$$

So the integers $a_1, \dots, a_r$ of Grothendieck's splitting are all zero, for the line ℓ_∞.

1.2.A. Beilinson monad

The usual approach to study the moduli space of semistable sheaves on $\mathbb{P}^2$ goes through the use of monads, which afford a translation of the abstract moduli problem into a concrete problem in linear algebra. The idea of a monad was introduced by Horrocks in [Hor64] in a slightly different and more general situation, and developed by Barth, Hulek, Le Potier, Drézet; see [Bar77, BH78, Hul79, LP79, DLP85] for the study of vector bundles on $\mathbb{P}^2$. The approach with derived categories is due to Beilinson, [Bei78].

Here we follow closely [Nak99, Chapter 2], where the moduli problem of framed instantons is first reduced to a monad-theoretic description and, in a second step, further reduced to the ADHM data.

Consider a torsion-free sheaf $\mathcal{E}$ on $\mathbb{P}^2$ such that $\mathrm{rk}(\mathcal{E}) = r$, $c_2(\mathcal{E}) = k$. Consider a fixed vector space W and a framing $\Phi \colon \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} W \otimes \mathcal{O}_{\ell_\infty}$. We start with the following result.

Lemma 1.9. *The sheaf* $\mathcal{E}$ *is the cohomology of a complex*

$$I_{-1} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{a} I_0 \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{b} I_1 \otimes \mathcal{O}_{\mathbb{P}^2}(1), \tag{7}$$

with

$$\dim(I_{-1}) = \dim(I_1) = k, \qquad \dim(I_0) = r + 2k.$$

Proof. We apply Beilinson's theorem to $\mathcal{E}(-1)$ and take

$$I_{-1} = \mathrm{H}^1(\mathcal{E}(-2)), \qquad I_0 = \mathrm{H}^1(\mathcal{E} \otimes \Omega_{\mathbb{P}^2}), \qquad I_1 = \mathrm{H}^1(\mathcal{E}(-1)).$$

Then we have to prove that the cohomology table (6) takes the form

p	$q=2$	$q=1$	$q=0$
2	$\mathrm{H}^2(\mathcal{E}(-2)) = 0$	$\mathrm{H}^2(\mathcal{E} \otimes \Omega_{\mathbb{P}^2}) = 0$	$\mathrm{H}^2(\mathcal{E}(-1)) = 0$
1	$\mathrm{h}^1(\mathcal{E}(-2)) = k$	$\mathrm{h}^1(\mathcal{E} \otimes \Omega_{\mathbb{P}^2}) = r + 2k$	$\mathrm{h}^1(\mathcal{E}(-1)) = k$
0	$\mathrm{H}^0(\mathcal{E}(-2)) = 0$	$\mathrm{H}^0(\mathcal{E} \otimes \Omega_{\mathbb{P}^2}) = 0$	$\mathrm{H}^0(\mathcal{E}(-1)) = 0$

(8)

We look at the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\ell_\infty} \to 0.$$

Tensoring it with $\mathcal{E}(-1)$ we get

$$0 \to \mathcal{E}(-2) \to \mathcal{E}(-1) \to \mathcal{O}_{\ell_\infty}(-1)^{\oplus r} \to 0.$$

Therefore, $\mathrm{h}^0(\mathcal{E}(-2)) = \mathrm{h}^0(\mathcal{E}(-1))$ and by induction $\mathrm{h}^0(\mathcal{E}(-1)) = \mathrm{h}^0(\mathcal{E}(-t))$ for all $t \geq 1$, which gives $\mathrm{h}^0(\mathcal{E}(-1)) = 0$. Tensoring the Euler sequence (3) with $\mathcal{E}(-1)$ we deduce $\mathrm{H}^0(\mathcal{E} \otimes \Omega_{\mathbb{P}^2}) = 0$. This gives the bottom row of (8). A similar reasoning applies to show the top row of (8). The middle row is obtained similarly, by tensoring the previous displayed sequence with $\mathcal{O}_{\mathbb{P}^2}(-1)$ or $\mathcal{O}_{\mathbb{P}^2}(1)$ as needed, and using the Euler exact sequence again. □

Remark 1.10. A few remarks about this lemma are in order.

1. The morphism a is injective and b is surjective. This follows from the lemma, as the only cohomology of the complex (7) takes place in degree zero.
2. Starting from a sheaf $\mathcal{E}$ trivial along ℓ_∞, we get the complex (7). The matrices in this complex are determined up to the action of $G = \mathrm{GL}(I_{-1}) \times \mathrm{GL}(I_0) \times \mathrm{GL}(I_1)$, acting on the matrices by left and right multiplication, namely, $g \in G$ is of the form (g_{-1}, g_0, g_1). It sends (a, b) to $(g_0 a g_{-1}^{-1}, g_1 b g_0^{-1})$.

3. The subgroup $\Bbbk^*$ of G consisting of matrices of automorphisms of the form $\lambda(\mathrm{id}_{I_{-1}}, \mathrm{id}_{I_0}, \mathrm{id}_{I_1})$ acts trivially on any pair (a, b). So we consider the action of $G_0 = G/\Bbbk^*$ instead. This group acts on pairs (a, b) and on the quasi-affine locus consisting of pairs (a, b) such that $ba = 0$, a is injective and b is surjective.

   It turns out that G_0 acts with finite stabilizers on the stable locus, consisting of pairs (a, b) such that the cohomology of the associated complex is a stable sheaf. Therefore, on the stable locus, we get a geometric quotient for the action of G_0. This quotient is a coarse moduli space for the moduli space of stable sheaves $\mathcal{E}$ on $\mathbb{P}^2$ such that $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = k$ and $\mathcal{E}$ restricts to a general line ℓ as the trivial bundle of rank r.

1.2.B. ADHM data

As a next step, we need to find the relevant ADHM data starting from the complex (7).

Step 1 (Fix some notation)**.** Fix homogeneous coordinates (z_0, z_1, z_2) on $\mathbb{P}^2$, where the line ℓ_∞ is $\mathbb{V}(z_0)$. We start from the matrices a and b which we write

$$a = z_0 a_0 + z_1 a_1 + z_2 a_2, \qquad b = z_0 b_0 + z_1 b_1 + z_2 b_2.$$

Here, for $0 \le i \le 2$, we have linear maps $a_i : I_{-1} \to I_0$ and $b_i : I_0 \to I_1$. These maps satisfy

$$\begin{aligned} b_0 a_0 &= 0 & b_1 a_1 &= 0, & b_2 a_2 &= 0 \\ b_0 a_1 + b_1 a_0 &= 0, & b_1 a_2 + b_2 a_1 &= 0, & b_0 a_2 + b_2 a_0 &= 0, \\ a|_{\ell_\infty} &= z_1 a_1 + z_2 a_2, & b|_{\ell_\infty} &= z_1 b_1 + z_2 b_2. \end{aligned}$$

We put $\mathcal{K} = \ker(b|_{\ell_\infty})$. By definition, we have an exact sequence

$$0 \to \mathcal{K} \to I_0 \otimes \mathcal{O}_{\ell_\infty} \xrightarrow{z_1 b_1 + z_2 b_2} I_1 \otimes \mathcal{O}_{\ell_\infty}(1) \to 0. \tag{9}$$

Here, $\mathcal{K}$ is a vector bundle of rank $r + k$ on ℓ_∞. The framing identifies $\mathrm{H}^0(\mathcal{E}|_{\ell_\infty})$ with the fixed vector space W. Via the exact sequence below, we view this W as a subspace of I_0:

$$W \simeq \mathrm{H}^0(\mathcal{E}|_{\ell_\infty}) \simeq \mathrm{H}^0(\mathcal{K}) \subset I_0.$$

This is justified by the exact sequence

$$0 \to I_{-1} \otimes \mathcal{O}_{\ell_\infty}(-1) \to \mathcal{K} \to \mathcal{E}|_{\ell_\infty} \to 0. \tag{10}$$

By (9), $W \subset I_0$, more precisely, $W = \ker(b_1) \cap \ker(b_2)$. We also set $I = I_{-1}$.

Step 2 (Identify $I_{-1} \simeq I_1$ and $I_0 \simeq I \oplus I \oplus W$)**.** Let $p = (0 : 1 : 0) \in \ell_\infty$. Note that $\mathrm{H}^0(\mathcal{E}|_{\ell_\infty}) \to \mathrm{H}^0(\mathcal{E}|_p)$ is an isomorphism. Since $p = \mathbb{V}(z_0, z_2)$, we specialize a and b by setting $z_0 = z_2 = 0$ to look at $\mathrm{H}^0(\mathcal{E}|_p)$ and we get $\mathrm{H}^0(\mathcal{E}|_p) = \mathcal{E}|_p = \mathrm{Ker}(b_1)/\mathrm{Im}(a_1)$. On the other hand, restricting (10) to p, we get an isomorphism. Therefore,

$$\mathrm{Im}(a_1) \cap \mathrm{Ker}(b_2) = \mathrm{Im}(a_1) \cap \mathrm{Ker}(b_2) \cap \mathrm{Ker}(b_1) = \mathrm{Im}(a_1) \cap W = 0.$$

Therefore, $b_2 a_1$ is injective, and thus bijective. This allows us to identify I_{-1} with I_1. Another way to recover this is to note that restriction to ℓ_∞ induces an exact sequence:

$$0 \to \mathcal{E}(-2) \to \mathcal{E}(-1) \to \mathcal{O}_{\ell_\infty}(-1)^{\oplus r} \to 0.$$

So multiplication by z_0 induces an isomorphism. With the above normalization, this isomorphism corresponds, up to the harmless sign convention in the Euler sequence, to $b_2 a_1$:

$$I_{-1} = \mathrm{H}^1(\mathcal{E}(-2)) \to \mathrm{H}^1(\mathcal{E}(-1)) = I_1.$$

On the other hand, multiplication by z_1 and z_2 gives linear maps $I_{-1} \to I_1$. Composing with the inverse of the above isomorphism, however, gives endomorphisms B_1 and B_2 of $I = I_{-1}$.

The next observation is that, since $\mathcal{K} \simeq W \otimes \mathcal{O}_{\ell_\infty} \oplus I_{-1} \otimes \mathcal{O}_{\ell_\infty}(-1)$, we have an exact sequence

$$0 \to I_{-1} \otimes \mathcal{O}_{\ell_\infty}(-1) \to (I_0/W) \otimes \mathcal{O}_{\ell_\infty} \to I_1 \otimes \mathcal{O}_{\ell_\infty}(1) \to 0. \tag{11}$$

Recall that $\mathrm{Ext}^1_{\ell_\infty}(\mathcal{O}_{\ell_\infty}(1), \mathcal{O}_{\ell_\infty}(-1)) = \Bbbk\langle\xi\rangle$, where ξ is the Euler sequence

$$0 \to \mathcal{O}_{\ell_\infty}(-1) \xrightarrow{\left(\begin{smallmatrix} -z_1 \\ -z_2 \end{smallmatrix}\right)} \mathcal{O}_{\ell_\infty}^{\oplus 2} \xrightarrow{\left(\begin{smallmatrix} z_2 & -z_1 \end{smallmatrix}\right)} \mathcal{O}_{\ell_\infty}(1) \to 0.$$

Since the middle term of (11) contains no direct summand of the form $\mathcal{O}_{\ell_\infty}(-1)$ or $\mathcal{O}_{\ell_\infty}(1)$, the extension (11) corresponds to an element of

$$\mathrm{Ext}^1_{\ell_\infty}(I_1 \otimes \mathcal{O}_{\ell_\infty}(1), I_{-1} \otimes \mathcal{O}_{\ell_\infty}(-1)) = \mathrm{Hom}_{\Bbbk}(I_1, I_{-1}) \otimes \Bbbk\langle\xi\rangle,$$

which must induce an isomorphism $I_1 \to I_{-1}$, which is just $(b_2 a_1)^{-1}$. Identifying I_1 and $I = I_{-1}$ with this isomorphism and choosing a splitting of the resulting vector-space exact sequence gives $I_0 \simeq I \oplus I \oplus W$, and the restriction to ℓ_∞ of (7) reads

$$I \otimes \mathcal{O}_{\ell_\infty}(-1) \xrightarrow{\begin{pmatrix} -z_1 \mathrm{id}_I \\ -z_2 \mathrm{id}_I \\ 0 \end{pmatrix}} (I \oplus I \oplus W) \otimes \mathcal{O}_{\ell_\infty} \xrightarrow{\begin{pmatrix} z_2 \mathrm{id}_I & -z_1 \mathrm{id}_I & 0 \end{pmatrix}} I \otimes \mathcal{O}_{\ell_\infty}(1),$$

Step 3 (Extract the ADHM data and the closed conditions)**.** Now, to obtain the ADHM data, we look at the extra data needed to obtain (7) from the previous complex on $\mathbb{P}^2$. This amounts precisely to the matrices a_0 and b_0, which we write under the decomposition $I_0 \simeq I \oplus I \oplus W$. So we write

$$a_0 = \begin{pmatrix} B_1 \\ B_2 \\ j \end{pmatrix}, \qquad \text{for some } B_1, B_2 \in \mathrm{End}_{\Bbbk}(I) \text{ and } j \in \mathrm{Hom}_{\Bbbk}(I, W).$$

The equations $ba = 0$ are already satisfied except for those involving a_0 and b_0. Looking at the mixed coefficients $z_0 z_1$ and $z_0 z_2$ in $ba = 0$, namely at the conditions $b_0 a_1 + b_1 a_0 = 0$ and $b_0 a_2 + b_2 a_0 = 0$, one obtains that b_0 takes the form

$$b_0 = \begin{pmatrix} -B_2 & B_1 & i \end{pmatrix}, \qquad \text{for some } i \in \mathrm{Hom}_{\Bbbk}(W, I).$$

The equation $b_0 a_0 = 0$ then gives:

$$B_1 B_2 - B_2 B_1 + ij = 0.$$

Step 4 (Check the open conditions on ADHM data)**.** To check the stability condition, we prove the statement

$$\Big(\exists S \subsetneq I \mid B_1(S) \subset S, B_2(S) \subset S, \mathrm{Im}(i) \subset S\Big) \iff \Big(b \text{ is not surjective at some point of } \mathbb{P}^2\Big).$$

Note that the form of b already implies that b is surjective along ℓ_∞, so the open ADHM condition is related to the surjectivity of b along $\mathbb{A}^2 = \mathbb{P}^2 \setminus \ell_\infty$. Set $x_1 = z_1/z_0$ and $x_2 = z_2/z_0$ and write the evaluation of b at $x = (x_1, x_2) \in \mathbb{A}^2$ as:

$$b|_x : (v_1, v_2, w) \mapsto (x_2 - B_2)v_2 + (B_1 - x_1)v_1 + i(w).$$

Now, $b|_x$ is not surjective if and only if there is a non-zero linear form $\varphi : I \to \Bbbk$ such that

$$\varphi \circ i = 0, \qquad \varphi \circ (x_2 - B_2) = 0, \qquad \varphi \circ (B_1 - x_1) = 0, \tag{12}$$

which gives $\varphi B_2 = x_2 \varphi$ and $\varphi B_1 = x_1 \varphi$.

Assume that $b|_x$ is not surjective. Then we set $S = \mathrm{Ker}(\varphi)$ and find

$$S \neq I, \qquad \mathrm{Im}(i) \subset S, \qquad B_1(S) \subset S, \qquad B_2(S) \subset S.$$

Thus failure of surjectivity produces a forbidden subspace. This proves the implication from the right-hand condition to the left-hand condition.

Conversely, assume that S exists. Then write $\overline{I} = I/S$. The endomorphisms B_1 and B_2 of I descend to endomorphisms $\overline{B}_1$ and $\overline{B}_2$ of $\overline{I}$ because $B_1(S) \subset S$ and $B_2(S) \subset S$. Also, $\overline{B}_1$ and $\overline{B}_2$ commute, because of the equation $[B_1, B_2] + ij = 0$ and the condition $\mathrm{Im}(i) \subset S$. Since $\Bbbk$ is algebraically closed, the commuting dual endomorphisms have a common eigenvector. Equivalently, there is a common eigenform $\overline{\varphi}$, namely a non-zero linear form $\overline{\varphi} \in \overline{I}^\vee$ satisfying $\overline{\varphi}\overline{B}_1 = x_1 \overline{\varphi}$ and $\overline{\varphi}\overline{B}_2 = x_2 \overline{\varphi}$. Letting φ be the composition of $\overline{\varphi}$ with the quotient map $I \to \overline{I}$, we get $\varphi B_1 = x_1 \varphi$ and $\varphi B_2 = x_2 \varphi$, as well as $\varphi \circ i = 0$. This shows (12) and thus implies that $b|_x$ is not surjective.

What we have done so far shows that there is a correspondence between the quasi-affine set of ADHM data, consisting of quadruples (B_1, B_2, i, j) satisfying $[B_1, B_2] + ij = 0$ and having no proper subspace S as above, and the moduli space of framed instantons, which we think of as pairs $(\mathcal{E}, \Phi)$ where Φ is an isomorphism from $\mathcal{E}|_{\ell_\infty}$ to $W \otimes \mathcal{O}_{\ell_\infty}$.

Let us spell this out once again and clarify what remains to be done. Starting from $(\mathcal{E}, \Phi)$, the framing induces an isomorphism $\mathrm{H}^0(\Phi) : \mathrm{H}^0(\mathcal{E}|_{\ell_\infty}) \to W$. We then choose an isomorphism $\Psi : \mathrm{H}^1(\mathcal{E}(-2)) \to I$. Then all the remaining identifications are canonical and we find an ADHM quadruple (B_1, B_2, i, j). However, Ψ is not part of the datum $(\mathcal{E}, \Phi)$, which is the reason why our construction only determines (B_1, B_2, i, j) up to the action of $\mathrm{GL}(I)$, see the next step.

Conversely, given (B_1, B_2, i, j), we define the maps a, b as above and obtain an instanton $\mathcal{E}$ as $\mathrm{Ker}(b)/\mathrm{Im}(a)$, which is trivial along ℓ_∞. On ℓ_∞, the projection onto the W-summand identifies the cohomology of the restricted complex with $W \otimes \mathcal{O}_{\ell_\infty}$; this gives the framing $\Phi : \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} W \otimes \mathcal{O}_{\ell_\infty}$.

Step 5 (Study the quotient)**.** We consider the reductive group $\mathrm{GL}_{\Bbbk}(I)$ and let it act on these data. More precisely, once I and W are fixed vector spaces of dimensions k and r, the ADHM space is

$$\mathbf{M} = \mathrm{End}(I) \oplus \mathrm{End}(I) \oplus \mathrm{Hom}(W, I) \oplus \mathrm{Hom}(I, W),$$

with points denoted by (B_1, B_2, i, j). The complex moment map is

$$\mu(B_1, B_2, i, j) = [B_1, B_2] + ij \in \mathrm{End}(I).$$

We consider the closed subscheme

$$\mu^{-1}(0) \subset \mathbf{M}.$$

The "gauge" group $\mathrm{GL}(I)$, acts by

$$g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1}).$$

The stability condition is

$$\Big(\text{there is no proper subspace } S \subsetneq I \text{ such that } B_1(S) \subset S, B_2(S) \subset S, \mathrm{Im}(i) \subset S\Big).$$

We denote the stable locus by

$$\mu^{-1}(0)^{\mathrm{st}}.$$

This stability condition is the GIT stability condition for the corresponding character of $\mathrm{GL}(I)$, in the sense of King. If $x = (B_1, B_2, i, j) \in \mu^{-1}(0)^{\mathrm{st}}$ and $g \in \mathrm{GL}(I)$ fixes (B_1, B_2, i, j), then

$$gB_1 = B_1g, \quad gB_2 = B_2g, \qquad gi = i.$$

Hence $S = \ker(g - \mathrm{id}_I)$ is stable under B_1, B_2 and contains $\mathrm{Im}(i)$. By stability we must have $S = I$, hence $g = \mathrm{id}_I$. Thus the stabilizers are trivial. Moreover, since $\mu^{-1}(0)^{\mathrm{st}}$ is the GIT-stable locus, the $\mathrm{GL}(I)$-orbits in $\mu^{-1}(0)^{\mathrm{st}}$ are closed in the semistable locus. Therefore the GIT quotient

$$\mathfrak{MF}(r, k) = \mu^{-1}(0)^{\mathrm{st}}/\mathrm{GL}(I)$$

is a geometric quotient. This means in particular that the fibres of the quotient morphism

$$\mu^{-1}(0)^{\mathrm{st}} \longrightarrow \mathfrak{MF}(r, k)$$

are exactly the $\mathrm{GL}(I)$-orbits. Consequently, closed points of $\mathfrak{MF}(r, k)$ are precisely isomorphism classes of stable ADHM data. In other words, the quotient is isomorphic to $\mathfrak{MF}(r, k)$.

2. Twistor spaces: the projective space and the flag threefold

In this section we will review instanton bundles on some twistor spaces, starting from the original formulation arising from Penrose's transform. Unfortunately, we will not mention anything about the Atiyah–Ward correspondence between connections and holomorphic structures. We refer to [Don24].

2.1. Instantons on projective space and the twistor fibration

2.1.A. The twistor fibration

The twistor fibration $\pi : \mathbb{P}^3_{\mathbb{C}} \to S^4$ is a real analytic submersion, whose fibres are 2-dimensional spheres. These spheres correspond to complex lines contained in $\mathbb{P}^3_{\mathbb{C}}$, i.e., copies of $\mathbb{P}^1_{\mathbb{C}}$ embedded linearly in $\mathbb{P}^3_{\mathbb{C}}$, invariant under the real structure. These lines contain no real point.

Quaternionic notation. We start with the vector space of quaternions $\mathbb{H} = \langle 1, i, j, k\rangle_{\mathbb{R}}$, with $i^2 = j^2 = k^2 = -1$, and $ij = k, jk = i, ki = j$. We also have $ji = -k, kj = -i, ik = -j$. We identify $\mathbb{C}^4$ with $\mathbb{H}^2$ as follows. If $z_0, \dots, z_3$ are the complex coordinates on $\mathbb{C}^4$, set

$$w_0 = z_0 + z_1 j, \qquad w_1 = z_2 + z_3 j.$$

Equivalently, if $z_p = x_p + iy_p$, then $w_0 = x_0 + iy_0 + jx_1 + ky_1$, and similarly for w_1. Left multiplication by j gives an anti-linear map $j \colon \mathbb{H}^2 \longrightarrow \mathbb{H}^2$, namely

$$(w_0, w_1) \longmapsto (jw_0, jw_1) = \big(-\overline{z}_1 + \overline{z}_0 j, -\overline{z}_3 + \overline{z}_2 j\big).$$

Thus $j(\lambda v) = \overline{\lambda}\, j(v)$, for all $\lambda \in \mathbb{C}$, $v \in \mathbb{H}^2$, and $j^2 = -1$ on $\mathbb{H}^2$. On $\mathbb{P}^3_{\mathbb{C}}$, however, the induced map satisfies $j^2 = \mathrm{id}$. We see that j has no fixed points.

The twistor map. Define the projective line over the quaternions

$$\mathbb{P}^1_{\mathbb{H}} := \big\{[w_0, w_1]_{\mathbb{H}} \mid (w_0, w_1) \in \mathbb{H}^2 \setminus \{0\}\big\},$$

where $[w_0, w_1]_{\mathbb{H}} = [w_0', w_1']_{\mathbb{H}}$ if and only if there is $\alpha \in \mathbb{H}^\times$ such that $(w_0', w_1') = (\alpha w_0, \alpha w_1)$. Then we have a well-defined map

$$\mathbb{P}^3_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{H}}, \qquad [z_0 : \cdots : z_3]_{\mathbb{C}} \longmapsto [w_0, w_1]_{\mathbb{H}},$$

Indeed, if $z_i' = \lambda z_i$ for some $\lambda \in \mathbb{C}^\times$, then $w_0' = \lambda z_0 + \lambda z_1 j = \lambda w_0$, and $w_1' = \lambda w_1$. On the chart $w_0 \neq 0$, one writes

$$[w_0, w_1]_{\mathbb{H}} = [1, w_0^{-1} w_1]_{\mathbb{H}},$$

so this is the usual identification of $\mathbb{P}^1_{\mathbb{H}}$ with the one-point compactification $\mathbb{H} \cup \{\infty\} \simeq S^4$. The fibres of π are projective lines in $\mathbb{P}^3_{\mathbb{C}}$, called *twistor lines.* More precisely, if $L \subset \mathbb{H}^2$ is fixed, then the fibre over L is the set of complex lines contained in L, namely

$$\pi^{-1}(L) = \mathbb{P}_{\mathbb{C}}(L) \simeq \mathbb{P}^1.$$

Parametrizing complex structures. Let us recall how the twistor fibration is related to complex structures. The idea is that the fibre of π at $x \in S^4$ parametrizes the oriented orthogonal complex structures on the tangent space of the sphere at x. $\mathrm{T}_x S^4$ at a point $x \in S^4$.

More generally, let (M, g) be an oriented Riemannian four-manifold. Its twistor space is the smooth fibre bundle

$$\mathrm{Tw}(M) = \{(x, J) : x \in M,\ J \in \mathcal{J}_x\},$$

where $\mathcal{J}_x$ is the set of fibrewise complex structures, i.e.

$$\mathcal{J}_x = \{J \in \mathrm{End}(\mathrm{T}_x M) : J^2 = -1,\ J \in \mathrm{SO}(g_x)\}.$$

Here, the notation $J \in \mathrm{SO}(g_x)$ means that J is orthogonal for g_x and orientation-compatible. This is to say that $g_x(Ju, Jv) = g_x(u, v)$ for all $u, v \in \mathrm{T}_x M$. The condition $J^2 = -1$ makes $\mathrm{T}_x M$ into a complex vector space of complex dimension 2. Such a complex vector space has a natural real orientation: if e_1, e_2 is a complex basis, then (e_1, Je_1, e_2, Je_2) is a real basis defining this orientation. We say that J is orientation-compatible if this orientation coincides with the given orientation of

T_xM. Equivalently, for any real basis u_1, u_2 such that u_1, Ju_1, u_2, Ju_2 is a basis of T_xM, this basis is positively oriented.

Thus the fibre of $\mathrm{Tw}(M) \to M$ over x is the space of all orthogonal complex structures on the Euclidean vector space T_xM which induce the chosen orientation. Since $T_xM \simeq \mathbb{R}^4$, this fibre is

$$\mathcal{J}_x \simeq \mathrm{SO}(4)/\mathrm{U}(2) \simeq S^2 \simeq \mathbb{P}^1.$$

Here, $\mathrm{SO}(4) \simeq \mathrm{SU}(2)\times\mathrm{SU}(2)/\{\pm(1,1)\}$ and $\mathrm{U}(2) \simeq \mathrm{SU}(2)\times S^1/\{\pm(1,1)\}$ so the quotient $\mathrm{SO}(4)/\mathrm{U}(2)$ is $\mathrm{SU}(2)/S^1$, which is identified with S^2, by the Hopf fibration.

Moreover, using the Levi-Civita connection, the tangent space of $\mathrm{Tw}(M)$ splits into horizontal and vertical directions. At a point (x, J), the horizontal directions are identified with T_xM, on which we use the complex structure J, while the vertical directions are tangent to the fibre $\mathcal{J}_x \simeq \mathbb{P}^1$, hence carry the standard complex structure of the Riemann sphere. This defines a natural almost-complex structure on $\mathrm{Tw}(M)$. For the round four-sphere this almost-complex structure is integrable, and the resulting complex threefold can be described explicitly as follows.

Once we identify the sphere with the quaternionic projective line $S^4 \simeq \mathbb{P}^1_{\mathbb{H}}$, the two descriptions of the twistor fibres agree. Indeed, if $V \simeq \mathbb{H}$ is an oriented Euclidean real vector space of dimension 4, then every unit imaginary quaternion $q \in \mathrm{Im}\mathbb{H}$, $q^2 = -1$, defines an orthogonal complex structure on V by left multiplication, $J_q(v) = qv$. The unit imaginary quaternions form a sphere S^2 which is the fibre of π.

Let us summarize how the different descriptions of the fibre match. Fix a point $L \in \mathbb{P}^1_{\mathbb{H}}$, where $L \subset \mathbb{H}^2$ is a quaternionic line. Then

$$\pi^{-1}(L) = \mathbb{P}_{\mathbb{C}}(L).$$

After identifying L with $\mathbb{H}$, every complex line in L is of the form $h\mathbb{C}$, with $h \in \mathbb{H}^\times$, where we use the fixed copy $\mathbb{C} = \langle 1, i\rangle_{\mathbb{R}} \subset \mathbb{H}$. The corresponding unit imaginary quaternion is $q = hih^{-1}$. This is well-defined modulo the right action of $\mathbb{C}^\times$, because elements of $\mathbb{C}^\times$ commute with i. Thus a point of $\mathbb{P}_{\mathbb{C}}(L)$ determines the complex structure J_q on the tangent space, and the three descriptions

$$\mathbb{P}_{\mathbb{C}}(L), \qquad \{q \in \mathrm{Im}\mathbb{H} \mid q^2 = -1\}, \qquad \mathcal{J}_L$$

are naturally identified, up to the usual left/right multiplication and orientation convention.

2.1.B. Atiyah–Ward instantons

Quoting [AW77], *The term instanton or pseudo-particle has been coined for the minimum action solutions of* $\mathrm{SU}(2)$ *Yang-Mills fields in Euclidean 4-space* $\mathbb{R}^4$*. Conditions at infinity are imposed which are tantamount to working on the 4-sphere* S^4. Let us follow [Har78] to see the algebraic construction. Let $\mathcal{E}$ be a holomorphic vector bundle of rank r on $\mathbb{P}^3_{\mathbb{C}}$. The Atiyah–Ward conditions, at least for $r = 2$, saying that $\mathcal{E}$ corresponds to a *minimum action instanton* on S^4, are the following.

A.i) Topological invariants:

$$c_1(\mathcal{E}) = 0, \qquad c_2(\mathcal{E}) = k > 0, \qquad c_3(\mathcal{E}) = 0.$$

A.ii) Stability: the bundle $\mathcal{E}$ is stable. This happens if we have the vanishing:

$$\mathrm{H}^0\big(\wedge^p\mathcal{E}\big) = 0, \qquad \text{for } 0 < p < r.$$

To see this, note that, if a vector bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = 0$ satisfies the above vanishing, then a destabilising subsheaf $\mathcal{K} \subset \mathcal{E}$ of rank p, with $1 \le p \le r$ and $c_1(\mathcal{K}) = tH$ must have $t \ge 0$. This gives a non-zero map $\wedge^p\mathcal{K} \to \wedge^p\mathcal{E}$. Taking reflexive hulls, $\mathcal{O}_{\mathbb{P}^n}(t) \hookrightarrow \wedge^p\mathcal{E}$, hence $\mathrm{H}^0(\wedge^p\mathcal{E}(-t)) \ne 0$, a contradiction.

The converse is not true, for instance there are stable symplectic instantons $\mathcal{E}$ of rank $r \ge 3$, and these bundles carry a symplectic form $\omega \in \mathrm{H}^0(\wedge^2\mathcal{E})$. However, if $\mathcal{E}$ has full $\mathrm{SU}(r)$ holonomy, then it satisfies the vanishing conditions on exterior powers.

A.iii) For each $p \in S^4$, the restriction of $\mathcal{E}$ to the twistor line $\ell_p = \pi^{-1}(p) \simeq \mathbb{P}^1$ is trivial:

$$\mathcal{E}|_{\ell_p} \simeq \mathcal{O}_{\ell_p}^{\oplus r}.$$

A.iv) There is a conjugate-linear lift of the real structure, $\widetilde{j}\colon \mathcal{E} \longrightarrow j^*\mathcal{E}$, such that

$$(\widetilde{j})^2 = -\mathrm{id}_{\mathcal{E}}.$$

The prefix A here stands for Atiyah.

Remark 2.1. Yang-Mills instantons satisfy the crucial cohomology vanishing

$$\mathrm{H}^1(\mathcal{E}(-2)) = 0.$$

This goes back to the Penrose transform, indeed, this cohomology group is identified with the space of solutions of the coupled conformal Laplace equation

$$\nabla^*\nabla s + \frac{R}{8}s = 0.$$

Since the round S^4 has positive scalar curvature, the only solution is zero. This argument goes back to Penrose's twistor theory and its Atiyah–Ward/ADHM application. It is explained in [Ati79] and discussed in the more recent [Don24].

2.1.C. Serre correspondence

Let X be a projective manifold and let $\mathcal{E}$ be a rank 2 vector bundle on X. Let $s \in \mathrm{H}^0(X, \mathcal{E})$ be a global section of $\mathcal{E}$ vanishing in codimension 2, and set

$$Z = \mathbb{V}(s).$$

Locally, on an open subset $U \subset X$ trivializing $\mathcal{E}$, we have $\mathcal{E}_{|U} \simeq \mathcal{O}_U^2$, so the section is given by two functions $s_{|U} = (f_U, g_U)$. Hence $Z \cap U = \mathbb{V}(f_U) \cap \mathbb{V}(g_U)$. This is the expected behaviour. It happens, for instance, if $\mathcal{E}$ is globally generated and s is generic.

Then there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} \mathcal{E} \longrightarrow \mathcal{I}_Z\big(c_1(\mathcal{E})\big) \longrightarrow 0. \tag{13}$$

By adjunction,

$$\omega_Z \simeq \big(\omega_X \otimes \det \mathcal{E}\big)|_Z.$$

Thus a pair $(\mathcal{E}, s)$ gives a codimension 2 locally complete intersection subscheme $Z \subset X$, together with this adjunction isomorphism.

Conversely, let $\mathcal{L}$ be a line bundle on X, and suppose that $Z \subset X$ is a locally complete intersection subscheme of codimension 2. Assume, in the threefold case, that

$$\mathrm{H}^2(\mathcal{I}_Z \otimes \mathcal{L} \otimes \mathrm{H}^i(X, \mathcal{L}^\vee) = 0, \qquad i = 1, 2.$$

Then extension classes for

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Z \otimes \mathcal{L} \longrightarrow 0$$

are controlled by

$$\mathrm{Ext}^1(\mathcal{I}_Z \otimes \mathcal{L}, \mathcal{O}_X).$$

Using Serre duality, and the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_X \to \mathcal{O}_Z \to 0,$$

and vanishing $\mathrm{h}^i(\mathcal{L} \otimes \omega_X) = \mathrm{h}^{3-i}(\mathcal{L}^\vee) = 0$ for $i \in \{1, 2\}$, one obtains

$$\mathrm{Ext}^1_X(\mathcal{I}_Z \otimes \mathcal{L}, \mathcal{O}_X) \simeq \mathrm{H}^2(\mathcal{I}_Z \otimes \mathcal{L} \otimes \omega_X)^\vee \simeq \mathrm{H}^1\big(\mathcal{L} \otimes \omega_X \otimes \mathcal{O}_Z\big)^\vee \simeq \mathrm{H}^0\big((\mathcal{L} \otimes \omega_X)^\vee \otimes \omega_Z\big).$$

Given a nowhere vanishing generator $t \in \mathrm{H}^0\big(Z, (\mathcal{L} \otimes \omega_X)^\vee \otimes \omega_Z\big)$, one obtains a rank 2 vector bundle $\mathcal{E}$ fitting into (13), with $c_1(\mathcal{E}) = \mathcal{L}$.

Example 2.2. We consider the union of three lines disjoint in $\mathbb{P}^3$, and take $\mathcal{L} = \mathcal{O}_{\mathbb{P}^3}(2)$. In other words, take $Z = \ell_1 \cup \ell_2 \cup \ell_3$. Then we have:

$$(\mathcal{O}_{\mathbb{P}^3}(2) \otimes \omega_{\mathbb{P}^3})^\vee \otimes \omega_Z \simeq \mathcal{O}_Z.$$

Hence one may choose

$$\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathrm{H}^0(Z, \mathcal{O}_Z).$$

If $\lambda_i \neq 0$ for all $1 \leq i \leq 3$, then the sheaf $\mathcal{E}$ is a vector bundle.

Instantons of charge one, null-correlation bundles. For instantons of rank 2 and charge 1, one has a very explicit description of the moduli space. The result is

$$\mathfrak{M}^s(2,1) \simeq \mathbb{P}^5 \setminus \mathrm{Gr}(2,4),$$

where $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$ is the Plücker quadric. More precisely, let $\mathbb{P}^3 = \mathbb{P}(V)$, so $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(1)) = V$, so that $\mathrm{Gr}(2,V) \subset \mathbb{P}(\wedge^2 V) \simeq \mathbb{P}^5$. To see how this goes, take $\mathcal{E} \in \mathfrak{M}^s(2,1)$. The first observation is that $\mathcal{E}$ is a vector bundle. This is not hard to see, indeed, it is enough to see that $\mathcal{E}$ is reflexive, for otherwise $\mathcal{E}^{\vee\vee}$ would have vanishing Chern classes, hence $\mathcal{E}^{\vee\vee} \simeq \mathcal{O}_{\mathbb{P}^3}^{\oplus 2}$ and $\mathcal{E}$ would be only semistable. Then, by Grauer-Mülich, for a sufficiently general line ℓ in $\mathbb{P}^3$ we have $\mathcal{E}|_\ell \simeq \mathcal{O}_\ell^{\oplus 2}$. A line ℓ is jumping if $\mathcal{E}|_\ell \simeq \mathcal{O}_\ell(a) \oplus \mathcal{O}_\ell(-a)$, for some $a > 0$. Therefore, the jumping lines of an instanton bundle of rank 2 and charge k are described as

$$Z_{\mathcal{E}} := \big\{[\ell] \in \mathrm{Gr}(2,4) \mid \mathrm{h}^1(\ell, \mathcal{E}_{|\ell}(-1)) \neq 0\big\}.$$

It turns out that this is a divisor of $\mathrm{Gr}(2,4)$. Its class is kH, where H is the hyperplane section of $\mathrm{Gr}(2,4)$. Thus, when $k=1$, $Z_{\mathcal{E}}$ is just a hyperplane, giving a point $[Z_{\mathcal{E}}] \in \mathbb{P}^5$. This hyperplane corresponds to a non-zero element

$$v \in \wedge^2 V = \mathrm{H}^0\big(\Omega_{\mathbb{P}^3}(2)\big).$$

The associated null-correlation bundle is defined by

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \xrightarrow{v} \Omega_{\mathbb{P}^3}(2) \longrightarrow \mathcal{E}(1) \longrightarrow 0.$$

Equivalently, $\mathcal{E}$ is the cohomology of the monad

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow 0,$$

where the two maps are induced by the skew form $v \in \wedge^2 V$. The condition $v \notin \mathrm{Gr}(2,V)$ is precisely the condition that the map to $\mathcal{O}_{\mathbb{P}^3}(1)$ is surjective.

Instantons of charge two. It is possible to describe the moduli space $\mathfrak{MI}_{\mathbb{P}^3}(2,2)$ quite explicitly. First of all, Example 2.2 affords a rank-2 vector bundle $\mathcal{E}$ fitting into:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{E} \to \mathcal{I}_Z(1) \to 0.$$

We compute immediately that $\mathrm{H}^*(\mathcal{E}(-2)) = 0$. Indeed, this follows essentially from $\mathrm{H}^*(\mathcal{O}_Z(-1)) = 0$. We see that

$$\mathrm{h}^0(\mathcal{I}_Z(2)) \geq \mathrm{h}^0(\mathcal{O}_{\mathbb{P}^3}(2)) - \mathrm{h}^0(\mathcal{O}_Z(2)) = 10 - 9 = 1.$$

Therefore Z is contained in a quadric surface Q. Hence this surface must be smooth, otherwise it could not contain 3 disjoint lines. Also, the three lines belong to a single ruling of Q, for otherwise at least two of the three lines would meet at a point. Hence Z determines an identification of Q with $\mathbb{P}^1 \times \mathbb{P}^1$ and a choice of a divisor $Z \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,0)|$. We also get $\mathrm{h}^0(\mathcal{E}(1)) = 2$ and thus

$$0 \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \to \mathcal{E}(1) \to \mathcal{L} \to 0.$$

Here $\mathcal{L}$ is a line bundle on Q and the evaluation map $\mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \to \mathcal{E}(1)$ is injective, as its determinant is an equation of Q itself. The dual of this sequence is

$$0 \to \mathcal{E}(-1) \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,0) \to 0. \tag{14}$$

Here, the map $\mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,0)$ defines a base-point-free linear system of degree 3 on $\mathbb{P}^1$, aka a g^1_3 with no base point. From this, Grothendieck duality gives $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,2)$.

This construction is reversible and identifies $\mathfrak{MI}_{\mathbb{P}^3}(2,2)$ with an open subset of a $\mathrm{Gr}(2,4)$-fibration of a double cover of $\mathbb{P}^9 \setminus \Delta$, where $\mathbb{P}^9$ parametrizes quadric surfaces and Δ is the set of singular quadric surfaces. The double cover arises by considering the two rulings of a smooth quadric surface. Finally, the $\mathrm{Gr}(2,4)$-fibration occurs because, for a given ruled surface $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ corresponding to a given point of $\mathbb{P}^9 \setminus \Delta$, we choose a base-point-free pencil in $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,0)|$.

One can compute the resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,0) \to 0.$$

Combining it with (14) we get the monad-theoretic description of $\mathcal{E}$:

$$\mathcal{E} \simeq \mathcal{H}\Big(0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2} \to 0\Big)$$

Construction of 't Hooft bundles. More generally, we consider the union of $k+1$ lines disjoint in $\mathbb{P}^3$, and take $\mathcal{L} = \mathcal{O}_{\mathbb{P}^3}(2)$. In other words, take $Z = \ell_1 \cup \cdots \cup \ell_{k+1}$. Then we have:

$$(\mathcal{O}_{\mathbb{P}^3}(2) \otimes \omega_{\mathbb{P}^3})^\vee \otimes \omega_Z \simeq \mathcal{O}_Z.$$

Hence one may choose

$$\lambda = (\lambda_1, \cdots, \lambda_{k+1}) \in \mathrm{H}^0(Z, \mathcal{O}_Z).$$

If $\lambda_i \neq 0$ for all $1 \leq i \leq k+1$, then the sheaf $\mathcal{E}$ is a vector bundle, actually an instanton bundle of rank 2 and charge k. These vector bundles, called 't Hooft instantons, form a smooth closed subvariety of the moduli space $\mathfrak{MI}_{\mathbb{P}^3}(2,k)$. For $k \geq 3$, the dimension of this subvariety is $5k+4$ and its codimension in $\mathfrak{MI}_{\mathbb{P}^3}(2,k)$ is $3k-7$. The caveat for $k \leq 2$ is that $\mathrm{h}^0(\mathcal{E}(1)) = 5$ for $k=1$, whereas $\mathrm{h}^0(\mathcal{E}(1)) = 2$ for $k=2$; hence the same bundle $\mathcal{E}$ determines many choices of disjoint lines Z. In any case, for $k=1$ and $k=2$ the variety of 't Hooft bundles fills the whole $\mathfrak{MI}_{\mathbb{P}^3}(2,k)$.

The special 't Hooft bundles arise when the lines of Z are contained in a (ruling of a smooth) quadric in $\mathbb{P}^3$. In this case $\mathcal{E}$ is equivariant for the action of SL_2. For $k=2$, all instantons are special 't Hooft instantons. For $k=1$, they are all null-correlation bundles as we have seen, and are thus equipped with the action of Sp_4.

2.2. Properties of instanton moduli on projective three-space

2.2.A. Monads

The other definition of instanton sheaf $\mathcal{E}$ on $\mathbb{P}^3$ is the following.

M.i) Topological invariants:

$$c_1(\mathcal{E}) = 0, \qquad c_2(\mathcal{E}) = k \geq 0, \qquad c_3(\mathcal{E}) = 0.$$

M.ii) Stability: $\mathcal{E}$ is semistable.

M.iii) Cohomological vanishing: $\mathrm{H}^1(\mathcal{E}(-2)) = 0$.

This implies that $\mathcal{E}$ is the cohomology of a monad

$$\mathcal{E} \simeq \mathcal{H}\Big(\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus k} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus(r+2k)} \to \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus k}\Big). \tag{15}$$

Note that, if $\mathcal{E}$ is an instanton bundle of rank r and charge k, then $\mathrm{ch}(\mathcal{E}) = r - kH^2$. The moduli space of instanton sheaves on $\mathbb{P}^3$ is the open subset

$$\mathfrak{M}(r,k) := \mathfrak{M}_{\mathbb{P}^3}(r - kH^2) \supset \mathfrak{MI}_{\mathbb{P}^3}(r,k) = \{\mathcal{E} \in \mathfrak{M}_{\mathbb{P}^3}(r,k) \mid \mathrm{H}^1(\mathcal{E}(-2)) = 0.\}$$

Let I_{-1} and I_1 be vector spaces of dimension k and let I_0 be a vector space of dimension $r+2k$. Write $V = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(1))$. Consider matrices A, B of the form

$$B \in I_{-1}^\vee \otimes I_0 \otimes V, \qquad A \in I_0^\vee \otimes I_1 \otimes V,$$

so that we write

$$B : I_{-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to I_0 \otimes \mathcal{O}_{\mathbb{P}^3}, \qquad A : I_0 \otimes \mathcal{O}_{\mathbb{P}^3} \to I_1 \otimes \mathcal{O}_{\mathbb{P}^3}(1).$$

Then the result about monads gives (up to some identifying stability conditions, which holds generically) a rational map

$$\mathfrak{M}_{\mathbb{P}^3}(r,k) \dashrightarrow \{(A,B) \mid AB = 0, A \text{ is surjective}, B \text{ is injective}\} /\!\!/_\theta \mathrm{GL}(I_{-1}) \times \mathrm{GL}(I_1) \times \mathrm{GL}(I_0).$$

Here, we use the standard character θ of $\mathrm{GL}(I_{-1}) \times \mathrm{GL}(I_1) \times \mathrm{GL}(I_0)$ operating on the pairs of matrices A and B by left and right multiplication.

One can also study orthogonal or symplectic instantons, which are defined as instanton bundles having an everywhere non-degenerate inner product which is symmetric or skew-symmetric. This means that there is an isomorphism $\alpha : \mathcal{E} \to \mathcal{E}^\vee$ such that $\alpha^\vee : \mathcal{E} \to \mathcal{E}^{\vee\vee} \to \mathcal{E}^\vee$ equals α, in the symmetric case, or $-\alpha$, in the skew-symmetric case (in this last case, r is even). Usually, one writes $\mathfrak{MI}(\mathbf{v})^{\mathfrak{sp}}$ and $\mathfrak{MI}(\mathbf{v})^{\mathfrak{so}}$ for the moduli spaces of symplectic and orthogonal instantons, respectively. Incidentally, rank-2 instantons are necessarily symplectic.

The tangent spaces of these moduli spaces at a stable bundle $\mathcal{E}$ are:

$$\mathrm{T}_{\mathfrak{MI}(\mathbf{v})^{\mathfrak{sp}},[\mathcal{E}]} \simeq \mathrm{H}^1(S^2\mathcal{E}), \qquad \mathrm{T}_{\mathfrak{MI}(\mathbf{v})^{\mathfrak{o}},[\mathcal{E}]} \simeq \mathrm{H}^1(\wedge^2\mathcal{E}).$$

2.2.B. Jumping lines

We discussed a bit earlier the idea of jumping lines and we said that a rank-2 stable bundle on $\mathbb{P}^3$ with $c_2(\mathcal{E}) = k$ has a divisor of jumping lines of class $kH_{\mathrm{Gr}(2,4)}$ in $\mathrm{Gr}(2,4)$.

Let us recover this for instanton sheaves. Let $\mathcal{E}$ be a torsion-free slope-semistable instanton sheaf of rank r and charge k on $\mathbb{P}^3$, so that $\mathcal{E}$ is the cohomology of a monad, see (15). For $r = 2$, we know by Grauert-Mülich that, for a general line $\ell \subset \mathbb{P}^3$, the sheaf $\mathcal{E}$ splits trivially on ℓ. For $r > 2$, let us assume that a line ℓ with trivial splitting exists. The set of lines $\ell \subset \mathbb{P}^3$ is identified with $\mathrm{Gr}(2,4)$ and a point p of $\mathrm{Gr}(2,4)$ corresponds to a jumping line ℓ_p if and only if $\mathrm{H}^1(\mathcal{E}|_{\ell_p}(-1)) \neq 0$. Therefore, the set of jumping lines is the support of some first direct image sheaf.

To see this, consider the incidence variety

$$\mathrm{F}(1,2,4) = \{(z,p) \in \mathbb{P}^3 \times \mathrm{Gr}(2,4) \mid z \in \ell_p\}.$$

Let us denote by $\mathcal{U}$ the tautological sub-bundle of rank 2 on $\mathrm{Gr}(2,4)$ and let us write p_1 and p_2 for the projection maps from $\mathrm{F}(1,2,4)$ to $\mathbb{P}^3$ and $\mathrm{Gr}(2,4)$. The fibre at p of $p_2 : \mathrm{F}(1,2,4) \to \mathrm{Gr}(2,4)$ is the line ℓ_p. Thus, after pulling back $\mathcal{E}(-1)$ to $\mathrm{F}(1,2,4)$, the condition $\mathrm{H}^1(\mathcal{E}|_{\ell_p}(-1)) \neq 0$ means exactly that p lies in the support of $\mathrm{R}^1 p_{2*}(p_1^*(\mathcal{E}(-1)))$.

It turns out that $\mathrm{F}(1,2,4)$ is cut in $\mathbb{P}^3 \times \mathrm{Gr}(2,4)$ by a tautological section of $\mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathcal{U}^\vee$. So we have a Koszul-type resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \boxtimes \mathcal{O}_{\mathrm{Gr}(2,4)}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{U} \to \mathcal{O}_{\mathbb{P}^3 \times \mathrm{Gr}(2,4)} \to \mathcal{O}_{\mathrm{F}(1,2,4)} \to 0.$$

From this, using Künneth's formula and the projection formula, we get:

$$\mathrm{R}p_{2*}(p_1^*(\mathcal{O}_{\mathbb{P}^3})) \simeq \mathcal{O}_{\mathrm{Gr}(2,4)}, \quad \mathrm{R}p_{2*}(p_1^*(\mathcal{O}_{\mathbb{P}^3}(-1))) = 0, \quad \mathrm{R}p_{2*}(p_1^*(\mathcal{O}_{\mathbb{P}^3}(-2))) \simeq \mathcal{O}_{\mathrm{Gr}(2,4)}(-1)[-1].$$

So the subset of jumping lines, which is the support of

$$\mathrm{R}^1 p_{2*}(p_1^*(\mathcal{E}(-1))) \simeq \mathrm{Coker}\Big(I_{-1} \otimes \mathcal{O}_{\mathrm{Gr}(2,4)}(-1) \xrightarrow{\varphi} I_1 \otimes \mathcal{O}_{\mathrm{Gr}(2,4)}, \Big)$$

for some map φ determined by the monad defining $\mathcal{E}$. Since there is $p \in \mathrm{Gr}(2,4)$ such that $\mathcal{E}$ restricts trivially to ℓ_p, the map φ is invertible at p. Hence the support of $\mathrm{R}^1 p_{2*}(p_1^*(\mathcal{E}(-1)))$ is a divisor in $\mathrm{Gr}(2,4)$. We see also that it has class $kH_{\mathrm{Gr}(2,4)}$.

2.2.C. Smoothness

A breakthrough in the study of $\mathfrak{MI}_{\mathbb{P}^3}(r,k)$ was the result of Jardim and Verbitsky, see [JV11, JV14]. They proved that the moduli space of framed instantons is smooth. In their definition, an instanton is a locally free sheaf $\mathcal{E}$ satisfying

$$\mathrm{H}^0(\mathcal{E}(-1)) = \mathrm{H}^1(\mathcal{E}(-2)) = \mathrm{H}^2(\mathcal{E}(-2)) = \mathrm{H}^3(\mathcal{E}(-3)) = 0. \tag{16}$$

When referring to this notion, we say that $\mathcal{E}$ is a *cohomological instanton*.

To state the main smoothness result, we consider a line $\ell \subset \mathbb{P}^3$ and introduce the moduli space of framed instantons along the line ℓ:

$$\mathfrak{MIF}(r,k) = \{(\mathcal{E}, \Phi) \mid \mathcal{E} \text{ satisfies (16)}, \Phi : \mathcal{E}|_\ell \xrightarrow{\simeq} \mathcal{O}_\ell^{\oplus r}.\}$$

Theorem 2.3 (Jardim-Verbitsky). *The moduli space $\mathfrak{MIF}(r,k)$ is smooth of dimension $4rk$.*

In particular, for rank 2, take $\mathcal{E}$ to be an instanton bundle in our sense, i.e., a slope-semistable locally free sheaf of rank 2 with $c_1(\mathcal{E}) = 0$ and $\mathrm{H}^1(\mathcal{E}(-2)) = 0$. Then, playing with the restriction on a given plane, we see that $\mathcal{E}$ satisfies (16). Also, there is a line ℓ such that $\mathcal{E}$ is trivial along ℓ, by Grauert-Mülich. The choices of framings Φ over ℓ identify the fibre of the tautological map $\mathfrak{MIF}(2,k) \to \mathfrak{MI}(2,k)$ that forgets the framing, which is a principal PGL_2 bundle, locally around $[\mathcal{E}]$, with image the open subset of instanton bundles trivial along ℓ. This implies that the open piece of $\mathfrak{MI}(2,k)$ consisting of vector bundles is smooth of dimension $8k - 3$.

Their strategy of proof is to use the hyperkähler structure. They consider the pencil of planes $L_t \subset \mathbb{P}^3$ containing the fixed line ℓ, so t varies in a projective line $\mathbb{P}^1$. For general t, the restriction $\mathcal{E}|_{L_t}$ is an instanton on L_t, framed along ℓ, in the sense of [Nak99]. The moduli space $\mathfrak{MF}_{\mathbb{P}^2}(r,k)$ is a hyperkähler manifold of dimension $4rk$. It turns out that $\mathbb{P}^1$ is the base of the hyperkähler twistor map of this manifold:

$$\mathrm{Tw}(\mathfrak{MF}_{\mathbb{P}^2}(r,k)) \to \mathbb{P}^1.$$

Jardim-Verbitsky prove that $\mathfrak{MIF}(r,k)$ is a space of sections of this twistor map. The section sends $t \in \mathbb{P}^1$ to $\mathcal{E}|_{L_t}$, together with its framing along ℓ. They proved that this space of sections is smooth by showing that it admits an extra structure, which they called trisymplectic.

2.2.D. Irreducibility and rationality

The moduli space of instanton bundles of rank 2 on $\mathbb{P}^3$ has also been proved to be irreducible and rational. Irreducibility is due to Tikhomirov, [Tik13, Tik12].

Rationality was claimed by Markushevich and Tikhomirov, [MT24]. But this paper has been withdrawn.

There is also a proof of rationality in [HT19]. This paper is not published yet. The definition of Halic and Tajarod is that $\mathcal{E}$ is an instanton if $\mathcal{E}$ is locally free of rank r, satisfies (M.i)), (M.iii)) and moreover there is a line $\ell \subset \mathbb{P}^3$ such that $\mathcal{E}|_\ell$ is trivial.

2.2.E. Components of sheaves

Already in rank 2, if one removes the condition that $\mathcal{E}$ is locally free, using the definition of instanton as a coherent sheaf satisfying (16), one gets many components of the moduli space of instantons, see for instance [JMT17]. There are many papers by Jardim and collaborators about these components of the moduli space. One conjecture is that the moduli space of instantons (or maybe just of semistable sheaves) is *connected*, though perhaps reducible and badly singular. This would support the idea that these moduli spaces behave in a somewhat similar way as the Hilbert scheme.

2.3. Other twistor spaces

Let us say a word about the other case where the twistor space can be used to relate connections to holomorphic bundles on a smooth threefold via the twistor transform. There is some work of Nagatomo concerning different twistor spaces, related to adjoint varieties of simple Lie group over $\mathbb{C}$. However, the situation in higher dimension, besides some constructions involving homogeneous bundles, is quite mysterious. We only discuss the three-dimensional situation here.

2.3.A. The point-line flag in the plane

One may wonder whether there are other smooth projective threefolds that have a structure of twistor space of a smooth 4-manifold. The answer is a theorem of Hitchin: it asserts that there is one and only one such threefold, namely the full flag of the linear group SL_3, which can be described as the projective tangent bundle of $\mathbb{P}^2$, or the point-line incidence:

$$\mathrm{F}(1,2,3) = \{(p,q) \in \mathbb{P}(V) \times \mathbb{P}(V^\vee) \mid p \in \ell_q\}.$$

Here, $\ell_q \subset \mathbb{P}(V)$ denotes the line corresponding to the point q of the dual projective plane $\mathbb{P}(V^\vee)$ and V is just a 3-dimensional vector space. Although this is a projective homogeneous variety, the twistor projection is not one of the two holomorphic projections

$$\mathrm{F}(1,2,3) \longrightarrow \mathbb{P}^2, \qquad \mathrm{F}(1,2,3) \longrightarrow (\mathbb{P}^2)^\vee.$$

Indeed, after choosing the standard Hermitian form on $\mathbb{C}^3$, the twistor map is the smooth map

$$\pi \colon \mathrm{F}(1,2,3) \longrightarrow \mathbb{P}^2, \qquad (P \subset Q) \longmapsto Q \cap P^\perp.$$

Here, Q is the 2-dimensional vector subspace of V corresponding to q. The subspace $P^\perp$ is the 2-dimensional vector subspace of V determined by p as follows. If the line $P \subset V$ corresponds to p, then the Hermitian product gives an antilinear map $V \to V^\vee$, and we set

$$P^\perp = \mathrm{Ker}(V \longrightarrow V^\vee \longrightarrow P^\vee).$$

Thus $\pi(P \subset Q)$ is the line $Q \cap P^\perp$. Since the operation $L \mapsto L^\perp$ involves complex conjugation, the map π is not holomorphic for the natural complex structure on the flag variety. Nevertheless, its fibres are projective lines: if $M \in \mathbb{P}^2$, then

$$\pi^{-1}(M) = \{\, L \subset M^\perp \,\} \simeq \mathbb{P}(M^\perp) \simeq \mathbb{P}^1.$$

In this way $\mathrm{F}(1,2,3)$ is an algebraic variety carrying the twistor fibration of $\mathbb{P}^2$.

2.3.B. Cohomology ring of the point-line flag

We denote by

$$p_1 \colon \mathrm{F}(1,2,3) \to \mathbb{P}^2, \qquad p_2 \colon \mathrm{F}(1,2,3) \to (\mathbb{P}^2)^\vee$$

the two projections, and set $H_1 = p_1^*\mathcal{O}_{\mathbb{P}^2}(1)$, and $H_2 = p_2^*\mathcal{O}_{(\mathbb{P}^2)^\vee}(1)$. By abuse of notation we also denote by H_1, H_2 their first Chern classes. Then

$$\mathrm{H}^*(\mathrm{F}(1,2,3),\mathbb{Z}) \simeq \mathbb{Z}[H_1,H_2]/\left(H_1^3,\, H_2^3,\, H_1^2 - H_1H_2 + H_2^2\right).$$

Equivalently, the only relation in degree 2 is $H_1^2 - H_1H_2 + H_2^2 = 0$. The top intersections are normalized by

$$\int_{\mathrm{F}(1,2,3)} H_1^2 H_2 = \int_{\mathrm{F}(1,2,3)} H_1 H_2^2 = 1,$$

and therefore $H_1^2H_2 = H_1H_2^2$ generates $\mathrm{H}^6(\mathrm{F}(1,2,3),\mathbb{Z})$.

2.3.C. Instanton bundles on the point-line flag

The study of Yang-Mills instantons on $\mathbb{P}^2$ via the twistor manifold $\mathrm{F}(1,2,3)$ was carried out in [Buc93]. More recently, the monad-theoretic approach and the construction of 't Hooft bundles were taken up again in [MMPL20, AMMPL25].

They set $H = H_1 + H_2$ and defined k-instantons on $\mathrm{F}(1,2,3)$ as slope-semistable vector bundles $\mathcal{E}$ of rank 2 with

$$\mathrm{H}^1(\mathcal{E}(-H)) = 0, \qquad c_1(\mathcal{E}) = 0, \qquad c_2(\mathcal{E}) = kH_1H_2.$$

Then the moduli space $\mathfrak{MI}_{X,H}(2,k)$ of these instantons contains a component of dimension $8k-3$. This contains (at least) two components of special 't Hooft bundles, which are defined as instantons $\mathcal{E}$ having $\mathrm{h}^0(\mathcal{E}(H_1)) \neq 0 \neq \mathrm{h}^0(\mathcal{E}(H_2))$. These components are smooth of dimension $4k+4$ and $2k+7$, for $k \geq 2$.

3. Fano threefolds: monads and semiorthogonal decompositions

In Lecture 2 we discussed instanton bundles on $\mathbb{P}^3$ and on the flag variety $\mathrm{F}(1,2,3)$. These varieties are twistor spaces, respectively of S^4 and $\mathbb{P}^2$. They are also Fano threefolds.

The goal of this lecture is to ask whether similar structures exist on other Fano threefolds. We shall see that the answer is closely related to two features of the geometry of a Fano threefold X: the existence of special semiorthogonal decompositions of $\mathrm{D}^{\mathrm{b}}(X)$, and the possibility of constructing vector bundles as cohomology of explicit monads.

3.1. Fano threefolds and their derived categories

3.1.A. Fano varieties

For a comprehensive reference about Fano manifolds, see [IP99]. Let X be a smooth connected projective variety of dimension n. The sheaf of differentials Ω^1_X is then a vector bundle of rank n, and the canonical bundle of X is

$$\omega_X = \bigwedge^n \Omega^1_X = \det(\Omega^1_X).$$

Equivalently, if K_X denotes a canonical divisor, then

$$\omega_X \simeq \mathcal{O}_X(K_X).$$

Negative canonical bundle. The positivity of the canonical bundle gives a first rough classification of projective varieties. This is particularly clear in dimension 1, where a smooth projective curve X has negative, trivial, or positive canonical bundle according to whether the genus g of X is 0, 1, or ≥ 2. More generally we have:

- if ω_X is trivial, then X has trivial canonical class. Under the additional usual conditions, such varieties are called Calabi-Yau varieties;
- if ω_X is ample, then X is canonically polarized, in particular of general type;
- if $\omega_X^\vee$ is ample, then X is called a Fano variety.

A smooth projective manifold X is Fano if and only if its anticanonical bundle $\omega_X^\vee \simeq \mathcal{O}_X(-K_X)$ is ample.

Example 3.1. Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of hypersurfaces of degrees $d_1, \ldots, d_c$. By adjunction,

$$\omega_X \simeq \mathcal{O}_X\big(-N-1+d_1+\cdots+d_c\big).$$

Hence $-K_X = \big(N+1-d_1-\cdots-d_c\big)H$, where H is the hyperplane class on X. Therefore X is Fano if and only if

$$d_1+\cdots+d_c \leq N.$$

Indeed, this condition is equivalent to $N+1-d_1-\cdots-d_c > 0$, so that $-K_X$ is a positive multiple of the hyperplane class.

First properties of Fano manifolds. Let us list some nice properties of Fano manifols.

a) Any Fano manifold over $\mathbb{C}$ is simply connected.

b) Let X be a Fano manifold over a field of characteristic 0. Then the outer diagonals of the Hodge diamond of X vanish. Indeed, Kodaira vanishing gives $\mathrm{H}^q(\mathcal{L}\otimes\omega_X) = 0$ for $q > 0$ for any ample line bundle $\mathcal{L}$, so choosing $\mathcal{L} = \omega_X^\vee$ gives $\mathrm{H}^q(\mathcal{O}_X) = 0$ for $q > 0$. In particular, $\chi(X, \mathcal{O}_X) = 1$. For a smooth Fano threefold X, Kodaira vanishing gives

$$\mathrm{H}^1(X,\mathcal{O}_X) = \mathrm{H}^2(X,\mathcal{O}_X) = \mathrm{H}^3(X,\mathcal{O}_X) = 0.$$

Hence the Hodge diamond has the form

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & \rho_X & & 0 & \\ 0 & & \mathrm{h}^{1,2}(X) & & \mathrm{h}^{1,2}(X) & & 0 \\ & 0 & & \rho_X & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

where

$$\rho_X = \mathrm{h}^{1,1}(X).$$

Thus the only numerical Hodge-theoretic invariants left are essentially ρ_X and $\mathrm{h}^{1,2}(X)$.

c) The Picard group of X is discrete; it is isomorphic to the Néron–Severi group of X. The rank of this group, called the Picard rank and often denoted by ρ_X, equals $\mathrm{h}^{1,1}(X)$.

d) Any Fano manifold is rationally connected, which means that given any two closed points p, q of X, there is a rational curve contained in X passing through p and q. This implies that X is uniruled and actually also that X is simply connected.

e) The Mori cone $\overline{\mathrm{NE}}(X)$ is rational polyhedral. This is the closure of the cone of effective curves inside $N_1(X)_{\mathbb{R}}$. The extremal rays correspond to Mori contractions of X.

As for invariants of Fano manifolds, the first one is the Fano index. This is defined as the divisibility of K_X, namely:

$$i_X = \max\{j \geq 1 \mid \text{there exists } H \in \mathrm{Pic}(X) \text{ ample, with } -K_X = jH\}.$$

If X is not Fano, we just let i_X be $-\infty$.

A result of Kobayashi-Ochiai says that, if X is a smooth connected complex manifold of dimension n, then $i_X \leq n+1$ and

- $i_X = n+1$ if and only if $X \simeq \mathbb{P}^n$,
- $i_X = n$ if and only if $X \simeq Q_n$, a smooth quadric in $\mathbb{P}^{n+1}$.

Boundedness. A fundamental property of Fano manifolds is that they form a bounded class.

Theorem 3.2. *Fix an integer $n > 0$ and let $\mathbb{k}$ be an algebraically closed field of characteristic zero. Then smooth Fano varieties of dimension n over $\mathbb{k}$ form a bounded family. More precisely, there exist a quasi-projective scheme B and a projective morphism $\mathcal{X} \longrightarrow B$ such that, for every smooth Fano variety X of dimension n over $\mathbb{k}$, there exists a point $b \in B$ with*

$$X \simeq \mathcal{X}_b.$$

Let us spell out what this means. First, B is quasi-projective, so that B is of finite type and Noetherian over $\mathbb{k}$, in particular B has finitely many irreducible components and each of them is of finite dimension. The theorem says that any Fano manifold of dimension n determines a point $b \in B$ in such a way that X is the fibre of $\mathcal{X} \to B$ at b. So B is the parameter space of all Fano manifolds of dimension n. Note that the point b is not necessarily unique: different points of B may parametrize isomorphic Fano varieties. The scheme B may have many irreducible components (but only finitely many of them), each of them is usually called a deformation type of Fano manifolds.

The deep input behind this statement is the following boundedness result. First, for every $n > 0$, there exists an integer $m > 0$ such that, for every smooth Fano variety X of dimension n, the divisor $-mK_X$ is very ample. Moreover, analogous boundedness results hold for Fano varieties with mild singularities; this is one of the major consequences of Birkar's work. Assuming this very ampleness statement, the construction of a parameter space is conceptually simple. Embed X by the complete linear system $|-mK_X|$. Then X becomes a smooth subvariety of some projective space $\mathbb{P}^N$. Once the relevant Hilbert polynomial

$$\mathbf{P}_X(t) = \chi\big(\mathcal{O}_X(tH_X)\big), \qquad \text{with } H = -mK_X.$$

is fixed, such subvarieties are parametrized by a Hilbert scheme, which is a projective scheme by Grothendieck's result. In fact, not the whole Hilbert scheme parametrizes Fano manifolds; rather, a locally closed subset of it does. Indeed, the condition of parametrizing irreducible subschemes isolates some open subset of some irreducible components of the Hilbert scheme, and smoothness determines some open subset of them, while the Fano condition is also open in smooth projective families. In any case, we end up with a quasi-projective parameter space B. Hence smooth Fano varieties with this fixed Hilbert polynomial form an open subscheme of a Hilbert scheme.

Finally, the possible Hilbert polynomials are finite in number. Indeed, the boundedness theorem gives uniform numerical bounds, notably on intersection numbers $H^n, c_1(X)H^{n-1}, c_2(X)H^{n-2}$, where $H = -mK_X$. Thus the Hilbert polynomial is allowed to vary only in a finite set.

For more details on this point of view, see for instance Debarre's lecture notes on higher-dimensional algebraic geometry.

3.1.B. Del Pezzo manifolds

Del Pezzo manifolds of dimension n are varieties of high Fano index, namely $i_X = n - 1$. This is just one less than the range classified by the theorem of Kobayashi-Ochiai. In other words:

$$-K_X = (n-1)H_X$$

for an ample divisor H_X. We define the degree of X by

$$d = H_X^n.$$

The classification of Del Pezzo manifolds is due to Fujita. It generalizes the classification of Del Pezzo surfaces, i.e. Fano surfaces, due to the Italian school of the late XIX century. Del Pezzo surfaces over an algebraically closed field $\Bbbk$ are rational over $\Bbbk$ and can actually be described in a very explicit way. Indeed, if X is a smooth projective surface and $H_X = -K_X$ is ample with $d = H_X^2$, then $1 \leq d \leq 9$ and X is isomorphic to the blow-up of $\mathbb{P}^2$ at $9 - d$ points in sufficiently general position, or to $\mathbb{P}^1 \times \mathbb{P}^1$. In higher dimension, we have the following result.

Theorem 3.3 (Fujita). *Let X be a smooth Fano variety of dimension $n \geq 2$ with $-K_X = (n-1)H_X$. Set $d = H_X^n$. Then X is one of the following varieties:*

d	X
1	*a sextic hypersurface in* $\mathbb{P}(1^n, 2, 3)$
2	*a double cover of* $\mathbb{P}^n$ *branched along a quartic hypersurface*
3	*a cubic hypersurface in* $\mathbb{P}^{n+1}$
4	*a complete intersection of two quadrics in* $\mathbb{P}^{n+2}$
5	*a smooth linear section of* $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$
6	$(\mathbb{P}^1)^3$, $\mathrm{F}(1,2,3)$, $\mathbb{P}^2 \times \mathbb{P}^2$
7	$\mathrm{Bl}_p(\mathbb{P}^3)$

In particular, in the above table, only the classes above the horizontal line occur in arbitrary dimension. For dimension $n \geq 3$, we have the following restrictions on the dimension: in degree 5, one has $2 \leq n \leq 6$. In degree 6, the examples are $(\mathbb{P}^1)^3$ and $\mathrm{F}(1,2,3)$ in dimension 3, and $\mathbb{P}^2 \times \mathbb{P}^2$ in dimension 4. In degree 7, the example occurs only in dimension 3.

For us, the main case is $n = 3$. A Del Pezzo threefold is therefore a smooth Fano threefold X such that $-K_X = 2H_X$. Its degree is $d = H_X^3$. In Picard number one, the possible degrees are $1 \leq d \leq 5$. The corresponding models are:

d	X
1	a sextic hypersurface in $\mathbb{P}(1,1,1,2,3)$
2	a double cover of $\mathbb{P}^3$ branched along a quartic surface
3	a cubic threefold in $\mathbb{P}^4$
4	a complete intersection of two quadrics in $\mathbb{P}^5$
5	a codimension 3 linear section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$.

There are also Del Pezzo threefolds of higher Picard number. As we just mentioned, in degree 6, one has $(\mathbb{P}^1)^3$ and $\mathrm{F}(1,2,3)$, and in degree 7, one has $\mathrm{Bl}_p(\mathbb{P}^3)$.

3.1.C. Prime Fano threefolds of Picard number one

Let us now move to the next case. Namely, we consider smooth Fano threefolds X with

$$\rho_X = 1, \qquad i_X = 1.$$

Thus $\mathrm{Pic}(X) = \mathbb{Z}H_X$, and $-K_X = H_X$. Such a Fano threefold is called a *prime Fano threefold.* The degree of X is H_X^3. It is convenient to write

$$H_X^3 = 2g - 2.$$

The integer g is called the *genus* of X. This terminology comes from the following observation. If $S \in |H_X|$ is a general anticanonical divisor, then S is a K3 surface. If $C \subset S$ is a general hyperplane section, equivalently $C = H_1 \cap H_2$ for general $H_1, H_2 \in |H_X|$, then by adjunction $K_C = H_X|_C$. Moreover $\deg K_C = H_X^3 = 2g - 2$. Thus C is a canonical curve of genus g.

The possible genera are

$$g \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12\}.$$

In particular, genus 11 does not occur.

The classification of prime Fano threefolds is as follows.

g	X
Classical complete-intersection models	
2	sextic hypersurface $V_6 \subset \mathbb{P}(1,1,1,1,3)$
3	quartic hypersurface $V_4 \subset \mathbb{P}^4$
4	complete intersection $V_{2,3} \subset \mathbb{P}^5$
5	complete intersection $V_{2,2,2} \subset \mathbb{P}^6$
Gushel-Mukai model	
6	Gushel-Mukai threefold: $X = \mathrm{Cone}(\mathrm{Gr}(2,5)) \cap Q \cap \mathbb{P}^7$
Homogeneous linear-section models	
7	linear section of $\mathrm{OGr}_+(5,10) \subset \mathbb{P}^{15}$
8	linear section of $\mathrm{Gr}(2,6) \subset \mathbb{P}^{14}$
9	linear section of $\mathrm{LGr}(3,6) \subset \mathbb{P}^{13}$
10	linear section of the adjoint G_2-variety $\mathrm{G}_2/P \subset \mathbb{P}^{13}$
The exceptional genus 12 *model*	
12	zero locus on $\mathrm{Gr}(3,7)$

To explain the range $7 \le g \le 10$, we should describe the rational homogeneous manifolds where X embeds as a linear section.

g	Σ_g	$\dim \Sigma_g$	K_{Σ_g}	$X \subset \Sigma_g$
7	$\mathrm{OGr}_+(5,10) \subset \mathbb{P}^{15}$	10	$-8H$	$\Sigma_7 \cap \mathbb{P}^8$
8	$\mathrm{Gr}(2,6) \subset \mathbb{P}^{14}$	8	$-6H$	$\Sigma_8 \cap \mathbb{P}^9$
9	$\mathrm{LGr}(3,6) \subset \mathbb{P}^{13}$	6	$-4H$	$\Sigma_9 \cap \mathbb{P}^{10}$
10	$\mathrm{G}_2^{\mathrm{ad}} \subset \mathbb{P}^{13}$	5	$-3H$	$\Sigma_{10} \cap \mathbb{P}^{11}$

Here $H = \mathcal{O}_{\Sigma_g}(1)$ is the ample generator of $\mathrm{Pic}(\Sigma_g)$. The varieties appearing in the table are described briefly as follows.

- $\Sigma_7 = \mathrm{OGr}_+(5,10)$ is one connected component of the orthogonal Grassmannian of maximal isotropic 5-planes in a 10-dimensional quadratic vector space. It is the spinor variety.
- $\Sigma_8 = \mathrm{Gr}(2,6)$ is the Grassmannian of 2-planes in a 6-dimensional vector space, embedded by the Plücker embedding.
- $\Sigma_9 = \mathrm{LGr}(3,6)$ is the Lagrangian Grassmannian of 3-dimensional Lagrangian subspaces in a symplectic vector space of dimension 6.
- $\Sigma_{10} = \mathrm{G}_2^{\mathrm{ad}}$ is the adjoint G_2-variety, equivalently the closed G_2-orbit in $\mathbb{P}(\mathfrak{g}_2)$.

More generally, one can consider Fano manifolds X of dimension n and index $i_X = n-2$. Equivalently, their coindex is equal to 3. These are called *Mukai varieties*. Thus, if

$$-K_X = (n-2)H_X,$$

one defines the genus of X by $H_X^n = 2g-2$. The homogeneous varieties Σ_g appearing in the table below are Mukai varieties. Indeed, they have Picard number one and $-K_{\Sigma_g} = (\dim \Sigma_g - 2)H$. Taking a smooth linear section of dimension 3 gives a prime Fano threefold: if $X = \Sigma_g \cap \mathbb{P}^{g+1}, \quad \dim X = 3$, then by adjunction $K_X = \big(K_{\Sigma_g} + (\dim \Sigma_g - 3)H\big)|_X = -H_X$. This explains the homogeneous models of prime Fano threefolds of genera $7, 8, 9, 10$.

3.1.D. Derived categories of Fano threefolds

We now recall some semiorthogonal decompositions for Fano threefolds. The general philosophy is that projective space has a full exceptional collection, while more complicated Fano threefolds have a distinguished residual category. This residual category is usually called the *Kuznetsov component*.

Let us first recall the two basic homogeneous examples.

The projective space. For projective space $\mathbb{P}^n$, we mentioned Beilinson's theorem, which gives

$$\mathrm{D^b}(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}(-n), \mathcal{O}_{\mathbb{P}^n}(-n+1), \ldots, \mathcal{O}_{\mathbb{P}^n} \rangle .$$

Equivalently, using the dual exceptional collection,

$$\mathrm{D^b}(\mathbb{P}^n) = \big\langle \Omega^n_{\mathbb{P}^n}(n), \Omega^{n-1}_{\mathbb{P}^n}(n-1), \ldots, \Omega^1_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n} \big\rangle .$$

Quadrics. Let $Q_m \subset \mathbb{P}^{m+1}$ be a smooth quadric hypersurface. Kapranov constructed full exceptional collections on Q_m. If $m = 2l+1$ is odd, there is one spinor bundle $\mathcal{S}$. This is a vector bundle of rank 2^l, and $\mathcal{S}^\vee$ is generated by 2^{l+1} global sections, which are an irreducible representation of Spin_{2l+3} coming from the Clifford algebra. Kapranov's collection reads

$$\mathrm{D^b}(Q_m) = \langle \mathcal{S}(1-m), \mathcal{O}_{Q_m}(1-m), \ldots, \mathcal{O}_{Q_m} \rangle .$$

If $m = 2l$ is even, there are two half-spinor bundles $\mathcal{S}_+$ and $\mathcal{S}_-$, both of rank 2^{l-1}, with $\mathcal{S}_+^\vee$ and $\mathcal{S}_-^\vee$ generated by 2^l global sections. Then

$$\mathrm{D^b}(Q_m) = \langle \mathcal{S}_+(1-m), \mathcal{S}_-(1-m), \mathcal{O}_{Q_m}(1-m), \ldots, \mathcal{O}_{Q_m} \rangle .$$

For instance, for the quadric threefold Q_3, this gives

$$\mathrm{D^b}(Q_3) = \langle \mathcal{S}(-2), \mathcal{O}_{Q_3}(-2), \mathcal{O}_{Q_3}(-1), \mathcal{O}_{Q_3} \rangle .$$

Del Pezzo threefolds. Let $Y = Y_d$ be a del Pezzo threefold, so that

$$-K_Y = 2H_Y, \qquad d = H_Y^3.$$

For such a threefold, one has a natural semiorthogonal decomposition

$$\mathrm{D^b}(Y) = \langle \mathcal{A}, \mathcal{O}_Y(-1), \mathcal{O}_Y \rangle ,$$

where $\mathcal{A}$ is the full triangulated subcategory of $\mathrm{D^b}(Y)$ defined as

$$\mathcal{A} = \langle \mathcal{O}_Y(-1), \mathcal{O}_Y \rangle^\perp = \{\mathcal{F} \in \mathrm{D^b}(Y) \mid \mathrm{RHom}_Y(\mathcal{O}_Y(-1), \mathcal{F}) = \mathrm{RHom}_Y(\mathcal{O}_Y, \mathcal{F}) = 0\}.$$

For convenience, we often use slightly different semiorthogonal decompositions, mostly

$$\mathrm{D^b}(Y) = \langle \mathrm{Ku}(Y), \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle ,$$

We call the Kuznetsov component of Y the subcategory $\mathrm{Ku}(Y) = \langle \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle^\perp$. For small degrees, this component, or equivalently $\mathcal{A}$, is usually quite complicated.

In degrees $4, 5$, however, one has more explicit descriptions.

d	$\mathrm{D^b}(Y_d)$
5	$\langle \mathcal{U}_Y, \mathcal{O}_Y, \mathcal{U}_Y^\vee, \mathcal{O}_Y(1) \rangle$
4	$\big\langle \mathrm{D^b}(C_2), \mathcal{O}_Y, \mathcal{O}_Y(1) \big\rangle$

Here Y_5 is the degree 5 del Pezzo threefold, realized as a linear section of $\mathrm{Gr}(2,5)$, and $\mathcal{U}_Y, \mathcal{Q}_Y$ denote the restrictions to Y of the tautological sub-bundle of rank 2 and the tautological quotient bundle of rank 3 on $\mathrm{Gr}(2,5)$, respectively.

In degree 4, the threefold is an intersection of two quadrics, and C_2 is the associated hyperelliptic curve of genus 2.

Fractional Calabi-Yau categories from hypersurfaces. In degree 3, namely for cubic threefolds, the component $\mathrm{Ku}(Y)$ is genuinely non-trivial. To understand this, let us look more broadly at $X = X_d \subset \mathbb{P}^{n+1}$, a smooth hypersurface of degree d, so that $\dim X = n$. If X is Fano, then

$$-K_X = (n+2-d)H_X.$$

Writing $i_X = n+2-d$ for the index, one has the standard semiorthogonal decomposition

$$\mathrm{D^b}(X) = \langle \mathrm{Ku}(X), \mathcal{O}_X, \mathcal{O}_X(1), \ldots, \mathcal{O}_X(i_X-1)\rangle .$$

The residual category $\mathrm{Ku}(X)$ is the Kuznetsov component of X.

A remarkable feature is that $\mathrm{Ku}(X)$ is a fractional Calabi-Yau category. More precisely, if $\mathsf{S}_{\mathrm{Ku}(X)}$ denotes its Serre functor, then

$$\mathsf{S}^d_{\mathrm{Ku}(X)} \simeq [(n+2)(d-2)] .$$

In other words, $\mathrm{Ku}(X)$ has fractional Calabi-Yau dimension

$$\frac{(n+2)(d-2)}{d}.$$

For instance, if $Y \subset \mathbb{P}^4$ is a cubic threefold, then $n=3$ and $d=3$, hence

$$\mathsf{S}^3_{\mathrm{Ku}(Y)} \simeq [5],$$

so $\mathrm{Ku}(Y)$ is fractional Calabi-Yau of dimension $5/3$. By contrast, for a cubic fourfold one gets dimension 2, which is why its Kuznetsov component is often called a K3 category.

Prime Fano threefolds. Let now $X = X_g$ be a prime Fano threefold of genus g. Thus

$$\mathrm{Pic}(X) = \mathbb{Z}H_X, \qquad -K_X = H_X, \qquad H_X^3 = 2g-2.$$

For $g = 7, 8, 9, 10, 12$, the derived category admits especially useful semiorthogonal decompositions. The rough picture is as follows:

g	residual category
12	trivial: X has a full exceptional collection
10	$\mathrm{D^b}(C_2)$
9	$\mathrm{D^b}(C_3)$
8	$\mathrm{Ku}(Y_3)$, where Y_3 is an associated cubic threefold
7	$\mathrm{D^b}(C_7)$.

Here C_i denotes a smooth curve of genus i.

More explicitly, in genus 12, one has a full exceptional collection

$$\mathrm{D^b}(X_{12}) = \langle \mathcal{U}_X^\vee(-1), \mathcal{F}_0, \mathcal{U}_X, \mathcal{O}_X\rangle .$$

Here $\mathcal{U}_X$ and $\mathcal{Q}_X$ are the tautological sub-bundle of rank 3 and the tautological quotient bundle of rank 4 on $\mathrm{Gr}(3,7)$, respectively, restricted to X_{12}. Also, $\mathcal{F}_0$ is a minimal instanton, namely a rank 2 stable bundle on X with $c_1(\mathcal{F}_0) = -H_X$ and $c_2(\mathcal{F}_0)$ as small as possible, in this case $c_2(\mathcal{F}_0) = 8\ell_X$, where ℓ_X is the class of a line contained in X.

For $g = 10$ and $g = 9$, the residual category is the derived category of a curve:

$$\mathrm{D^b}(X_{10}) = \left\langle \mathcal{U}_X^\vee(-1), \mathrm{D^b}(C_2), \mathcal{O}_X \right\rangle ,$$

and

$$\mathrm{D^b}(X_9) = \left\langle \mathcal{U}_X^\vee(-1), \mathrm{D^b}(C_3), \mathcal{O}_X \right\rangle .$$

This is one of the reasons why these Kuznetsov components may be thought of as *curvilinear*: the interesting part of the derived category is controlled by a curve.

For $g = 7$, there is a similar phenomenon:

$$\mathrm{D^b}(X_7) = \left\langle \mathcal{U}_X^\vee(-1), \mathrm{D^b}(C_7), \mathcal{O}_X \right\rangle ,$$

where C_7 is the homological projective dual curve associated with X_7. In all these cases, the vector bundle $\mathcal{U}_X$ is the restriction to X of the tautological sub-bundle on the rational homogeneous variety Σ_g that contains X as a linear section.

A Pfaffian bridge between X_8 and cubic threefolds. The genus 8 case is different and will be especially important for instantons. If X_8 is a prime Fano threefold of genus 8, then we have

$$\mathrm{D}^{\mathrm{b}}(X_8) = \langle \mathcal{U}_X, \mathrm{Ku}(X_8), \mathcal{O}_X \rangle .$$

Now, X_8 has an associated cubic threefold Y_3, and one has an equivalence of Kuznetsov components

$$\mathrm{Ku}(X_8) \simeq \mathrm{Ku}(Y_3).$$

Here

$$\mathrm{D}^{\mathrm{b}}(Y_3) = \langle \mathrm{Ku}(Y_3), \mathcal{O}_{Y_3}, \mathcal{O}_{Y_3}(1) \rangle .$$

This equivalence is a manifestation of homological projective duality for $\mathrm{Gr}(2,6)$ and the Pfaffian cubic hypersurface. It is also the categorical explanation for several geometric links between X_8 and Y_3. Thus the genus 8 prime Fano threefold and the associated cubic threefold share the same non-trivial categorical core.

To see how this goes, let $Y = Y_3 \subset \mathbb{P}^4$ be a cubic threefold. A rank two instanton bundle $\mathcal{E}_0$ of minimal charge on Y, namely, with $c_2(\mathcal{E}_0) = 2\ell_Y$, is an Ulrich bundle. We will see more on them later on.

The bundle $\mathcal{E}_0$ gives a skew-symmetric resolution on $\mathbb{P}^4$ of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^4}^{\oplus 6} \longrightarrow \mathcal{E}_0(1) \longrightarrow 0.$$

The map φ is represented by a 6×6 skew-symmetric matrix of linear forms, and its Pfaffian cuts out the cubic threefold:

$$\mathrm{Pf}(\varphi) = Y.$$

Equivalently, if V_6 is a vector space of dimension 6, then such a skew-symmetric matrix of linear forms may be viewed as a linear subspace

$$V_5 \subset H^0\big(\mathbb{P}(\wedge^2 V_6), \mathcal{O}(1)\big) = \wedge^2 V_6^\vee .$$

The Grassmannian $\mathrm{Gr}(2, V_6) \subset \mathbb{P}(\wedge^2 V_6)$ has a codimension 5 dual (or "orthogonal") linear section

$$X_8 = \mathrm{Gr}(2, V_6) \cap \mathbb{P}^9,$$

which is a prime Fano threefold of genus 8.

This way, a Pfaffian description of the cubic threefold naturally produces the corresponding genus 8 Fano threefold. This can be used to show the equivalence

$$\mathrm{Ku}(X_8) \simeq \mathrm{Ku}(Y_3).$$

In terms of moduli space, we get a correspondence, originally observed in [MT01].

Mukai bundles and minimal instantons Let $X = X_g$ be a prime Fano threefold of genus g, so that

$$\mathrm{Pic}(X) = \mathbb{Z} H_X, \qquad -K_X = H_X.$$

On such threefolds, Mukai constructed special stable vector bundles with many sections. A caveat here is that the construction of Mukai is totally rigorous only when one restricts to a hyperplane section surface S of X. This is a K3 surface and the Mukai bundle exists on S. But the lifting to X is not entirely clear, and this is actually the main focus of [BKM25].

More precisely, when the integer g admits a factorization

$$g = rs,$$

then one finds a stable vector bundle $\mathcal{F}$ of rank r, with

$$c_1(\mathcal{F}) = -H_X,$$

and with $\mathcal{F}^\vee$ generated by $r+s$ global sections. The evaluation map gives a morphism from X to a Grassmannian, which is often a closed immersion. Some important examples are:

g	$\mathrm{rk}(\mathcal{F})$	$h^0(X, \mathcal{F}^\vee)$	Grassmannian model
9	3	6	$X \subset \mathrm{Gr}(3,6)$
10	2	7	$X \subset \mathrm{Gr}(2,7)$
12	2	8	$X \subset \mathrm{Gr}(2,8)$
12	3	7	$X \subset \mathrm{Gr}(3,7)$.

The genus 7 case is more special; it is related to the spinor variety

$$\mathrm{OGr}_+(5,10) \subset \mathbb{P}^{15}.$$

Warning 3.4. There are two common sign conventions here. Mukai's globally generated bundle is often written with

$$c_1(\mathcal{F}^\vee) = H_X.$$

In the semiorthogonal decompositions, however, one usually uses the normalized bundle $\mathcal{F}$. Thus $\mathcal{F}$ has $c_1(\mathcal{F}) = -H_X$, which is the normalization expected for an instanton bundle on an index-one Fano threefold.

In particular, in the rank two cases, the normalized Mukai bundle $\mathcal{F}$ is a basic example of an instanton bundle. It is the *minimal instanton* on X. We denote it by $\mathcal{F}_0$.

3.2. Instanton bundles on Fano threefolds

3.2.A. Instanton bundles on Fano threefolds

This part is based on [Kuz12, CF25, Fae14]. We now introduce instanton bundles on Fano threefolds. The guiding example is the classical notion on $\mathbb{P}^3$. One of the closest definitions to Atiyah–Ward's instantons is that a mathematical instanton bundle is a semistable rank two vector bundle $\mathcal{E}$ such that

$$c_1(\mathcal{E}) = 0, \qquad c_3(\mathcal{E}) = 0, \qquad H^1(\mathbb{P}^3, \mathcal{E}(-2)) = 0.$$

We would like to generalize this vanishing to arbitrary Fano threefolds. Let X be a smooth Fano threefold with Picard number one, and write $-K_X = i_X H_X$, where H_X is the ample generator of $\mathrm{Pic}(X)$. It is convenient to write

$$i_X = 2q_X + \epsilon_X, \qquad \epsilon_X \in \{0,1\}.$$

Therefore:

X	i_X	q_X	ϵ_X
$\mathbb{P}^3$	4	2	0
Q^3	3	1	1
Y_d	2	1	0
X_g	1	0	1

The integer q_X is the integral part of one half of the index. It gives the twist appearing in the instanton vanishing.

Definition 3.5. Let X be a smooth Fano threefold with Picard number one and let H_X be the ample generator of $\mathrm{Pic}(X)$. A coherent sheaf $\mathcal{E}$ on X is said to satisfy the *instanton vanishing* if

$$H^1(X, \mathcal{E}(-q_X)) = 0.$$

This vanishing alone is not enough. One also has to impose the correct normalization on the Chern character. The uniform way to write it is the following.

Definition 3.6. Let X be as above. A coherent sheaf $\mathcal{E}$ is called *numerically self-dual* if

$$\mathrm{ch}(\mathcal{E}) = \mathrm{ch}\big(\mathrm{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{O}_X(-\epsilon_X))\big).$$

If $\mathcal{E}$ is locally free, this condition becomes

$$\mathrm{ch}(\mathcal{E}) = \mathrm{ch}\big(\mathcal{E}^\vee(-\epsilon_X)\big).$$

Warning 3.7. The condition

$$\mathrm{ch}(\mathcal{E}) = \mathrm{ch}\big(\mathcal{E}^\vee(-\epsilon_X H_X)\big)$$

is only a numerical self-duality condition. It does not say that $\mathcal{E}$ is isomorphic to $\mathcal{E}^\vee(-\epsilon_X H_X)$. In many geometric situations one does have an actual self-duality, but this is an additional structure.

Let us spell out what this means in rank two. If $\mathcal{E}$ has rank 2, then the equality of first Chern classes gives

$$c_1(\mathcal{E}) = -\epsilon_X H_X.$$

Thus:

X	ϵ_X	$c_1(\mathcal{E})$
$\mathbb{P}^3$	0	0
Q^3	1	$-H_X$
Y_d	0	0
X_g	1	$-H_X$.

In particular, for even index Fano threefolds one obtains the familiar normalization $c_1(\mathcal{E}) = 0$, while for odd index one obtains $c_1(\mathcal{E}) = -H_X$.

Definition 3.8. An *instanton bundle* on X is a torsion-free sheaf $\mathcal{E}$ satisfying:

F.i) Topological classification: $\mathcal{E}$ is numerically self-dual, namely

$$\mathrm{ch}(\mathcal{E}) = \mathrm{ch}\big(\mathcal{E}^\vee(-\epsilon_X)\big);$$

F.ii) Semistability: $\mathcal{E}$ is slope-semistable with respect to H_S, when restricted to some surface S, disjoint from the non-locally-free locus of $\mathcal{E}$;

F.iii) Cohomological condition: $\mathcal{E}$ satisfies the instanton vanishing

$$\mathrm{H}^1(X, \mathcal{E}(-q_X)) = 0.$$

The letter F in the above numbering stands for Fano.

Remark 3.9. For $X = \mathbb{P}^3$, one has $q_X = 2$ and $\epsilon_X = 0$. Therefore the above conditions become

$$\mathrm{H}^1(\mathbb{P}^3, \mathcal{E}(-2)) = 0, \qquad \mathrm{ch}(\mathcal{E}) = \mathrm{ch}(\mathcal{E}^\vee).$$

The topological condition is equivalent to:

$$c_1(\mathcal{E}) = 0, \qquad c_3(\mathcal{E}) = 0.$$

Thus we recover the usual numerical conditions in the definition of classical instantons on $\mathbb{P}^3$.

Remark 3.10. For a del Pezzo threefold Y_d, one has $q_X = 1$ and $\epsilon_X = 0$. Thus a rank two instanton bundle has

$$c_1(\mathcal{E}) = 0, \qquad \mathrm{H}^1(Y_d, \mathcal{E}(-1)) = 0.$$

We recover here the definition given in [Kuz12].

Remark 3.11. For a prime Fano threefold $X = X_g$, one has $q_X = 0$ and $\epsilon_X = 1$. Thus a rank two instanton bundle has

$$c_1(\mathcal{E}) = -H_X, \qquad \mathrm{H}^1(X, \mathcal{E}) = 0.$$

In loc. cit., Kuznetsov states an interesting conjecture, to the effect that a rank 2 instanton bundle $\mathcal{E}$ on Y_d should have trivial generic splitting, which is to say, $\mathcal{E}$ should admit a line $\ell \subset Y$ such that $\mathcal{E}$ restricts to ℓ as the trivial bundle of rank 2. However, this is known only for small charge.

3.2.B. Topological classification of instantons

More generally, one can package the Chern character of an instanton bundle in terms of this minimal object. Let $\ell \subset X$ be a line. Then for every instanton bundle $\mathcal{E}$, there exist unique integers r and k such that

$$\mathrm{ch}(\mathcal{E}) = r\,\mathrm{ch}(\mathcal{F}_0) - k\,\mathrm{ch}\big(\mathcal{O}_\ell(q_X - 1)\big).$$

The integer k is called the *charge* of $\mathcal{E}$. Here $\mathcal{F}_0$ is the minimal instanton. When i_X is even, $\mathcal{F}_0$ is just $\mathcal{O}_X$, while when i_X is odd $\mathcal{F}_0$ is the unique instanton of rank 2, $c_1(\mathcal{F}_0) = -H_X$ and $c_2(\mathcal{F}_0)$ as low as possible.

$$\mathcal{F}_0 = \begin{cases} \mathcal{O}_X, & \text{if } i_X \text{ is even,} \\ \mathcal{S}, & \text{if } X = Q^3, \end{cases}$$

where $\mathcal{S}$ is the spinor bundle on the quadric threefold. Recall that in our convention, the slope of an instanton bundle is

$$\mu(\mathcal{E}) = -\frac{\epsilon_X}{2}.$$

Thus $\mu(\mathcal{E}) = 0$ if i_X is even, while $\mu(\mathcal{E}) = -\frac{1}{2}$ if i_X is odd. For example, on $\mathbb{P}^3$, this recovers the familiar formula

$$\mathrm{ch}(\mathcal{E}) = \mathrm{rk}(\mathcal{E}) - k\,\mathrm{ch}(\mathcal{O}_\ell(1)) = r - kH^2.$$

3.2.C. The categorical viewpoint

Let X be a smooth Fano threefold with $\rho_X = 1$, and let $\mathcal{E}$ be an (r,k)-instanton sheaf on X. Recall that this means, in particular, that

$$\mathrm{ch}(\mathcal{E}) = r\,\mathrm{ch}(\mathcal{F}_0) - k\,\mathrm{ch}\big(\mathcal{O}_\ell(q_X - 1)\big),$$

where $\ell \subset X$ is a line and $\mathcal{F}_0$ is the minimal instanton on X. Now, if $\mathcal{E}$ is slope-semistable and satisfies the instanton vanishing

$$\mathrm{H}^1(X, \mathcal{E}(-q_X)) = 0$$

The instanton vanishing and the orthogonal category. One first checks that, with the numerical invariants at hand, $\chi(\mathcal{E}(-q_X)) = 0$. Indeed, combining Serre duality with $\omega_X \simeq \mathcal{O}_X(-2q_X - \epsilon_X)$ and with the instanton condition on $\mathrm{ch}(\mathcal{E})$, we have

$$\chi(\mathcal{E}(-q_X)) = -\chi(\mathrm{R}\mathcal{H}om_X(\mathcal{E}(-q_X), \omega_X)) = -\chi(\mathrm{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{O}_X(-q_X - \epsilon_X))) = -\chi(\mathcal{E}(-q_X)).$$

Then, together with semistability and Serre duality, the instanton vanishing implies

$$\mathrm{H}^*(X, \mathcal{E}(-q_X)) = 0.$$

Equivalently,

$$\mathrm{RHom}_X(\mathcal{O}_X(q_X), \mathcal{E}) = 0.$$

Thus

$$\mathcal{E} \in \mathcal{O}_X(q_X)^\perp \subset \mathrm{D^b}(X).$$

This is the basic categorical observation: instanton sheaves do not live randomly in $\mathrm{D^b}(X)$; they naturally belong to the right orthogonal of the exceptional object $\mathcal{O}_X(q_X)$.

3.2.D. Lower bounds on the charge for existence

For Fano threefolds, we constructed in [CF25] instantons (for Definition 3.8) on smooth Fano threefolds of Picard number one for all possible values of r, k. One has to have $k \geq k_0(r)$ with:

- for $i_X = 4$, so for $X = \mathbb{P}^3$, $k_0(r) = \lceil r/2 \rceil$,
- for $i_X = 3$, so for $X = Q_3$, $k_0(r) = \lceil r/3 \rceil$,
- for $i_X = 2$, so X is a Del Pezzo threefold, $k_0(r) = r$,
- for $i_X = 1$, so X is a prime Fano threefold of genus g, $k_0(1) = 0$ and $k_0^r = 1$ when g is odd, while $k_0^r = r$ when g is even.

Proposition 3.12. *For all (r, k) with $k \geq k_0(r)$, the moduli space $\mathfrak{MI}_X(r, k)$ has an irreducible component which is generically smooth, of the expected dimension, with expected generic splitting.*

Remark 3.13. The restriction map $\mathfrak{MI}_X(n, k) \dashrightarrow \mathfrak{M}_S(\mathbf{v}(n, k))$ takes an (n, k)-instanton on X to a semistable sheaf on the anticanonical divisor S, which is a K3 surface. The Mukai vector $\mathbf{v}$ depends on (n, k). The image is a Lagrangian subvariety of $\mathfrak{M}_S(n, k)$. In some nice cases this is (generically) a Lagrangian fibration, e.g. when S is a quadric section of a cubic threefold X and $(n, k) = (2, 0)$ so $\mathbf{v} = (2, 0, 2)$.

3.2.E. Instanton monads on Fano threefolds.

Suppose now that $\mathrm{D^b}(X)$ has a full exceptional collection.

Remark 3.14. The existence of a full exceptional collection is a very restrictive property. If a smooth projective variety X has a full exceptional collection of length t, then

$$\mathrm{H}^{p,q}(X) = 0 \qquad \text{for } p \neq q,$$

and $t = \sum_p \mathrm{h}^{p,p}(X)$. For a Fano threefold this forces, in particular, $\mathrm{H}^3(X, \mathbb{C}) = 0$.

Thus only very special Fano threefolds can have full exceptional collections. For Picard number one, this happens precisely for

$$\mathbb{P}^3, \qquad Q^3, \qquad Y_5, \qquad X_{12}.$$

As we said, it turns out that for these four threefolds we do have a full exceptional collection. Let us make this more precise and provide some explicit collections of a very special form, which moreover satisfies nice symmetry property. We write

$$\mathrm{D^b}(X) = \langle \mathcal{O}_X(q_X)^\perp, \mathcal{O}_X(q_X) \rangle.$$

Then $\mathcal{O}_X(q_X)^\perp$ admits a full exceptional collection. There exists a decomposition of the form:

$$\mathcal{O}_X(q_X)^\perp = \langle \mathcal{F}_{-1}, \mathcal{F}_0, \mathcal{F}_1 \rangle .$$

where, for $-1 \leq i \leq 1$, we have

$$\mathcal{F}_i^\vee \simeq \mathcal{F}_{-i}(\epsilon_X). \tag{17}$$

We have the following description of $\mathcal{O}_X(q_X)^\perp$.

X	q_X	$\mathcal{O}_X(q_X)^\perp$
$\mathbb{P}^3$	2	$\langle \mathcal{O}_{\mathbb{P}^3}(-1), \mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}(1)\rangle$
Q^3	1	$\left\langle \mathcal{O}_{Q^3}(-1), \mathcal{S}, \mathcal{O}_{Q^3}\right\rangle$
Y_5	1	$\langle \mathcal{U}_Y, \mathcal{O}_Y, \mathcal{U}_X^\vee\rangle$
X_{12}	0	$\langle \mathcal{U}_X^\vee(-1), \mathcal{F}_0, \mathcal{U}_X\rangle$.

Here $\mathcal{S}$ is the spinor bundle on Q^3, the bundles $\mathcal{U}_Y$ is the restrictions of the tautological rank-2 sub-bundle from $\mathrm{Gr}(2,5)$, $\mathcal{F}_0$ is the minimal instanton, or Mukai bundle, of rank 2 (or rather its dual) on X_{12}, while $\mathcal{U}_X$ is the (dual of the) Mukai bundle of rank 3 on X_{12}, which is the restriction of the tautological sub-bundle of rank 3 on $\mathrm{Gr}(3,7)$.

Then every instanton sheaf $\mathcal{E}$ can be written in terms of the exceptional objects $\mathcal{F}_{-1}, \mathcal{F}_0, \mathcal{F}_1$. More precisely, we have a dual exceptional collection: $\langle \mathcal{G}_{-1}, \mathcal{G}_0, \mathcal{G}_1\rangle$. We will recall the precise form of these collection in a minute.

Then the Beilinson-type spectral sequence expresses $\mathcal{E}$ in terms of $\mathrm{H}^*(\mathcal{G}_i \otimes \mathcal{E})$. We write down the cohomology table of $\mathcal{E}$ with respect to the collection mentioned above:

$\mathcal{F}_{-1}$	$\mathcal{F}_0$	$\mathcal{F}_1$
$\mathrm{h}^3(\mathcal{E}\otimes\mathcal{G}_{-1})$	$\mathrm{h}^3(\mathcal{E}\otimes\mathcal{G}_0)$	$\mathrm{h}^3(\mathcal{E}\otimes\mathcal{G}_1)$
$\mathrm{h}^2(\mathcal{E}\otimes\mathcal{G}_{-1})$	$\mathrm{h}^2(\mathcal{E}\otimes\mathcal{G}_0)$	$\mathrm{h}^2(\mathcal{E}\otimes\mathcal{G}_1)$
$\mathrm{h}^1(\mathcal{E}\otimes\mathcal{G}_{-1})$	$\mathrm{h}^1(\mathcal{E}\otimes\mathcal{G}_0)$	$\mathrm{h}^1(\mathcal{E}\otimes\mathcal{G}_1)$
$\mathrm{h}^0(\mathcal{E}\otimes\mathcal{G}_{-1})$	$\mathrm{h}^0(\mathcal{E}\otimes\mathcal{G}_0)$	$\mathrm{h}^0(\mathcal{E}\otimes\mathcal{G}_1)$

We prove that here this takes the form

$\mathcal{F}_{-1}$	$\mathcal{F}_0$	$\mathcal{F}_1$
0	0	0
0	0	0
$\mathrm{h}^1(\mathcal{E}\otimes\mathcal{G}_{-1})$	$\mathrm{h}^1(\mathcal{E}\otimes\mathcal{G}_0)$	$\mathrm{h}^1(\mathcal{E}\otimes\mathcal{G}_1)$
0	0	0

The dimension of each of these vector spaces is responsible for the multiplicity of the object $\mathcal{F}_i$ in the Beilinson-type resolution of $\mathcal{E}$.

The dual collection. Let us recall how the dual collection is obtained. If $A, B \in \mathrm{D^b}(X)$, the left mutation $\mathbf{L}_A B$ is defined by the distinguished triangle

$$\mathrm{RHom}(A,B)\otimes A \to B \to \mathbf{L}_A B,$$

where the map is given by (derived) evaluation of morphisms. There is also a right mutation $\mathbf{R}$, which can be described as cone of the natural coevaluation of maps, namely

$$\mathbf{R}_B A \to \mathrm{RHom}(A,B)^\vee \otimes B \to A$$

For a length-3 exceptional collection

$$\langle \mathcal{F}_{-1}, \mathcal{F}_0, \mathcal{F}_1\rangle ,$$

the dual collection is obtained by iterated mutations. One has:

$$\mathcal{G}_1 = \mathcal{F}_1^\vee, \qquad \mathcal{G}_0 = \mathbf{L}_{\mathcal{F}_1^\vee}(\mathcal{F}_0^\vee)[-1], \qquad \mathcal{G}_{-1} = \mathbf{L}_{\mathcal{F}_1^\vee}\mathbf{L}_{\mathcal{F}_0^\vee}(\mathcal{F}_{-1}^\vee)[-2].$$

A nice property of this collection is that, for $-1 \le i \le 1$, we have

$$\mathcal{G}_{-i}^\vee(-q_X) \simeq \mathbf{L}_{\mathcal{O}_X}(\mathcal{G}_i(q_X))[-1]. \tag{18}$$

Explicitly, we have

X	$\mathcal{G}_{-1}$	$\mathcal{G}_0$	$\mathcal{G}_1$
$\mathbb{P}^3$	$\mathrm{T}_{\mathbb{P}^3}(-3)$	$\Omega_{\mathbb{P}^3}$	$\mathcal{O}_{\mathbb{P}^3}(-1)$
Q_3	$\mathrm{T}_{\mathbb{P}^4}\vert_{Q_3}(-2)$	$\mathcal{S}$	$\mathcal{O}_{Q_3}$
Y_5	$\mathcal{Q}_Y(-1)$	$\mathcal{R}_Y$	$\mathcal{U}_Y$
X_{12}	$\mathcal{Q}_X$	$\mathcal{R}_X$	$\mathcal{U}_X^\vee$

Here, $\mathcal{Q}_Y$ is the tautological quotient bundle of rank 3 on $\mathrm{Gr}(2,5)$, restricted to $Y = Y_5$, while $\mathcal{R}_Y$ is a rank-9 bundle fitting into

$$0 \to \mathcal{R}_Y \to \mathcal{U}_Y^{\oplus 5} \to \mathcal{O}_Y \to 0.$$

Similarly, for $X = X_{12}$, $\mathcal{Q}_X$ is the tautological quotient bundle of rank 4 on $\mathrm{Gr}(3,7)$, restricted to X, while $\mathcal{R}_X$ is a rank-10 bundle fitting into an exact sequence

$$0 \to \mathcal{R}_X^\vee \to (\mathcal{U}_X^\vee)^{\oplus 4} \to \mathcal{F}_0^\vee \to 0.$$

Monad-theoretic description of instanton moduli. For any admissible pair (r,k), i.e., $k \geq k_0(r)$, the moduli space of (r,k)-instantons can be described in terms of monads of the form

$$I_{-1} \otimes \mathcal{F}_1 \xrightarrow{B} I_0 \otimes \mathcal{F}_0 \xrightarrow{A} I_1 \otimes \mathcal{F}_1,$$

where

$$\dim(I_{-1}) = \dim(I_1) = k.$$

Theorem 3.15. *The moduli space of semistable instantons $\mathfrak{MI}_X^{\mathrm{s}}(r,k)$ has a rational map toward*

$$\{(A,B) \mid A \circ B = 0,\ B \textit{ is injective},\ A \textit{ is surjective}\} /\!\!/_\theta\, G,$$

where $G = \mathrm{GL}(I_{-1}) \times \mathrm{GL}(W) \times \mathrm{GL}(I_1)$ is acts naturally on the vector spaces I_{-1}, I_0, I_1.

The group G acts on the closed subset of

$$\mathbf{M} = I_{-1}^\vee \otimes I_0 \otimes V \oplus I_0^\vee \otimes I_1 \otimes V,$$

where $V = \mathrm{Hom}_X(\mathcal{F}_{-1}, \mathcal{F}_0) \simeq \mathrm{Hom}_X(\mathcal{F}_0, \mathcal{F}_1)$. The group G also acts on the closed subset consisting of pairs (A,B) with $A \in I_{-1}^\vee \otimes W \otimes V$ and $B \in W \otimes I_1^\vee \otimes V$. In the quotient appearing in the statement above, we are considering the G-semisstable locus for the usual character θ of G. Given matrices A, B as above, the corresponding instanton is the middle cohomology

$$\mathcal{E}_{A,B} := \mathrm{Ker}(A)/\mathrm{Im}(B).$$

The dimension of I_0, for an (r,k)-instanton on a Fano threefold of index i_X is

i_X	$\dim(I_0)$
4	$2k+r$
3	$k+r$
2	$4k+r$
1	$3k+r$

(19)

To have an isomorphism of these moduli spaces, one would need to check that the stability conditions are exactly the same, but this has not been studied in detail until now.

3.2.F. Curvilinear Kuznetsov categories

Let us look at Fano threefolds X whose Kuznetsov category is equivalent to the derived category of a curve. In this case we have a curve C and a fully faithful functor $\Phi : \mathrm{D^b}(C) \to \mathrm{D^b}(X)$. This functor is given by a Fourier-Mukai kernel, which means that there is an object $\mathcal{K}$ of $\mathrm{D^b}(C \times X)$ such that

$$\Phi(\mathcal{E}) = \mathrm{R}q_*(p^*(\mathcal{E}) \otimes \mathcal{K}),$$

where p and q are the projection maps from $C \times X$ to C and X. It turns out that $\mathcal{K}$ is a vector bundle. The functor Φ has a left adjoint Φ^* and a right adjoint $\Phi^!$. Recall that we wrote

$$\mathcal{O}_X(q_X)^\perp = \langle \mathcal{G}, \Phi(\mathrm{D^b}(C)) \rangle.$$

where $\mathcal{G}$ is an exceptional bundle, C is a smooth projective curve, and $\Phi : \mathrm{D^b}(C) \to \mathrm{D^b}(X)$ is fully faithful. We have $\mathcal{G} = \mathcal{O}_X$ when X has index 2 and $\mathcal{G} = \mathcal{U}_X^\vee(-1)$ when $i_X = 1$, where $\mathcal{U}_X$ is the restriction of the tautological bundle on a Grassmann manifold where X sits naturally.

Let $\mathcal{E}$ be a slope-stable (r,k)-instanton sheaf on X. Then we have

$$\mathrm{ext}^1_X(\mathcal{E}, \mathcal{G}) = k, \qquad \mathrm{Ext}^p_X(\mathcal{E}, \mathcal{G}) = 0, \qquad \text{for } p \neq 1.$$

Therefore, we have a tautological extension

$$0 \to \mathcal{G}^{\oplus k} \to \tilde{\mathcal{E}} \to \mathcal{E} \to 0. \tag{20}$$

The tautological extension can be thought of as the right mutation of $\mathcal{E}$ through $\mathcal{G}$, see § 3.2.E, or in other words, the projection of $\mathcal{E}$ to the subcategory ${}^{\perp}\mathcal{G}$, as the mutation sequence reads:

$$0 \to \mathcal{H}^{-1}\left(\mathrm{RHom}_X(\mathcal{E},\mathcal{G})^\vee \otimes \mathcal{G}\right) \to \mathbf{R}_{\mathcal{G}}(\mathcal{E}) \to \mathcal{E} \to 0,$$

which agrees with (20). So $\tilde{\mathcal{E}}$ lies in $\Phi(\mathrm{D^b}(C))$ and therefore we have $\tilde{\mathcal{E}} \simeq \Phi(E_{\mathcal{E}})$ for some $E_{\mathcal{E}}$ in $\mathrm{D^b}(C)$. However, the right adjoint functor $\Phi^!$ of Φ vanishes on $\mathcal{G}$, since for all objects a of $\mathrm{D^b}(C)$

$$\mathrm{Hom}_C(a, \Phi^!(\mathcal{G})) = \mathrm{Hom}_X(\Phi(a), \mathcal{G}) = 0.$$

Then we get $\Phi^!(\mathcal{E}) \simeq \Phi^!(\tilde{\mathcal{E}}) \simeq \Phi^!(\Phi(E_{\mathcal{E}})) \simeq E_{\mathcal{E}}$. In conclusion we have a canonical exact sequence:

$$0 \to \mathcal{G}^{\oplus k} \to \Phi(E_{\mathcal{E}}) \to \mathcal{E} \to 0, \qquad \text{with } E_{\mathcal{E}} = \Phi^!(\mathcal{E}). \tag{21}$$

Using the left adjoint Φ^* of Φ, write $\mathcal{V} = \Phi^*(\mathcal{G})$. Applying $\mathrm{Hom}(\mathcal{G}, -)$ to (21), we find

$$\begin{aligned} \mathrm{Hom}_C(\mathcal{V}, E_{\mathcal{E}}) &= \mathrm{Hom}_X(\mathcal{G}, \Phi(E_{\mathcal{E}})) \simeq \mathrm{Ext}^1_X(\mathcal{E},\mathcal{G})^\vee, \\ \mathrm{Ext}^1_C(\mathcal{V}, E_{\mathcal{E}}) &= \mathrm{Ext}^1_X(\mathcal{G},\mathcal{E}). \end{aligned}$$

Lemma 3.16. *The objects $E_{\mathcal{E}}$ and $\mathcal{V}$ are vector bundles on C.*

The proof of the lemma is rather formal and is based on mutations. Then $E_{\mathcal{E}}$ lies in the moduli space of vector bundles E on C with $\mathrm{ch}(E) = \mathrm{ch}(E_{\mathcal{E}})$ and with $\hom(\mathcal{V}, E) = \mathrm{ext}^1_X(\mathcal{E},\mathcal{G})$, which is a Brill–Noether locus. Given a vector bundle E in this locus, using the Fourier–Mukai kernel $\mathcal{K}$, we see that, if $\mathrm{H}^1(E \otimes \mathcal{K}_x) = 0$ for all $x \in X$, then $\Phi(E)$ gives rise canonically to an exact sequence as in (21) and the associated term $\mathcal{E}$ will be a sheaf having the correct Chern character and lying in $\mathcal{O}_X^{\perp}$. For rank 2, one can say a bit more, using the inner duality of rank-2 bundles, which allows one to see that E is a simple bundle.

This can be used to define a birational map from a component of $\mathfrak{M}_X(r,k)$ to a Brill-Noether locus of stable vector bundles E on the curve C with $\mathrm{ch}(E) = \mathrm{ch}(E_{\mathcal{E}})$ and with $\hom(\mathcal{V}, E) = \mathrm{ext}^1_X(\mathcal{E},\mathcal{G})$. However, this has been written only for rank 2.

4. Higher-dimensional Fano varieties and Kuznetsov components

4.1. Instanton sheaves on a projective variety

For more general varieties, there is a definition in [AC25].

Definition 4.1. Given a polarized variety (X, H), $\mathcal{E}$ is a k-instanton sheaf if

$$\begin{aligned} \mathrm{h}^{i-1}(\mathcal{E}(-i)) &= \mathrm{h}^{i+1}(\mathcal{E}(-i-1)) = 0, && \text{if } 1 \leq i \leq n-1, \\ \mathrm{h}^{1}(\mathcal{E}(-1)) &= \mathrm{h}^{n-1}(\mathcal{E}(-n)) = k. \end{aligned}$$

So the cohomology table of a k-instanton bundle reads:

values of $\mathrm{h}^p(\mathcal{E}(t))$

$p\backslash t$	$\cdots$	$-n$	$1-n$	$\cdots$	-1	0
n	$*$	0				
$n-1$	0	k	0	0	0	
$\vdots$	0	0	$\ddots$	$\ddots$	0	
1	0	0	0	0	k	
0					0	$*$

Question 4.2. *Determine for which values of a Mukai vector* $\mathbf{v} \in \mathrm{H}^*(X, \mathbb{Z})$ *there are instanton sheaves with* $\mathrm{ch}(\mathcal{E}) = \mathbf{v}$ *on a given variety* X*.*

Notice that there are instantons according to Definition 3.8 which do not satisfy Definition 4.1. Also k is not the same.

4.1.A. Reminders about Ulrich bundles

Ulrich sheaves are characterized by some extremal cohomological properties. Introduced by Eisenbud-Schreyer, they are related to Castelnuovo-Mumford regularity, Chow forms, determinantal representations of hypersurfaces, Boij-Söderberg theory. They are studied using deformation theory, representations of algebraic groups, push-forwards and other functorial techniques, Brill-Noether theory, Serre's correspondence and low-codimensional subvarieties. The main connection with instantons is that, according to Definition 4.1 and to point c) below, we have:

Ulrich sheaves are instantons of charge 0.

Let us defined Ulrich sheaves. Let X be an n-dimensional projective scheme on a field, polarized by a very ample divisor H and write $\mathcal{O}_X(H) = \mathcal{O}_X(1)$ so that $X \subset \mathbb{P}^N = \mathbb{P}(\mathrm{H}^0(X, \mathcal{O}_X(1)))$.

Definition 4.3. A coherent sheaf $\mathcal{E}$ on X is *Ulrich* with respect to H if

$$\mathrm{H}^*(X, \mathcal{E}(-q)) = 0 \quad \text{for all} \quad 1 \leq q \leq n.$$

Equivalent formulations. The following are equivalent to $\mathcal{E}$ being Ulrich:

a) if $\pi : X \to \mathbb{P}^n$ is finite, then $\pi_*(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^n}^{\oplus s}$ for some $s \in \mathbb{N}$. If X is irreducible, $\mathcal{E}$ has rank r on X and $d = H^n$, then $s = rd$.

b) the sheafified minimal free graded resolution of $\mathcal{E}$ on $\mathbb{P}^N$ is linear of length $N - n$;

c) $\mathrm{h}^i(\mathcal{E}(-i-1)) = 0$ for $0 \leq i \leq n-1$ and $\mathrm{h}^i(\mathcal{E}(-i)) = 0$ for $1 \leq i \leq n$:

table of $\mathrm{h}^p(\mathcal{E}(t))$

$p\backslash t$	$\cdots$	$-n$	$1-n$	$\cdots$	-1	0
n	$*$	0				
$n-1$		0	0			
$\vdots$			$\ddots$	$\ddots$		
1				0	0	
0					0	$*$

d) $\mathcal{E}$ is arithmetically Cohen-Macaulay and $\mathrm{h}^0(\mathcal{E}(-1)) = 0 = \mathrm{h}^n(\mathcal{E}(-n))$.

e) $\mathrm{h}^i(\mathcal{E}(-i-1)) = 0$ for $0 \leq i \leq n-1$ and $\mathrm{h}^i(\mathcal{E}(-i)) = 0$ for $1 \leq i \leq n$:

table of $\mathrm{h}^p(\mathcal{E}(t))$

$p\backslash t$	$\cdots$	$-n$	$1-n$	$\cdots$	-1	0
n	$*$	0				
$n-1$	0	0	0	0	0	
$\vdots$	0	0	$\ddots$	$\ddots$	0	
1	0	0	0	0	0	
0					0	$*$

Essential properties. Here are a couple of fundamental properties of Ulrich sheaves.

1. A H-Ulrich sheaf is H-semistable, destabilized only by H-Ulrich sheaves.
2. Setting $\deg(X) = \mathrm{H}^n$, the reduced Hilbert polynomial is

$$\mathbf{p}(t) = \frac{1}{r}\chi(\mathcal{E}(t)) = \frac{\deg(X)}{n!} \prod_{1 \leq j \leq n} (t+j).$$

Of course, the Hilbert polynomial $\mathbf{P}(t)$ is $\mathbf{P}(t) = r\mathbf{p}(t)$.

4.1.B. Ulrich ranks / Ulrich complexity

We define the set of Ulrich ranks, also called Ulrich geography, of (X, H) as

$$\mathrm{Ulr}(X, H) = \{r \in \mathbb{N}^* \mid \text{there is } \mathcal{E}\text{, an } H\text{-Ulrich of rank } r \text{ on } X.\}$$

Often one forgets about H in the notation. The main problem about Ulrich bundles is the following conjecture.

Conjecture 4.4. *For any (X, H), there is $\mathcal{E}$, an H-Ulrich sheaf on X.*

Thus $\mathrm{Ulr}(X)$ would always be well-defined. In any case, we define it on X if we know that there is an Ulrich sheaf on X. Also, one defines the Ulrich complexity of X as:

$$\mathrm{uc}(X, H) = \min(\mathrm{Ulr}(X, H)).$$

Again, we often omit H from the notation. This is well-defined for every X if the conjecture above holds true.

Question 4.5. *Can we determine* $\mathrm{Ulr}(X, H)$ *for some polarized variety* (X, H)*?*

Example 4.6. There are very few examples where we can determine $\mathrm{uc}(X)$, and even fewer where we know $\mathrm{Ulr}(X)$. Take $\Bbbk$ algebraically closed of any characteristic.

1. Curves. If X is a projective curve, then $\mathrm{Ulr}(X) = \mathbb{N}^*$.
2. Del Pezzo surfaces.
   a) If X is a smooth Del Pezzo surface, taking $H = -K_X$, then $\mathrm{Ulr}(X, H) = \mathbb{N}^*$ except when $(X, H) = (\mathbb{P}^2, 3\ell)$ or $(X, H) = (\widetilde{\mathbb{P}}^2, 3\ell - e_1)$. In these two cases $\mathrm{Ulr}(X) = \mathbb{N}_{\geq 2}$.
   b) More generally, if $X \subset \mathbb{P}^N$ is a non-degenerate integral, perhaps singular or even non-normal, surface of degree $N > 2$, then $\mathrm{Ulr}(X) \supset 2\mathbb{N}^*$.
   c) If $X \subset \mathbb{P}^3$ is an integral, normal cubic surface, then

   $$\mathrm{uc}(X) = 1 \Leftrightarrow X \text{ does not have a singularity of type } E_6,$$

   as proved by Dolgachev, see [Dol12].
   d) According to work in progress with J. Pons-Llopis, an integral, normal Del Pezzo surface X of degree d over an algebraically closed field should support as many Ulrich sheaves of rank one as there are roots in $H_X^{\perp} \subset \mathrm{Pic}(X)$, which are not effective nor antieffective, namely classes L such that $L^2 = -2$, $L \cdot K_X = 0$ and $\mathrm{H}^0(\mathcal{O}_X(L)) = \mathrm{H}^0(\mathcal{O}_X(-L)) = 0$.
   For $2 \leq d = H_X^2 \leq 7$, this happens if and only if X does not have a singularity whose diagram is the full Picard lattice, E_7 for $d = 2$, D_5 for $d = 4$, etc. Note that when $d = 2$, the meaning of Ulrich sheaf is a bit different, as it refers to base-point-free ample divisors, rather than very ample.
   e) What happens over non-closed fields?

3. K3 surfaces. If X is a very general K3 surface, then $\mathrm{Ulr}(X) = 2\mathbb{N}^*$. If X is any (smooth) K3 surface, then $\mathrm{Ulr}(X) \supset 2\mathbb{N}^*$. This is proved in [Fae19].
4. Veronese varieties. Let $(X, H) = (\mathbb{P}^n, dH_0)$, where $H_0 = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$, $n, d \geq 2$.
   a) In any case, $n!\mathbb{N}^* \subset \mathrm{Ulr}(X)$.
   b) If $n!$ divides d, then $n!\mathbb{N}^* = \mathrm{Ulr}(X)$.
   c) Let $n = 3$. Assume 6 does not divide d and write $d \equiv \pm e \mod 6$, with $e \in \{1, 2, 3\}$. Then we proved in [FP25]:
   $$\mathrm{Ulr}(X) = e\mathbb{N}^* \setminus \{1\}.$$

   Item 4.a is easy, a clever proof was given in [ES09]. Indeed, a finite Galois cover of $\mathbb{P}^n$ is given by $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. Then the push-forward of $\mathcal{O}(d-1, 2d-1, \ldots, nd-1)$ is an Ulrich bundle of rank $n!$. For item 4.b, we write, for an Ulrich bundle $\mathcal{E}$ of rank r:
   $$\chi(\mathcal{E}(tH_0)) = \frac{r}{n!} \prod_{1 \leq j \leq n} (dj + t),$$
   so $n!\chi(\mathcal{E}(1)) = r \prod_{1 \leq j \leq n}(dj + 1)$, hence, when $n!$ divides d, we get that r should be a multiple of $n!$.
   On the other hand, item 4.c is quite tricky. Say we look at $r = 2$, i.e. $e \in \{1, 2\}$. Notice that $\mathcal{F}_0 = \mathcal{E}(-2d + 2)$ should be an instanton bundle of rank 2 on $\mathbb{P}^3$, namely $\mathrm{H}^*(\mathcal{F}_0(-2)) = 0$ and $c_1(\mathcal{F}_0) = 0$. Note that $\mathcal{F}_0 \simeq \mathcal{F}_0^\vee$. Then $\mathrm{h}^p(\mathcal{F}_0(d-2)) = \mathrm{h}^{3-p}(\mathcal{F}_0(-d-2))$, hence checking $\mathrm{H}^*(\mathcal{F}_0(d-2)) = 0$ is enough. Also, $\chi(\mathcal{F}_0(d-2)) = 0$ by choosing conveniently $c_2(\mathcal{F}_0)$, which is possible when $e \in \{1, 2\}$, namely $c_2(\mathcal{F}_0) = \frac{1}{3}(d^2 - 1)$. So really one has to check $\mathrm{h}^p(\mathcal{F}_0(d-2)) = 0$ for $p = 0, 1$. This is due to results about naturality of cohomology by Hartshorne and Hirschowitz.
   For $r = 3$, the construction is along the same line but we use $\mathcal{E} = S^2\mathcal{F}$, where $\mathcal{F}$ is an instanton of rank 2. This gives a morphism:
   $$\mathfrak{M}_{\mathbb{P}^3}^{\mathfrak{sl}_2}(\mathrm{ch}(\mathcal{F})) \to \mathfrak{M}_{\mathbb{P}^3}^{\mathfrak{o}_3}(\mathrm{ch}(\mathcal{E})), \qquad \mathcal{F} \mapsto S^2\mathcal{F}.$$
   In some sense, this is a vector-bundle version of the usual $2:1$ cover $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{C})$. Interestingly, this gives an identification of the tangent spaces of these moduli spaces as follows:
   $$\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \simeq \mathrm{H}^1(S^2\mathcal{F}) \simeq \mathrm{H}^1(\mathcal{E}^\vee) \simeq \mathrm{H}^1(\wedge^2\mathcal{E}) \simeq \mathrm{H}^1(\mathrm{ad}(\mathcal{E})).$$
   So the image of the moduli space of instantons of rank 2 on $\mathbb{P}^3$ is dense in a component of the moduli space of orthogonal instantons of rank 3 on $\mathbb{P}^3$, for the appropriate charges.
   More precisely, for $e \in \{0, 1, 2\}$, one writes $d = 6h + 2e + 1$, $m = 3h + e$, $k = \binom{m+1}{2}$ and finds that $\mathcal{F} \mapsto S^2\mathcal{F}$ gives a local isomorphism
   $$\mathfrak{MI}_{\mathbb{P}^3}^{\mathfrak{sl}_2}(k) \dashrightarrow \mathfrak{MI}_{\mathbb{P}^3}^{\mathfrak{o}_3}(4k), \qquad \mathcal{F} \mapsto S^2\mathcal{F}.$$
   Use $\mathcal{F} = \mathcal{F}_m$ defined by an SL_2-equivariant resolution $\mathcal{R}_m \to \mathcal{F}_m$, with $\mathcal{R}_m$
   $$\mathcal{O}_{\mathbb{P}(V_3)}(-2m-1) \to V_{3m} \otimes \mathcal{O}_{\mathbb{P}(V_3)}(-m-1) \to V_{3m+1} \otimes \mathcal{O}_{\mathbb{P}(V_3)}(-m)$$
   d) It is also known that, when $n = 4$ or $n = 5$, $\mathrm{uc}(\mathbb{P}^n, dH_0) \geq 4$, [LR24].
5. Hypersurfaces. Let $X_d \subset \mathbb{P}^{n+1}$ be an integral hypersurface of degree d, $n \geq 1$.
   a) If $n = 1$ then any X, even reducible or non-reduced has $\mathrm{Ulr}(X) = \mathbb{N}^*$.
   b) Take $n = 2$. Then, $\mathrm{Ulr}(X) = \mathbb{N}^*$ for $d = 3$ and X smooth. For X general enough and $1 \leq d \leq 15$, we have $\mathrm{uc}(X) = 2$. For $d \geq 16$, $\mathrm{uc}(X) \geq 3$, see [Bea00]. I don't have an estimate of $\mathrm{uc}(X)$ as a function of d.
   c) Take $n = 3$. Then $\mathrm{Ulr}(X) = \mathbb{N}_{\geq 2}$ for X smooth and $d = 3$. For $d \in \{4, 5\}$ and X general enough, $\mathrm{uc}(X) = 2$. For $d \geq 6$, we have $\mathrm{uc}(X) \geq 4$, [RT19].
   d) Take $d \geq 3$, $n = 7$ and X smooth or $n = 6$, X very general. Then $\mathrm{uc}(X) \geq 6$.
   e) Take $n \geq 9$ and X smooth or $n = 8$ and X very general. Then $\mathrm{uc}(X) \geq 8$.

   The last two items have been proved recently, see [LR25].
6. For $r = 2$ and X irreducible, having an Ulrich sheaf on X amounts to having a linear Pfaffian presentation of the equation of X, namely:
   $$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-1)^{\oplus 2d} \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^{n+1}}^{\oplus 2d} \to \mathcal{E} \to 0.$$
   With ${}^{\mathrm{t}}\varphi = -\varphi$ and $X = \mathbb{V}(\mathrm{Pf}(\varphi))$. If we want X smooth, then $n \leq 4$.

4.1.C. Moduli spaces of Ulrich sheaves

Fix a polarized variety (X, H). Any H-Ulrich sheaf is H-semistable. Then we can speak of the moduli space $\mathfrak{U}_X(r)$ of Ulrich sheaves of rank r. We see this as a subset of Maruyama's moduli space $\mathfrak{M}_X(\mathbf{P})$ of semistable sheaves with Hilbert polynomial $\mathbf{P}$. Its closure $\overline{\mathfrak{U}}_X(r)$ is taken within $\mathfrak{M}_X(\mathbf{P})$. An S-equivalence class lies in this space if one representative has stable Ulrich factors.

Question 4.7. *Describe the moduli space $\mathfrak{U}_X(r)$ of Ulrich sheaves of rank r on (X, H).*

Notice that $\mathfrak{U}_X(r)$ is open (maybe empty) inside $\mathfrak{M}_X(\mathbf{P})$.

Example 4.8. If $X \subset \mathbb{P}^4$ is a smooth cubic threefold $\mathrm{uc}(X) = 2$ and:

$$\mathfrak{U}_X(2) = J(X) \setminus F(X),$$
$$\overline{\mathfrak{U}}_X(2) = \mathrm{Bl}_{F(X)}(J(X)).$$

Here $J(X) = \mathrm{H}^{2,1}(X)/H_3(X, \mathbb{Z})$ is the intermediate Jacobian, $F(X) = \mathrm{Hilb}_{t+1}(X)$ is the Fano surface of lines in X. It sits in $J(X)$ by the Abel-Jacobi map.

For Del Pezzo threefolds $Y = Y_d$ ode degree d with $4 \leq d \leq 5$, we have very nice descriptions, in terms of the Kuznetsov component of Y. Note that, if $\mathcal{E}$ is an Ulrich bundle on Y, then

$$\mathcal{E}(-1) \in \langle \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle^{\perp} = \mathrm{Ku}(Y).$$

Let us check what this gives for the various values of d. Let $r \geq 2$.

- When $d = 5$, $\mathcal{E}(-1) \in \langle \mathcal{U}_Y, \mathcal{Q}_Y^{\vee} \rangle$. Therefore $\mathcal{E}(-1)$ corresponds to a representation of the Kronecker quiver K_3 with two vertices and 3 arrows, as $\hom_Y(\mathcal{U}_Y, \mathcal{Q}_Y^{\vee}) = 3$. We get that the moduli space of stable Ulrich sheaves is isomorphic to the moduli space of stable representations $\mathfrak{M}^s_{\Upsilon_3}(r, r)$ of the Kronecker quiver Υ_3 having dimension vector (r, r). This was proved in [LP21]:

  $$\mathfrak{U}^s_Y(r) \simeq \mathfrak{M}^s_{\Upsilon_3}(r, r).$$

  In particular this moduli space is smooth and irreducible of dimension $r^2 + 1$.
- When $d = 4$, $\mathcal{E}(-1) \in \Phi(\mathrm{D^b}(C))$. So $\mathcal{E}(-1)$ comes from a vector bundle E on the curve C, which is given as $\Phi^!(\mathcal{E}(-1))$, where $\Phi^!$ is the right adjoint of Φ, and $\Phi(\mathrm{D^b}(C)) = \langle \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle^{\perp}$. We will recall later on what this gives for (r, k)-instantons. However, for Ulrich sheaves of rank r, this vector bundle E is stable, of rank r and, up to twisting with a fixed line bundle on C, we have $\deg(E) = 0$. The result of [CKL21] says

  $$\mathfrak{U}^s_Y(r) \simeq \mathrm{U}^s_C(r).$$

  Here, we use the traditional notation $\mathrm{U}^s_C(r)$ for the moduli space of stable bundles on C of rank r and degree 0. In particular this moduli space is smooth and irreducible of dimension $r^2 + 1$.
- When $d = 3$, $\mathcal{E}(-1) \in \mathrm{Ku}(Y)$, which is a fractional Calabi-Yau category of dimension 5/3. Using this, Pertusi and Zhao [FP23], showed that $\mathfrak{U}_X(r)$ is irreducible dimension $r^2 + 1$.

Here is a result in dimension 4.

Theorem 4.9 (D.F. & Y. Kim)**.** *Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold.*

i) We have $\mathrm{uc}(X) \leq 6$. In other words there is a 18×18 linear determinantal representation of X. If $\mathrm{H}^{2,2}(X, \mathbb{Z}) = \mathbb{Z}h_X^2$, then $\mathrm{uc}(X) = 6$.

ii) Take $\mathbf{v} = \frac{1}{2}\mathrm{ch}(\mathcal{U})$ with $\mathcal{U}$ of rank 6 constructed in item i) Then $\mathfrak{U}_X(k\mathbf{v})$ is a (non-compact) hyperkähler manifold of dimension $6k^2 + 2$, $k \geq 2$.

Idea of the proof. Start with a twisted cubic $C \subset X$. The Hilbert scheme of these cubics forms a 10-dimensional family. Each C spans a $\mathbb{P}^3$ which cuts X along a cubic surface Y, where we have $\mathrm{h}^0(\mathcal{O}_Y(C)) = 3$. So we have a $\mathbb{P}^2$ of curves in the same class as C, contained in Y. Contracting all these projective planes amounts to taking the maximal rationally connected quotient of the Hilbert scheme. We obtain a hyperkähler 8-fold, called the Lehn-Lehn-Sorger-van Straten 8-fold.

The LLSvS 8-fold can be obtained (up to a small birational modification) as moduli space of sheaves $\mathcal{G}$ arising as:

$$0 \to \mathcal{G}_C \to \mathcal{O}_X^{\oplus 3} \to \mathcal{O}_Y(2H - C) \to 0.$$

We should use two twisted cubics C, D contained in the same surface and look at

$$0 \to \mathcal{G}_C \to \mathcal{F} \to \mathcal{G}_D \to 0.$$

Note that $C \cdot D \in \{1, \dots, 5\}$, for instance if $C \equiv \ell$ and $D \equiv 5\ell - 2e_1 - \cdots - 2e_6 = 2H - C$ we get $C \cdot D = 5$. For some reason we are able to control the cohomology of a deformation of $\mathcal{F}$ when $C \cdot D = 2$. For instance $D \equiv 2\ell - e_1 - \cdots - e_5$ will work. Then one shows that a generic element $\mathcal{G}'$ of the 26-deformation space of $\mathcal{G}$ is not an extension of any sheaves of the form $\mathcal{G}_{C'}$ and therefore that it is stable. From there one shows that $\mathcal{G}'(1)$ is an Ulrich bundle. □

Let us take $\mathbf{v} = \mathrm{ch}(\mathcal{U})$, where $\mathcal{U}$ is an Ulrich bundle.

Question 4.10. *Is the moduli space of Ulrich bundles $\mathfrak{U}_X(\mathbf{v})$ irreducible?*

In [FP23], it is proved that $\mathfrak{U}_X(\mathbf{v})$ is irreducible if X is a cubic threefold. Their result relies on the fact that $\mathrm{Ku}(Y)$ is a fractional CY category of dimension $5/3$.

4.1.D. Back to instantons: the push-forward

Let X be an n-dimensional polarized variety equipped with an ample, globally generated line bundle $\mathcal{O}_X(1)$. Let $\mathcal{E}$ be a k-instanton on X. Choose a finite projection $\pi : X \to \mathbb{P}^n$. Then we have

$$\pi_*(\mathcal{E}) \simeq \mathcal{H}\Big(0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus k} \to \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1+\chi(\mathcal{E}))} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus k} \to 0\Big)$$

In other words, the direct image of $\mathcal{E}$ on $\mathbb{P}^n$ is the cohomology of a monad of the same form of a basic instanton bundle on $\mathbb{P}^3$, as we discussed in the framework of the Atiyah–Ward correspondence. This is just because π is a finite map and $\pi^*(\mathcal{O}_{\mathbb{P}^n}(1)) \simeq \mathcal{O}_X(H)$, so

$$\mathrm{H}^p(\pi_*(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^n}(t)) \simeq \mathrm{H}^p(\mathcal{E}(tH)), \qquad \text{for all } t \in \mathbb{Z}.$$

Then $\pi_*(\mathcal{E})$ is an instanton bundle on $\mathbb{P}^n$. In turn, this gives the monad presentation above, via Beilinson's theorem.

4.1.E. Rank-2 instantons and Steiner-Pfaffians

According to [AC23], a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d carries a k-instanton of rank 2 iff $X = \mathbb{V}(\mathrm{Pf}(\varphi))$ with $\varphi \in \mathrm{H}^0(\wedge^2 \mathcal{F}^\vee(1))$ so

$$\varphi : \mathcal{F}(-1) \to \mathcal{F}^\vee, \qquad \varphi \text{ is alternating i.e. } {}^{\mathrm{t}}\varphi = -\varphi.$$

Here $\mathcal{F}$ is a Steiner bundle of a certain form, namely $\mathcal{F}$ has a presentation:

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-1)^{\oplus k} \to \mathcal{O}_{\mathbb{P}^{n+1}}^{\oplus 2d+3k} \to \mathcal{F} \to 0.$$

We say that X is Steiner-Pfaffian of type $\mathcal{F}^\vee$.

More generally, we say that S is Pfaffian of type $(\mathcal{F}^\vee, \ell)$ if X can be written as Pfaffian of some φ, i.e. $X = \mathbb{V}(\mathrm{Pf}(\varphi))$, for some vector bundle $\mathcal{F}$, and

$$\varphi : \mathcal{F}(-\ell) \to \mathcal{F}^\vee, \qquad \varphi \text{ is alternating i.e. } {}^{\mathrm{t}}\varphi = -\varphi.$$

Example 4.11. Take $\mathcal{F} = \mathrm{T}_{\mathbb{P}^{n+1}}(-1)$ and $\varphi \in \mathrm{H}^0\big(\wedge^2 \Omega(3)\big) = \wedge^3 V_{n+2}$. Then $k = 1$ and

$$2d + 3 = n + 2 \qquad \Longleftrightarrow \qquad n = 2d + 1.$$

This gives some interesting Steiner-Pfaffians.

n	d	moduli	variety
5	2	-14	smooth quadric with G_2-action
7	3	3	Coble cubic
9	4	$\gg 0$	?

The Coble cubic was found around 1900; see [Cob17]. Gruson–Sam–Weyman [GS15] found that it is related to Pfaffians using Vinberg's theta representations.

Remark 4.12. If X is a smooth Pfaffian hypersurface of type $(\mathcal{F}^\vee, \ell)$, it supports a rank-2 vector bundle $\mathcal{E} = \mathrm{coker}(\varphi)$. Then we have a distinguished sublattice

$$\langle h_X^2, c_2(E)\rangle \subset \mathrm{H}^{2,2}(X, \mathbb{Z}).$$

- The discriminant δ of this sublattice is computed directly from $\mathrm{ch}(\mathcal{F})$ and ℓ.
- In particular when $\mathcal{E}$ is a k-instanton on a fourfold hypersurface $X \subset \mathbb{P}^5$ of degree d, $\mathcal{F}$ is Steiner, $\ell = 1$ and δ depends on k, d.

4.2. Instantons and Ulrich bundles on a cubic fourfold

4.2.A. Sheaves on cubic fourfolds and the Kuznetsov category

Cubic fourfolds are Fano manifolds of dimension 4 and index 3, and have been studied from many points of view. Their rationality problem is a central motivation. A prominent approach goes through the Kuznetsov component of a cubic fourfold, which is a K3 category.

Hodge diamond of a cubic fourfold. For a Fano fourfold, the Hodge diamond looks as follows.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & \rho_X & & 0 & & \\
 & 0 & & \mathrm{h}^{1,2}(X) & & \mathrm{h}^{1,2}(X) & & 0 & \\
0 & & \mathrm{h}^{1,3}(X) & & \mathrm{h}^{2,2}(X) & & \mathrm{h}^{1,3}(X) & & 0 \\
 & 0 & & \mathrm{h}^{1,2}(X) & & \mathrm{h}^{1,2}(X) & & 0 & \\
 & & 0 & & \rho_X & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

For a cubic fourfold, this really looks like

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

A cubic fourfold is Fano, but its Hodge diamond is not of the same type as that of a Fano threefold. The middle cohomology $\mathrm{H}^4(X)$ contains a non-trivial piece of type $(3,1)$ and $(1,3)$. This is one reason why cubic fourfolds have a rich period theory, closer in spirit to K3 surfaces than to ordinary Fano threefolds.

4.2.B. Kuznetsov categories of hypersurfaces

The derived category $\mathrm{D^b}(X)$ of bounded complexes of coherent sheaves has a decomposition

$$\mathrm{D^b}(X) = \langle \mathrm{Ku}(X), \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle.$$

Here the Kuznetsov component $\mathrm{Ku}(X)$ is

$$\mathrm{Ku}(X) = \{\mathcal{F} \in \mathrm{D^b}(X) \mid \mathrm{H}^*(\mathcal{F}) = \mathrm{H}^*(\mathcal{F}(-1)) = \mathrm{H}^*(\mathcal{F}(-2)) = 0\}.$$

This is a K3 category, namely for $\mathcal{E}, \mathcal{F}$ in $\mathrm{Ku}(X)$ we have, functorially:

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{E}[2])^\vee.$$

An object $\mathcal{E}$ of $\mathrm{Ku}(X)$ is not a priori a sheaf. If it is, $\mathcal{E}(1)$ is Ulrich iff $\mathrm{H}^*(\mathcal{E}(-3)) = 0$.

4.2.C. Ulrich bundles on very general/special fourfolds

We have Ulrich bundles of rank 6 on X. But what about other ranks?

Remark 4.13. If X is very general, then $\mathrm{uc}(X) = 6$. More precisely, if $r = \mathrm{rk}(\mathcal{U})$ is not a multiple of 6, the equality $\chi(\mathcal{U}(-t)) = \frac{r}{8}(t+1)(t+2)(t+3)(t+4)$ gives constraints on $c_2(\mathcal{U})$ and $c_4(\mathcal{U})$. Let

$$\delta = \text{discriminant of } \langle h_X^2,\, c_2(\mathcal{U})\rangle, \qquad m = \dim(\mathfrak{M}_X(\mathrm{ch}(\mathcal{U}))), \qquad a = c_2(\mathcal{U})^2.$$

Then we have

- $r = 2 \Rightarrow (\delta, m) = (14, 0)$. We have only one possibility for a.
- $r = 3 \Rightarrow (\delta, m) = (18, 2)$. We have only one possibility for a.
- $r = 4 \Rightarrow$

δ	8	14	20	26	32	38
m	10	8	6	4	2	0

- $r = 5 \Rightarrow$

δ	8	14	20	26	32	38	44	50	56
m	16	14	12	10	8	6	4	2	0

Some results have been otbained for special fourfolds, for instance:

- In [Kim24] it is shown, among other things, that X generic in $\mathcal{C}_8 \Rightarrow \mathrm{uc}(X) = 4$.
- In [TY24] it is shown that X generic in $\mathcal{C}_{18} \Rightarrow \mathrm{uc}(X) = 3$. The divisor $\mathcal{C}_{18}$ has been studied in detail in [AHTVA19].

4.2.D. Instanton bundles on very general/special fourfolds

Let X be a smooth cubic fourfold and $\mathcal{E}$ a k-instanton bundle of rank 2 on X. The charge k and the discriminant δ cannot be arbitrary, actually:

- $k \in \{0, \dots, 7\}$ and for each k, only finitely many values of δ are allowed.
- For any instanton bundle $\mathcal{E}$ (of any rank), we have:

$$\mathrm{ext}^2_X(\mathcal{E}, \mathcal{E}) \geq \mathrm{end}_X(\mathcal{E}) \geq 1.$$

We call $\mathcal{E}$ *unobstructed* if

$$\mathrm{ext}^2_X(\mathcal{E}, \mathcal{E}) = 1.$$

Note: if $\mathcal{E}$ is unobstructed then $\mathcal{E}$ is simple.

Theorem 4.14 (D. F. & G. Casnati & F. Galluzzi)**.** *Let $\mathcal{E}$ be an unobstructed instanton on X. Then the moduli of simple sheaves on X is smooth at $[\mathcal{E}]$.*

The proof of this relies on the deformation theory developed by Buchweitz & Flenner and subsequent developments by Kuznetsov & Markushevich. On considers:

$$\sigma^{\mathcal{E}} = \sigma = \sum_{q \geq 0} \sigma_q : \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) \to \bigoplus \mathrm{H}^{q+2}(X, \Omega_X^q) = \mathrm{H}^3(X, \Omega_X^1) \simeq \mathbb{C}$$

This map σ_q relies on the Atiyah class $\mathrm{At}(\mathcal{E}) \in \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega_X^1)$ and its exterior powers. It first sends a class $\xi \in \mathrm{Ext}^2(\mathcal{E}, \mathcal{E})$ to

$$\xi \otimes \wedge^q \mathrm{At}(\mathcal{E}) \in \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) \otimes \mathrm{Ext}^q(\mathcal{E}, \mathcal{E} \otimes \Omega_X^q)$$

and then maps $\xi \otimes \wedge^q \mathrm{At}(\mathcal{E})$ to $\mathrm{H}^{q+2}(X, \Omega_X^q)$ via the composition

$$\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) \otimes \mathrm{Ext}^q(\mathcal{E}, \mathcal{E} \otimes \Omega_X^q) \to \mathrm{Ext}^{2+q}(\mathcal{E}, \mathcal{E} \otimes \Omega_X^q) \xrightarrow{\mathrm{tr}} \mathrm{H}^{q+2}(X, \Omega_X^q)$$

If σ is injective then we are done. We will see that $\sigma = \sigma^{\mathcal{E}}$ is injective, see below...

4.2.E. Back to Ulrich bundles

There is a connection between instanton and Ulrich bundles on a cubic fourfold.

The acyclic extension. Take $\mathcal{E}$ a k-instanton on a smooth cubic fourfold X. Then $\mathrm{h}^1(\mathcal{E}(-1)) = k$. This defines

$$0 \to \mathcal{E} \xrightarrow{\zeta} \mathcal{G} \to \mathcal{O}_X(1)^{\oplus k} \to 0. \tag{22}$$

The sheaf $\mathcal{G}$ is called *the acyclic extension* of $\mathcal{E}$, it satisfies

$$\mathrm{H}^*(\mathcal{G}(-1)) = 0, \qquad \text{and actually} \qquad \mathcal{G}(-1) \in \mathrm{Ku}(X).$$

Theorem 4.15 (D. F. & G. Casnati & F. Galluzzi)**.** *Let $\mathcal{E}$ be an unobstructed instanton of rank s and charge k on a smooth cubic fourfold X. Then the acyclic extension $\mathcal{G}$ deforms to an Ulrich bundle of rank $r = s + k$ on X.*

One proves Theorem 4.15 and finishes the proof of Theorem 4.14 at the same time. Steps:

(a) One uses (22) & some vanishing results to define a natural injective map

$$\psi = (\zeta^1_*)^{-1} \circ \zeta^*_1 : \mathrm{Ext}^1_X(\mathcal{G}, \mathcal{G} \otimes \Omega_X) \to \mathrm{Ext}^1_X(\mathcal{E}, \mathcal{E} \otimes \Omega_X).$$

Also, $\psi(\mathrm{At}(\mathcal{G})) = \mathrm{At}(\mathcal{E})$, by the functoriality of the Atiyah class.

(b) Again using (22) we get a surjection

$$\mathrm{Ext}^2_X(\mathcal{E}, \mathcal{E}) \xrightarrow{\zeta^2_* \circ (\zeta^*_2)^{-1}} \mathrm{Ext}^2_X(\mathcal{G}, \mathcal{G}).$$

This surjection is actually responsible for $\mathrm{ext}^2_X(\mathcal{E}, \mathcal{E}) \geq 1$. Since $\mathcal{E}$ is unobstructed, this is an isomorphism and $\mathcal{G}$ is simple.

(c) The deformation theory of $\mathcal{G}$ is well understood by [KMM10, KM09, BLM+21], because $\mathcal{G}(-1)$ is a simple object in $\mathrm{Ku}(X)$. As an outcome, the deformation space of $\mathcal{G}$ is smooth at $[\mathcal{G}]$ and $\sigma^{\mathcal{G}}$ is injective.

(d) One sees that $\sigma^{\mathcal{E}}$ is injective too. Let $\xi \in \mathrm{Ext}^2_X(\mathcal{E}, \mathcal{E})$ satisfy $\sigma^{\mathcal{E}}_1(\xi) = 0$. Then $\eta = \zeta^2_*(\zeta^*_2)^{-1}(\xi) \in \mathrm{Ext}^2_X(\mathcal{G}, \mathcal{G})$ vanishes if and only if ξ vanishes. We have:

$$\begin{aligned}
0 = \sigma^{\mathcal{E}}_1(\xi) =& \mathrm{tr}(\Upsilon(\xi \otimes \mathrm{At}(\mathcal{E}))) = \\
=& \mathrm{tr}(\Upsilon(\zeta^*_2((\zeta^2_*)^{-1}(\eta)) \otimes \mathrm{At}(\mathcal{E}))) = \\
=& \mathrm{tr}(\Upsilon(\zeta^*_2((\zeta^2_*)^{-1}(\eta)) \otimes (\zeta^1_*)^{-1}(\zeta^*_1(\mathrm{At}(\mathcal{G}))))) = \\
=& \mathrm{tr}(\Upsilon((\zeta^2_*)^{-1}(\eta) \otimes \zeta^*_1(\mathrm{At}(\mathcal{G})))) = \\
=& \mathrm{tr}(\Upsilon(\eta \otimes \mathrm{At}(\mathcal{G}))) = \sigma^{\mathcal{G}}_1(\eta),
\end{aligned}$$

so $\eta = 0$ by the injectivity of $\sigma^{\mathcal{G}}$ and thus $\xi = 0$. So $\sigma^{\mathcal{E}}$ is injective. Theorem 4.14 is proved and, moreover, we get the dimension of the moduli space of simple sheaves on X at $\mathcal{E}$ and $\mathcal{G}$.

(e) To prove that $\mathcal{G}$ deforms to an Ulrich bundle we check that a generic deformation $\mathcal{G}'$ of $\mathcal{G}$, which automatically satisfies $\mathcal{G}'(-1) \in \mathrm{Ku}(X)$, also has $\mathrm{H}^3(\mathcal{G}'(-4)) = 0$.

(f) Equivalently, we want $\mathrm{Hom}_X(\mathcal{G}', \mathcal{O}_X(1)) = 0$. If not, generically we would have a surjection $\mathcal{G}' \to \mathcal{O}_X(1)^{\oplus \ell}$ for some $\ell > 0$. The kernel would be another unobstructed instanton $\mathcal{E}'$, of different rank. But then, the dimension of the moduli space of simple sheaves at $\mathcal{E}'$ is under control (as a function of ℓ) and one checks the dimension of all families of the form

$$0 \to \mathcal{E}' \xrightarrow{\zeta} \mathcal{G}' \to \mathcal{O}_X(1)^{\oplus \ell} \to 0, \qquad \text{with } \ell > 0,$$

is not large enough to cover the dimension of the deformation space of $\mathcal{G}$.

4.2.F. Two specific Hassett divisors

One can combine these ideas to say a bit more about two Hassett divisors of cubic fourfolds, namely $\mathcal{C}_{18}$ and $\mathcal{C}_{20}$.

Discriminant 18. It is shown in [TY24] that, if a smooth cubic fourfold X carries a rank-3 Ulrich bundle, then X lies in $\mathcal{C}_{18}$ and also that a generic cubic in $\mathcal{C}_{18}$ carries a 2-dimensional family of Ulrich bundles. Their proof uses a lot of computer algebra.

Theorem 4.16 (D. F. & G. Casnati & F. Galluzzi)**.** *Let X be a general cubic of $\mathcal{C}_{18}$.*

- *a) X has a 1-dimensional family of unobstructed instanton 2-bundles of charge 1;*
- *b) any such instanton deforms to an Ulrich bundle of rank 3;*
- *c) X is a linear section of a Coble cubic;*
- *d) X is Steiner-Pfaffian of type $\Omega_{\mathbb{P}^5}(1) \oplus \mathcal{O}_{\mathbb{P}^5}^{\oplus 3}$.*

A Coble cubic is defined by an alternating 3-form $\omega \in \wedge^3 V_9$ with $V_9 \simeq \Bbbk^9$. If ω is general enough, Γ_ω is singular along an abelian surface $A_\omega \simeq J(C_\omega)$ with $C = C_\omega$ a curve of genus 2.

We choose $V_6 \subset V_9$, i.e. a $\mathbb{P}^5$ in $\mathbb{P}^8$. Then we decompose $V_9 = V_6 \oplus V_3$ and:

$$\wedge^3(V_6 \oplus V_3) \simeq \wedge^3 V_6 \oplus V_3 \otimes \wedge^2 V_6 \oplus \wedge^2 V_3 \otimes V_6 \oplus \wedge^3 V_3.$$

The first 3 pieces show up as the constituents of a skew-symmetric map:

$$\mathrm{T}_{\mathbb{P}(V_6)}(-2) \oplus V_3^\vee \otimes \mathcal{O}_{\mathbb{P}(V_6)}(-1) \to \Omega_{\mathbb{P}(V_6)}(1) \oplus V_3 \otimes \mathcal{O}_{\mathbb{P}(V_6)}$$

This gives the equivalence of items c) and d). Also, item b) follows from item a) using Theorem 4.15, which also says that the families have the desired dimensions. Item d) holds once item a) is proved, by the main result from [AC25].

Note that the surface S parametrizing Ulrich bundles here is the twisted K3 surface naturally associated with X by the structure fibration in Del Pezzo surfaces of degree 6, which gives $\mathrm{Ku}(X) \simeq \mathrm{D}^{\mathrm{b}}(S, \alpha)$, where α is a Brauer class of order 3.

Item a), in turn, is easy to show, as it suffices to use one of the Del Pezzo surfaces of degree 6 to construct the rank-2 1-instanton bundle $\mathcal{E}$ by the Serre correspondence.

If one takes a generic form ω, then Γ_ω is singular precisely along A_ω, where the meaning of generic is very precise: the form should lie outside the union of reflecting hyperplanes for the complex reflection group associated with a 3-cyclic grading of the Lie algebra $\mathfrak{e}_8$. There is a 3-parameter family of Coble cubics given by the quotient $\wedge^3 V_9 / SL_9$. Choosing V_6 in V_9 gives 18 dimensions. The fibre over the image in the moduli space of cubic fourfolds is 2-dimensional (one dimension for $\mathcal{E}$, plus one scalar). So this is consistent with the image being dense in the divisor $\mathcal{C}_{18}$.

This is another way to see that the divisor is unirational.

Discriminant 20. The case of discriminant 20 is mentioned among the experiments in [Kim24], but the situation below is not analyzed, I think.

Theorem 4.17 (D. F. & G. Casnati & F. Galluzzi)**.** *Let X be a general cubic of $\mathcal{C}_{20}$.*

- *a) X has a 2-dimensional family of unobstructed instanton 2-bundles of charge 2;*
- *b) any such instanton deforms to an Ulrich bundle of rank 4;*
- *c) X carries a 6-dimensional symplectic family of Ulrich bundles of rank 4;*
- *d) X is Steiner-Pfaffian of type $\Omega_{\mathbb{P}^5}(1)^{\oplus 2}$.*

This is connected to Pfaffian surfaces in $\mathbb{P}^5$ studied by Catanese. A general section of $\mathcal{E}(1)$ vanishes along a surface $Y \subset X$ with a skew-symmetric resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-4) \to \mathrm{T}_{\mathbb{P}^5}(-2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^5}(-1) \to \Omega_{\mathbb{P}^5}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^5} \to \mathcal{I}_{Y/\mathbb{P}^5}(3) \to 0.$$

This surface is *canonical*, i.e. $\omega_Y \simeq \mathcal{O}_Y(1)$. It has degree 16 and irregularity 0.

Here we use the computer to describe a generic Steiner-Pfaffian $\mathrm{Pf}(\varphi)$ with

$$\varphi : A^\vee \otimes \mathrm{T}_{\mathbb{P}^5}(-2) \to A \otimes \Omega_{\mathbb{P}^5}(1), \quad \text{skew-symmetric}, \quad A \cong \Bbbk^2.$$

We note that this is parametrized by $\varphi \in S^2 A \otimes \mathrm{H}^0(\Omega^2_{\mathbb{P}^5}(3)) \simeq S^2 A \otimes \wedge^3 B$, with $B \cong \Bbbk^6$.

The computer is needed to check that the associated instanton is actually unobstructed. This is equivalent to asking that $\mathrm{h}^1(N_{Y/X}) = 1$. Indeed, one considers

$$0 \to \mathcal{O}_X \to \mathcal{E}(1) \to \mathcal{I}_{Y/X}(4) \to 0,$$

arising from Y being the zero-locus of a section of $\mathcal{E}(1)$, and

$$0 \to \mathcal{I}_{Y/X} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

Take $\mathrm{Hom}_X(\mathcal{E}(1), -)$, $\mathrm{Hom}_X(\mathcal{E}(-3), -)$, use $\mathcal{E}^\vee \simeq \mathcal{E}(-2)$ and $\mathcal{E}(1)|_Y \simeq N_{Y/X}$ to get

$$\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) \to \mathrm{Ext}^2_X(\mathcal{E}, \mathcal{I}_{Y/X}(3)) \leftarrow \mathrm{H}^1(N_{Y/X}) = \mathrm{H}^1(\mathcal{E}^\vee(3)|_Y) = \mathrm{Ext}^1_X(\mathcal{E}(-3), \mathcal{O}_Y).$$

This identifies $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) \simeq \mathrm{H}^1(N_{Y/X})$, so if the latter is 1-dimensional then $\mathcal{E}$ is unobstructed. Then we can construct an Ulrich bundle of rank 4 from $\mathcal{E}$.

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