EVEN AND ODD INSTANTON BUNDLES ON FANO THREEFOLDS OF PICARD NUMBER ONE

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ABSTRACT. We consider an analogue of the notion of instanton bundle on the projective 3-space, consisting of a class of rank-2 vector bundles defined on smooth Fano threefolds X of Picard number one, having even or odd determinant according to the parity of K_X.

We first construct a well-behaved irreducible component of their moduli spaces. Then, when the intermediate Jacobian of X is trivial, we look at the associated monads, hyperdeterminants and nets of quadrics. We also study one case where the intermediate Jacobian of X is non-trivial, namely when X is the intersection of two quadrics in $\mathbb{P}^5$, relating instanton bundles on X to vector bundles of higher rank on a the curve of genus 2 naturally associated with X.

CONTENTS

Introduction

Let X be a smooth complex projective threefold of Picard number 1. Let H_X be the ample generator of the Picard group, and write:

$$K_X = -i_X H_X,$$

for some integer i_X. Let us assume that i_X is positive, so that X is a *Fano* threefold. Under these assumptions, each of the cohomology groups $\mathrm{H}^{p,p}(X)$ is 1-dimensional, more specifically $\mathrm{H}^{2,2}(X)$ is generated by the class L_X of a line contained in X (we will thus write the Chern classes of a coherent sheaf on X as integers). We have $i_X \in \{1, 2, 3, 4\}$, and $i_X = 4$ implies $X = \mathbb{P}^3$, while $i_X = 3$ implies that X is a smooth quadric hypersurface in $\mathbb{P}^4$. In case $i_X = 2$, the variety X is called a *Del Pezzo* threefold, while for $i_X = 1$ the variety X is called a *prime* Fano threefold, and in this case its *genus g_X* is defined as the genus of a general double hyperplane section curve.

Let now F be a rank-2 algebraic vector bundle on X, and suppose that F is stable (with respect to H_X). Let us assume that the following conditions hold:

$$F \simeq F^* \otimes \omega_X, \qquad \mathrm{H}^1(X, F) = 0.$$

Set E for the twist $F(t)$ of F having $c_1(E) = 0$ or -1, and let $c_2(E) = k$. Clearly, when $i_X = 4$, i.e. when $X = \mathbb{P}^3$, the sheaf $E = F(2)$ is an instanton bundle on $\mathbb{P}^3$, as first described in [AHDM78]. By this reason we say that E is a *k-instanton* on X. We say that E is an *even* or *odd instanton* depending on the parity of $c_1(E)$, and we denote the (coarse) moduli space of these bundles by $\mathrm{MI}_X(k)$. On $\mathbb{P}^3$, the moduli space of instanton bundles has long been conjectured to be smooth and irreducible, and smoothness has indeed been announced recently (see [JV11]) together with irreducibility at least for odd c_2 (see [Tik12]). Also, irreducible instanton moduli have been recently proved to be rational, see [MT10].

The purpose of this paper is to carry out a preliminary study of the moduli space of k-instantons E on Fano threefolds X besides $\mathbb{P}^3$. Our first result, based on previous work contained in [BF08], is to extend the non-emptiness statement for $\mathrm{MI}_X(k)$ to almost all Fano threefolds of Picard number 1. Namely, it holds for any X if $i_X > 1$ and, when $i_X = 1$, under the assumption that $-K_X$ is very ample (so X is *non-hyperelliptic*) and that X contains a line L with normal bundle $\mathscr{O}_L \oplus \mathscr{O}_L(-1)$ (so X is *ordinary*).

Theorem A. *The moduli space $\mathrm{MI}_X(k)$ has a generically smooth irreducible component whose dimension is the number δ below:*

i_X	4	3	2	1
δ	$8k - 3$	$6k - 6$	$4k - 3$	$2k - g_X - 2$

and $\mathrm{MI}_X(k)$ is empty when $i_X = 2, k = 1$ and when $i_X = 1, 2k < g_X + 2$.

Next, we focus on the case when the variety X satisfies $\mathrm{H}^3(X) = 0$, i.e. the intermediate Jacobian of X is trivial. This holds if the derived category $\mathrm{D}^b(X)$ of coherent sheaves on X admits a full strongly exceptional collection, and a case-by-case analysis shows that in fact the two conditions are equivalent. Indeed, there are only 4 classes of such varieties, one for each index, namely:

   i) the projective space $\mathbb{P}^3$, for $i_X = 4$;
  ii) a quadric hypersurface in $\mathbb{P}^4$, for $i_X = 3$;
 iii) a linear section $X = \mathbb{P}^6 \cap \mathbf{G}(2, \mathbb{C}^5) \subset \mathbb{P}^9$, with $H_X^3 = 5$, for $i_X = 2$;
 iv) a prime Fano threefold $X \subset \mathbb{P}^{13}$ of genus 12, in case $i_X = 1$.

In all these case, we have:

$$(1) \hspace{4cm} \mathrm{D}^b(X) = \langle \mathscr{E}_0, \mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3 \rangle,$$

for some vector bundles $\mathscr{E}_i$, with $i = 0, 1, 2, 3$, defined on X. Let q_X and r_X, be the quotient and the remainder of the division of i_X by 2. It turns out that one can choose the $\mathscr{E}_i$'s so that:

$$(2) \qquad \mathscr{E}_0 \simeq \mathscr{O}_X(-q_X - r_X), \qquad \mathscr{E}_3^*(-r_X) \simeq \mathscr{E}_1, \qquad \mathscr{E}_2^*(-r_X) \simeq \mathscr{E}_2.$$

Set $U = \mathrm{Hom}_X(\mathscr{E}_2, \mathscr{E}_3)$, and note that $U \simeq \mathrm{Hom}_X(\mathscr{E}_1, \mathscr{E}_2)$.

Now, given an integer k, we fix vector spaces I and W, and an isomorphism $D : W \to W^*$ with $D^{\mathrm{t}} = (-1)^{r_X + 1} D$ (an $(r_X + 1)$-symmetric duality). According to the values of i_X and k, we need to choose the dimension of I and W as follows:

(3)

i_X	k	$\dim(I)$	$\dim(W)$
4	$k \geq 1$	k	$2k + 2$
3	$k \geq 2$	$k - 1$	k
2	$k \geq 2$	k	$4k + 2$
1	$k \geq 8$	$k - 7$	$3k - 20$

The lower bounds for k appear in order to ensure non-emptiness of $\mathrm{MI}_X(k)$. Let us write $\mathrm{G}(W, D)$ for the symplectic group $\mathrm{Sp}(W, D)$, or for the orthogonal group $\mathrm{O}(W, D)$ depending on whether $r_X = 0$ or 1, so that $\eta \in \mathrm{G}(W, D)$ operates on W and satisfies:

$$\eta^{\mathrm{t}} D \eta = D.$$

We look at an element A of $I \otimes W \otimes U$ as a map:

$$A : W^* \otimes \mathscr{E}_2 \to I \otimes \mathscr{E}_3,$$

and, under the dualities (2), we can consider:

$$D\,A^{\mathrm{t}} : I^* \otimes \mathscr{E}_1 \to W^* \otimes \mathscr{E}_2.$$

We define the subvariety $\mathscr{Q}_{X,k}$ of $I \otimes W \otimes U$ by:

$$\mathscr{Q}_{X,k} = \{A \in I \otimes W \otimes U \mid A\,D\,A^{\mathrm{t}} = 0\},$$

and its open piece:

$$\mathscr{Q}_{X,k}^\circ = \{A \in \mathscr{Q}_{X,k} \mid A : W^* \otimes \mathscr{E}_2 \to I \otimes \mathscr{E}_3 \text{ is surjective}\}.$$

We also define the group:

$$G_k = \mathrm{GL}(I) \times \mathrm{G}(W, D),$$

acting on $I \otimes W \otimes U$ on the left by $(\xi, \eta).A = (\xi A \eta^{\mathrm{t}})$. The group G_k acts on the variety $\mathscr{Q}_{X,k}$, since, for all $A \in \mathscr{Q}_{X,k}$, we have $\xi A \eta^{\mathrm{t}} D \eta A^{\mathrm{t}} \xi^{\mathrm{t}} = \xi A D A^{\mathrm{t}} \xi^{\mathrm{t}} = 0$. Clearly, G_k acts also on $\mathscr{Q}_{X,k}^\circ$. Here is the first main result of this paper.

Theorem B. *Let X be a smooth complex Fano threefold of Picard number 1 and $\mathrm{H}^3(X) = 0$. Let I, W, D and the $\mathscr{E}_i$'s be as above. Then a k-instanton E on X is the cohomology of a monad of the form:*

$$I^* \otimes \mathscr{E}_1 \xrightarrow{D\,A^{\mathrm{t}}} W^* \otimes \mathscr{E}_2 \xrightarrow{A} I \otimes \mathscr{E}_3,$$

and conversely the cohomology of such a monad is a k-instanton. The moduli space $\mathrm{MI}_X(k)$ is isomorphic to the geometric quotient:

$$\mathscr{Q}_{X,k}^\circ / G_k.$$

More specific results in different directions will be given for each threefold, according to the index, leaving aside the well-known case of $\mathbb{P}^3$:

- for $i_X = 3$, i.e. when X is a quadric threefold Q, Theorem C will show that $\mathrm{MI}_Q(k)$ is affine, in analogy with the case of $\mathbb{P}^n$ proved in [CO03]. We also say some words on jumping lines in this case. Moreover, we show that 't Hooft bundles provide unobstructed instantons;

- for $i_X = 2$ a description of $\mathrm{MI}_X(k)$ in terms of nets of quadrics will be given in Theorem D;
- for $i_X = 1$, we provide a parametrization of $\mathrm{MI}_X(k)$ in terms of nets of quadrics, describe $\mathrm{MI}_X(k)$ when $k = 8$ (i.e. 1 higher than the minimal value 7), and we study the variety of jumping conics of an instanton. We refer to Theorem F.

When $\mathrm{H}^3(X) \neq 0$, there are not enough exceptional objects on X to generate $\mathrm{D}^{\mathrm{b}}(X)$. However in some cases, notably when X is rational, one can rely on a semiorthogonal decomposition of $\mathrm{D}^{\mathrm{b}}(X)$ containing a subcategory equivalent to $\mathrm{D}^{\mathrm{b}}(\Gamma)$, where Γ is a curve whose Jacobian is isomorphic to the intermediate Jacobian of X. We investigated $\mathrm{MI}_X(k)$ in terms of this curve already in [BF08] (for threefolds of index 1 and genus 7) and in [BF12] (for index 1 and genus 9). Here we carry out a similar analysis in the case of Del Pezzo threefolds of degree 4, see Theorem E.

Concerning smoothness and irreducibility of $\mathrm{MI}_X(k)$, it might be natural to conjecture that these properties hold when X is general in its moduli space. Although we do not tackle here this problem, a few comments are in order. First, in some cases these properties do hold, in particular for low values of k. For instance, this is the case for $i_X = 3$ (i.e. X is a quadric threefold) and $k = 2, 3$ (see [OS94]), for most Del Pezzo threefolds (i.e. $i_X = 2$) when $k = 2$, (see [MT01, MT03, Fae05]), and for many prime Fano threefolds (i.e. $i_X = 1$) when $\frac{g_X}{2} + 1 \leq k \leq \frac{g_X+1}{2} + 2$ (see for instance [IM00b, IM07a, BF08, BF12, BF11, IM07b]). We contribute to this with Theorem E, and Theorem F respectively for Del Pezzo threefolds of degree 4 and prime Fano threefolds of genus 12.

However, it should be clear that these properties do not necessarily hold when X is not general in its moduli space. For instance, for $i_X = 1$ and $g_X = 5$ (i.e. X is the intersection of 3 quadrics in $\mathbb{P}^6$), the moduli space $\mathrm{MI}_X(4)$ is isomorphic to a double cover of the discriminant septic, as proved in [BF11]. For special X, this septic can be singular and can have many irreducible components. Examples of threefolds X with $i_X = 1$ and $g_X = 7$ such that $\mathrm{MI}_X(6)$ is singular are given in [BF08]. Still $\mathrm{MI}_X(6)$ is always connected in this case. Finally, A. Langer outlined an argument based on [Lan08] that suggests that $\mathrm{MI}_X(k)$ cannot be smooth and irreducible for all k when X is a smooth quadric threefold.

Finally, I would like to mention the independent paper of Alexander Kuznetsov, [Kuz12], appeared shortly after submission of this manuscript. Kuznetsov's paper takes up many questions considered here; the results are particularly close for Fano threefolds of index 2 and degree 4 and 5. An interesting analysis of instantons in terms of their jumping lines is also carried out in *loc. cit.*

The paper is organized as follows. The next section contains some basic material, and the definition and first properties of even and odd instanton bundles on Fano threefolds. Next, we study $\mathrm{MI}_X(k)$ according to the value of the index i_X. In Section 2 we study (odd) instantons on a quadric threefold, i.e. when $i_X = 3$. In Section 3 we deal with Del Pezzo threefolds, i.e. with the case $i_X = 2$. We first give an existence result for instantons on any such threefold. Then, we focus on Del Pezzo threefolds of degree 5, in this case we have $\mathrm{H}^3(X) = 0$ and $\mathrm{MI}_X(k)$ is studied in terms of monads. Next we look at degree 4, in this case $\mathrm{H}^3(X) \neq 0$ and $\mathrm{MI}_X(k)$ is studied in terms of a curve of genus 2. In Section 4 we study $\mathrm{MI}_X(k)$ for $i_X = 1$, mainly focusing on the case prime Fano threefolds of genus 12 (corresponding to the condition $\mathrm{H}^3(X) = 0$). In Section 5 we carry out some basic considerations to check that the quotient describing $\mathrm{MI}_X(k)$ is geometric.

1. BASIC FACTS ABOUT INSTANTONS ON FANO THREEFOLDS

1.1. **Preliminaries.** We set up here some preliminary notions, mainly concerning Fano threefolds X of Picard number 1 with $\mathrm{H}^3(X) = 0$.

1.1.1. *Notations and conventions.* We work over the field $\mathbb{C}$ of complex numbers. Given a vector space V, we denote by $\mathbb{G}(k,V)$ the Grassmann variety of k-dimensional quotients of V, and we write $\mathbb{P}(V) = \mathbb{G}(1,V)$. If $\dim(V) = n$, we write $\mathbf{G}(k,V) = \mathbb{G}(n-k,V)$.

Let X be a smooth projective variety over $\mathbb{C}$. We already used the notation $\mathrm{H}^{k,k}(X)$ for the Hodge cohomology groups of X, and, given a coherent sheaf F on X, we let $c_k(F) \in \mathrm{H}^{k,k}(X)$ be the kth Chern class of F (we said that $c_k(F)$ is denoted by an integer as soon as $\mathrm{H}^{k,k}(X)$ is 1-dimensional and a generator is fixed). We denote by $\mathrm{H}^k(X,F)$, or simply by $\mathrm{H}^k(F)$, the kth cohomology group of F and by $\mathrm{h}^k(X,F)$ its dimension as a $\mathbb{C}$-vector space (the same convention will be applied to Ext groups).

We refer to [HL97] for the notion of stability of a coherent sheaf F on X with respect to a polarization H on X. Stability, for us, will be in the sense of Maruyama-Gieseker, we will talk about slope-stability when referring to the terminology of Mumford-Takemoto. If $\mathrm{Pic}(X) \simeq \mathbb{Z}$, the choice of H does not matter anymore and we will avoid mentioning it. We will denote by $\mathrm{M}_X(r, c_1, \ldots, c_s)$ the coarse moduli space of equivalence classes of semistable sheaves on X or rank r, with Chern classes $c_1, \ldots, c_s$, where s is the greatest non-vanishing Chern class (so $s \leq \dim(X)$). If H is an ample divisor class on X, the degree of X (with respect to H) is $H^{\dim(X)}$. The Hilbert scheme of connected Cohen-Macaulay curves of genus g and degree d contained in X will be denoted by $\mathscr{H}_g^d(X)$. The ideal and normal sheaves of a subscheme $Z \subset X$ will be denoted by $\mathscr{I}_{Z,X}$ and $\mathscr{N}_{Z,X}$.

We will use the derived category $\mathrm{D}^{\mathrm{b}}(X)$ of coherent sheaves on X. We refer to [Huy06] for basic material on this category. We refer to [Bon90, Gor90] for the notions of simple and exceptional bundle, full (strongly) exceptional collection in $\mathrm{D}^{\mathrm{b}}(X)$, (Koszul) dual collections, left and right mutations and helices.

1.1.2. *Fano threefolds admitting a full exceptional collection.* We denote by X a smooth Fano threefold of Picard number one, and by i_X its index, so that $K_X = -i_X H_X$. We have set $i_X = 2q_X + r_X$, where $q_X \geq 0$ and $0 \leq r_X \leq 1$.

It is well-known that X admits a full exceptional collection if and only if $\mathrm{H}^3(X) = 0$ i.e., X has trivial intermediate Jacobian. Further, the collection in this case is strongly exceptional. Up to our knowledge, the only available proof of this fact is a case-by-case analysis. Indeed, recall from the introduction that there are only 4 classes of such varieties, namely:

i) the projective space $\mathbb{P}^3$, for $i_X = 4$;
ii) a quadric hypersurface in $\mathbb{P}^4$, for $i_X = 3$;
iii) a linear section $X = \mathbb{P}^6 \cap \mathbf{G}(2, \mathbb{C}^5) \subset \mathbb{P}^9$, with $H_X^3 = 5$, for $i_X = 2$;
iv) a prime Fano threefold $X \subset \mathbb{P}^{13}$ of genus 12, in case $i_X = 1$.

So, for $i_X = 4$, this is Beilinson's theorem [Bei78]. For $i_X = 3$, the result is due to Kapranov [Kap88]. For $i_X = 2$ we refer to [Orl91], see also [Fae05]. Finally in case $i_X = 1$ we refer to [Kuz96, Kuz97, Fae07a]. For each of these cases, there is a full strongly exceptional collection $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathscr{E}_0, \mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3 \rangle$, where the $\mathscr{E}_i$'s are vector bundles satisfying (2). We let $\langle \mathscr{F}_0, \mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3 \rangle$ be the dual collection. Then, given a coherent sheaf $\mathscr{F}$ on X, we will write the cohomology table of $\mathscr{F}$ as the following display:

(1.4)

$\mathrm{h}^3(\mathscr{F} \otimes \mathscr{F}_0)$	$\cdots$	$\mathrm{h}^3(\mathscr{F} \otimes \mathscr{F}_3)$
$\vdots$		$\vdots$
$\mathrm{h}^0(\mathscr{F} \otimes \mathscr{F}_0)$	$\cdots$	$\mathrm{h}^0(\mathscr{F} \otimes \mathscr{F}_3)$
$\mathscr{E}_0$	$\cdots$	$\mathscr{E}_3$

The line containing the $\mathscr{E}_i$'s here is displayed to facilitate writing the Beilinson spectral sequence that converges to $\mathscr{F}$. This amounts to saying that $\mathscr{F}$ is the cohomology of a

complex $\mathscr{C}_{\mathscr{F}}^{\cdot}$ such that:

$$(1.5) \qquad \mathscr{C}_{\mathscr{F}}^{j} = \oplus_i \mathrm{H}^i(X, \mathscr{F} \otimes \mathscr{F}_{j-i+3}) \otimes \mathscr{E}_{j-i+3}.$$

where the index i in the sum runs between $\max(0,j)$ and $\min(3, j+3)$.

1.2. **The basic orthogonality relation.** Let F be a rank 2 algebraic vector bundle on X, and suppose that F is stable. Let us assume that the following conditions hold:

$$(1.6) \qquad F \simeq F^* \otimes \omega_X, \qquad \mathrm{H}^1(X, F) = 0.$$

The normalization E of F is defined as the twist $E = F(q_X)$, so:

$$c_1(E) \in \{-1, 0\}.$$

Definition 1. We say that E is an *instanton bundle* on X if E is the normalization of a stable rank-2 bundle F on X satisfying (1.6). We say that E is a *k-instanton* if moreover $c_2(E) = k$ (recall that this means $c_2(E) = kL_X$), and that E is *even* or *odd* according to whether $c_1(E)$ is even or odd. We denote by $\mathrm{MI}_X(k)$ the subscheme of $\mathrm{M}_X(2, c_1(E), k)$ consisting of (even or odd) instanton bundles.

The following elementary lemma summarizes the cohomology vanishing satisfied by an instanton bundle.

Lemma 1.1. *If E is an (even or odd) instanton on X, then we have the vanishing:*

$$(1.7) \qquad \mathrm{H}^i(X, E(-q_X)) = 0, \qquad \text{for all } i,$$

and also $\mathrm{H}^1(X, E(-q_X - t)) = 0$, *and* $\mathrm{H}^2(X, E(-q_X + t)) = 0$, *for all $t \geq 0$. Further, we have the relations in* $\mathrm{D}^{\mathrm{b}}(X)$:

$$(1.8) \qquad E \in \langle \mathscr{O}_X(q_X) \rangle^{\perp}, \qquad E \in {}^{\perp}\langle \mathscr{O}_X(-q_X - r_X) \rangle = \langle \mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3 \rangle.$$

Proof. We have defined E as the normalization $E = F(q_X)$, so $\mathrm{H}^1(X, E(-q_X)) = 0$ by definition. Clearly, we have $\mathrm{H}^0(X, E(-q_X)) = 0$ by stability of E. Using Serre duality, the relation $i_X = 2q_X + r_X$, and the fact that E^* is isomorphic to $E(r_X)$, we get:

$$\mathrm{H}^i(X, E(-q_X)) \simeq \mathrm{H}^{3-i}(X, E(-q_X))^*,$$

so (1.7) holds for $i = 3, 2$ as well. This proves $\mathrm{Ext}_X^i(\mathscr{O}_X(q_X), E)$ for all i, i.e. the first part of (1.8). For the second one, one just needs to observe that $\mathrm{Ext}_X^i(\mathscr{O}_X(q_X), E)$ is dual to $\mathrm{Ext}_X^{3-i}(E, \mathscr{O}_X(-q_X - r_X))$. The equality ${}^{\perp}\langle \mathscr{O}_X(-q_X - r_X) \rangle = \langle \mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3 \rangle$ is well-known, and has been mentioned already in the introduction, see (2). However, for the reader's convenience, we will review it in the sequel in a case-by-case fashion.

Concerning the vanishing $\mathrm{H}^1(X, E(-q_X - t)) = 0$, and $\mathrm{H}^2(X, E(-q_X + t)) = 0$, first note that the second is given by the first by the duality argument we already used, and remark that they both hold for $t = 0$ in view of (1.7). To prove the first vanishing for positive t, we consider the restriction $E|_S$ of E to a general hyperplane section S of X and we recall that $E|_S$ is slope-semistable by Maruyama's theorem, [Mar80]. This implies $\mathrm{H}^0(S, E|_S(-q_X - t)) = 0$ which gives $\mathrm{H}^1(X, E(-q_X - t - 1)) \subset \mathrm{H}^1(X, E(-q_X - t))$. So we deduce the desired vanishing for any t from the case $t = 0$, that we have already proved. $\qquad \square$

We will use this lemma to show that E is the cohomology of a *monad*, i.e. a complex:

$$(1.9) \qquad I^* \otimes \mathscr{E}_1 \to W^* \otimes \mathscr{E}_2 \to I \otimes \mathscr{E}_3,$$

where I and W are vector spaces, the first map is injective and the last map is surjective.

1.3. **Lower bound on the second Chern class of instantons.** The following lemma is almost entirely well-known, and proves the emptiness statement in Theorem A.

Lemma 1.2. *The moduli space* $M_X(2, -r_X, k)$ *is:*

1) *empty if* $i_X = 1$, $2k < g_X + 3$, *and reduced to a single sheaf lying in* $MI_X(k)$ *if* $i_X = 1$, $2k = g_X + 3$;
2) *reduced to the class of* $\mathscr{O}_X^2$ *if* $i_X = 2$, $k = 0$, *and empty if* $i_X = 2$, $k = 1$.
3) *empty if* $i_X = 3$, $k = 0$, *and reduced to a single sheaf* $\mathscr{S}$ *if* $i_X = 3$, $k = 1$.

Proof. The emptiness result for $i_X = 1$ appears in [BF08]. Uniqueness for $i_X = 1$ is essentially due to Mukai, however a proof can be found in [BF11].

We sketch the argument for the case $i_X = 2$. Let S be a general hyperplane section of X, and recall that S is a smooth Del Pezzo surface, polarized by $H_S = -K_S$. Let E be any slope-semistable sheaf on X with $c_1(E) = 0$ and $c_2(E) = k$. Set $E_S = E|_S$, and recall that E_S is slope-semistable (with respect to $H_S = H_X \cap S$) by Maruyamas's theorem. Note that this easily implies $H^2(S, E_S) = 0$ by Serre duality, indeed $H^2(S, E_S) \simeq \text{Hom}_S(E_S, \mathscr{O}_S(-1))^*$, and the image of a non-zero morphism $E_S \to \mathscr{O}_S(-1)$ would destabilize E_S.

Now, for $k \leq 1$, Riemann-Roch-Hirzebruch gives $\chi(E_S) > 0$, so there is a non-zero global section of E_S, hence E_S cannot be slope-stable. Then the Harder-Narasimhan (slope) filtration of E_S reads:

$$0 \to E_1 \to E_S \to E_2 \to 0,$$

where each of the E_i's is a torsion-free sheaf of rank 1 on S with $c_1(E_i) = 0$. So each E_i is the ideal sheaf of $c_2(E_i)$ points of S. Since $k \leq 1$, we have $c_2(E_i) \in \{0, 1\}$ for $i = 1, 2$. Then, either E_S is $\mathscr{O}_S^2$ (if $k = 0$) or one of the E_i's is $\mathscr{O}_S$ and the other is $\mathscr{I}_x$, where x is a point of S. When $k = 0$, it is easy to deduce $E \simeq \mathscr{O}_X^2$. When $k = 1$, if $E_1 \simeq \mathscr{I}_x$, then $E_S \simeq \mathscr{O}_S \oplus \mathscr{I}_x$ because $H^1(S, \mathscr{I}_x) = 0$. Then, no case allows E_S to be semistable (in the sense of Maruyama-Gieseker), so E cannot be semistable either.

The uniqueness result for $i_X = 3$, i.e. for the quadric threefolds, is very well known, the unique instanton in this case being the spinor bundle, see e.g. [OS94]. Emptiness for $i_X = 3$, $k = 0$ is clear by Bogomolov's inequality. $\square$

We note that the above proof show that, when $i_X = 2$, an indecomposable slope-semistable rank-2 sheaf E on X with $c_1(E) = 0$ and $c_2(E) = 1$ is a vector bundle fitting into:

$$0 \to \mathscr{O}_X \to E \to \mathscr{I}_L \to 0,$$

where L is a line contained in X.

We will need to use the well-known Hartshorne-Serre correspondence. Adapting [Har80, Theorem 4.1] to our setup, for fixed c_1, c_2, we have a one-to-one correspondence between the sets of:

i) equivalence classes of pairs (F, s), where F is a rank 2 vector bundle on X with $c_i(F) = c_i$ and s is a global section of F, up to multiplication by a non-zero scalar, whose zero locus has codimension 2,
ii) locally complete intersection curves $Z \subset X$ of degree c_2, with $\omega_Z \simeq \mathscr{O}_Z(c_1 - i_X)$.

If the pair (F, s) corresponds to Z in the above bijection, we have:

$$0 \to \mathscr{O}_X \xrightarrow{s} F \to \mathscr{I}_Z(c_1) \to 0.$$

For more details on reflexive sheaves we refer again to [Har80].

2. Odd instantons on a smooth three-dimensional quadric

Throughout this section, the ambient threefold X will be a smooth 3-dimensional quadric hypersurface in $\mathbb{P}^4$, and we will denote it by Q. An (odd) k-instanton E on Q is a rank-2 stable bundle E with $c_1(E) = -1$, $c_2(E) = k$ and $\mathrm{H}^1(Q, E(-1)) = 0$. The Hilbert polynomial of E is:

$$\chi(E(t)) = \frac{t+1}{3}(2t^2 + 4t - 3k + 3).$$

The manifold Q is homogeneous under $\mathrm{Spin}(5)$. Let $\mathscr{S}$ be the spinor bundle on Q. We have that $\mathscr{S}$ is the unique indecomposable bundle of rank 2 on Q with $c_1(\mathscr{S}) = -1$ and $c_2(\mathscr{S}) = 1$. Note that $\mathscr{S}^*(-1) \simeq \mathscr{S}$. We set $U = \mathrm{H}^0(Q, \mathscr{S}(1)) \simeq \mathrm{Hom}_Q(\mathscr{S}, \mathscr{O}_Q)$. Up to isomorphism, U is the unique indecomposable 4-dimensional $\mathrm{Spin}(5)$-module.

Fix an integer $k \geq 2$, a vector space I of dimension $k-1$, a vector space W of dimension k and a symmetric isomorphism $D : W \to W^*$. Recall from the introduction that, in order to study $\mathrm{MI}_Q(k)$, we are lead to consider the vector space $I \otimes W \otimes U$, and to view an element A of $I \otimes W \otimes U$ as a map:

$$A : W^* \otimes \mathscr{S} \to I \otimes \mathscr{O}_Q.$$

Transposing A and composing with D we get:

$$D\,A^{\mathrm{t}} : I^* \otimes \mathscr{O}_Q(-1) \to W^* \otimes \mathscr{S}.$$

We write $\mathscr{Q}_k = \mathscr{Q}_{Q,k}$ so that $\mathscr{Q}_k = \{A \in I \otimes W \otimes U \mid A\,D\,A^{\mathrm{t}} = 0\}$. Recall that we set $\mathrm{G}(W)$ for the orthogonal group $\mathrm{O}(W, D)$, and that $G_k = \mathrm{GL}(I) \times \mathrm{G}(W)$ acts on $\mathscr{Q}_k$ by $(\xi, \eta).A = \xi A \eta^{\mathrm{t}}$.

The goal of this section is to prove the following result, that shows Theorem A and Theorem B for the case $i_X = 3$. The fact that our moduli space is affine closely resembles the main result of [CO03].

Theorem C. *The moduli space* $\mathrm{MI}_Q(k)$ *is the geometric quotient:*

$$\mathscr{Q}_k^{\circ}/G_k \simeq \{A \in \mathscr{Q}_k \mid \mathscr{D}(A) \neq 0\}/G_k$$

where $\mathscr{D}$ *is a* $G_k \times \mathrm{Spin}(5)$-*invariant form on* $I \otimes W \otimes U$. *In particular,* $\mathrm{MI}_Q(k)$ *is affine. Moreover, there is a* $(6k-6)$-*dimensional component of* $\mathrm{MI}_Q(k)$, *smooth along the subvariety consisting of 't Hooft bundles.*

Here, we call *'t Hooft (odd) bundle* a rank-2 bundle E with $c_1(E) = -1$, such that $E(1)$ has a global section that vanishes along the disjoint union of k lines in Q. The proof of the above theorem occupies the rest of this section.

2.1. **Geometry of a smooth quadric threefold.** Let us review some basic facts on the geometry of a smooth three-dimensional quadric in $\mathbb{P}^4$. All the material contained in this part is well-known, we refer to [ES84, OS94, CF12] for more details.

2.1.1. *The spinor bundle on a quadric threefold.* The manifold $Q \subset \mathbb{P}^4 = \mathbb{P}(V)$, is homogeneous under the action of $\mathrm{Spin}(5)$. The Dynkin diagram of $\mathrm{Spin}(5)$ is of type B_2, and the corresponding Lie algebra has two positive roots, α_1 and α_2, where α_1 is the longest root. The quadric Q is identified with $\mathrm{Spin}(5)/\mathrm{P}(\alpha_1)$, and we have $\mathbb{P}^3 = \mathbb{P}(U) \simeq \mathrm{Spin}(5)/\mathrm{P}(\alpha_2)$. The group $\mathrm{Spin}(5)$ acts over V by the standard representation, and over U by the (irreducible) spin representation. Both V and U are equipped with an invariant duality, so we will silently identify U with U^* and V with V^*.

We denote by $\mathscr{S}$ the $\mathrm{Spin}(5)$-homogeneous vector bundle given by the semisimple part of $\mathrm{P}(\alpha_2)$, normalized to $c_1(\mathscr{S}) = -1$. The bundle $\mathscr{S}$ is called the spinor bundle on Q, it is just the restriction of the universal sub-bundle on the 4-dimensional quadric, isomorphic

to $\mathbb{G}(2, U)$. Note that the dual of the universal quotient bundle on $\mathbb{G}(2, U)$ also restricts to $\mathscr{S}$. We refer to [Ott88] for more details on spinor bundles. We have:

$$c_1(\mathscr{S}) = -1, \qquad c_2(\mathscr{S}) = 1, \qquad \mathrm{H}^i(Q, \mathscr{S}(t)) = 0,$$

for all t if $i = 1, 2$ and for $i = 0$, $t \leq 0$. Further, we have natural isomorphisms:

$$\mathrm{H}^0(Q, \mathscr{S}^*) \simeq U, \qquad \mathscr{S}^* \simeq \mathscr{S}(1).$$

Moreover, we have the universal exact sequence:

$$(2.1) \qquad 0 \to \mathscr{S} \to U \otimes \mathscr{O}_Q \to \mathscr{S}(1) \to 0.$$

We can view $\mathscr{S}$ also as the bundle associated with a line $L \subset Q$ via the Hartshorne-Serre correspondence, i.e. we have:

$$0 \to \mathscr{O}_Q(-1) \to \mathscr{S} \to \mathscr{I}_L \to 0.$$

The lines of $\mathbb{P}(V)$ contained in Q are parametrized by $\mathbb{P}^3 = \mathbb{P}(U)$. Let $\mathbb{L}$ denote the variety consisting of pairs (ℓ, x), where ℓ is a line contained in Q and x is a point of ℓ. Then, by the above description, we have $\mathbb{L} \simeq \mathbb{P}(\mathscr{S})$. Moreover, writing (2.1) in families with respect to global sections of $\mathscr{S}$, we get the description of the universal line $\mathbb{L}$ inside $\mathbb{P}(U) \times Q$:

$$(2.2) \qquad 0 \to \mathscr{O}_{\mathbb{P}(U)}(-2) \boxtimes \mathscr{O}_Q(-1) \to \mathscr{O}_{\mathbb{P}(U)}(-1) \boxtimes \mathscr{S} \to \mathscr{O}_{\mathbb{P}(U) \times Q} \to \mathscr{O}_{\mathbb{L}} \to 0.$$

It is well-known that $\mathscr{S}$ is stable, and that it is characterized by indecomposability and by Chern classes. Otherwise, $\mathscr{S}$ can be characterized up to twist as the only nontrivial indecomposable bundle on Q without intermediate cohomology. For a characterization of $\mathscr{S}$ in terms of unstable hyperplanes, see [CF12].

2.1.2. *Derived category of a smooth quadric threefold.* Let us briefly review the structure of the derived category of a quadric threefold, according to [Kap88]. The semiorthogonal decomposition (1) and the bundles $\mathscr{E}_i$'s and $\mathscr{F}_j$'s satisfying (2) reads:

$$(2.3) \qquad \mathrm{D}^b(Q) = \langle \mathscr{O}_Q(-2), \mathscr{O}_Q(-1), \mathscr{S}, \mathscr{O}_Q \rangle.$$

We have the dual decomposition:

$$(2.4) \qquad \mathrm{D}^b(Q) = \langle \mathscr{O}_Q(-1), \mathscr{T}_{\mathbb{P}^4}(-2)|_Q, \mathscr{S}, \mathscr{O}_Q \rangle.$$

2.2. **The monad of and odd instanton on a quadric threefold.** Let E be an odd instanton on Q. Then, according to Definition 1, E is a stable bundle of rank 2 with:

$$(2.5) \qquad c_1(E) = -1, \qquad \mathrm{H}^1(Q, E(-1)) = 0.$$

Let us note incidentally that instanton bundles on quadric hypersurfaces have been defined differently by Laura Costa and Rosa Maria Miró-Roig in [CMR09]. Our definition differs from theirs in that they consider bundles with even c_1. Our analysis starts with the next lemma.

Lemma 2.1. *Let E be a sheaf in $\mathrm{MI}_Q(k)$. Then E has the following cohomology table with respect to (2.4):*

0	0	0	0
0	0	0	0
0	$k-1$	k	$k-1$
0	0	0	0
$\mathscr{O}_Q(-2)$	$\mathscr{O}_Q(-1)$	$\mathscr{S}$	$\mathscr{O}_Q$

Proof. For $X = Q$, we have $q_X = 1$ and $r_X = 1$ so $F = E(1)$. The rightmost and leftmost columns are given by stability of E together with Lemma 1.1 and by Hirzebruch-Riemann-Roch. For the second column from the right, write the twisted Euler sequence:

$$0 \to \mathscr{O}_Q(-2) \to \mathscr{O}_Q(-1)^5 \to \mathscr{T}_{\mathbb{P}^4}(-2)|_Q \to 0,$$

and tensor it by E. Using Lemma 1.1, Serre duality, and $E^*(-1) \simeq E$ we get, for all i:

$$\mathrm{H}^{i-1}(Q, E \otimes \mathscr{T}_{\mathbb{P}^4}(-2)|_Q) \simeq \mathrm{H}^i(Q, E(-2)) \simeq \mathrm{H}^{3-i}(Q, E)^*.$$

These groups vanish for $i = 0, 2, 3$ as we know from the rightmost column, and we are left with $\mathrm{h}^1(Q, E \otimes \mathscr{T}_{\mathbb{P}^4}(-2)|_Q) = k - 1$.

It remains to look at the third column. We would like to prove:

$$\mathrm{H}^i(Q, E \otimes \mathscr{S}) = 0, \qquad \text{for } i \neq 1,$$

and this will finish the proof, for the dimension of $\mathrm{H}^2(Q, E \otimes \mathscr{S})$ will then be computed again by Hirzebruch-Riemann-Roch. The case $i = 0$ is clear by stability, and using Serre duality we see that the case $i = 3$ follows from stability too. To show the case $i = 2$, we tensor by $E(-1)$ the exact sequence (2.1), and we use $\mathscr{S}^*(-1) \simeq \mathscr{S}$ and $\mathrm{H}^i(Q, E(-1)) = 0$ for all i. We get $\mathrm{H}^2(Q, \mathscr{S} \otimes E) \simeq \mathrm{H}^3(Q, E \otimes \mathscr{S}(-1))$ and this group vanishes by Serre duality and by stability of $\mathscr{S}$ and E. $\qquad\square$

Lemma 2.2. *Let $k \geq 2$ and E be a sheaf in $\mathrm{MI}_Q(k)$. Then E is the cohomology of the following monad:*

$$(2.6) \qquad\qquad I^* \otimes \mathscr{O}_Q(-1) \xrightarrow{D\,A^{\mathrm{t}}} W^* \otimes \mathscr{S} \xrightarrow{A} I \otimes \mathscr{O}_Q,$$

where $I \simeq \mathbb{C}^{k-1}$, $W \simeq \mathbb{C}^k$, $D : W \to W^*$ is a symmetric duality, and A is surjective. Conversely, the cohomology of a monad of this form sits in $\mathrm{MI}_Q(k)$.

Proof. Let $k \geq 2$ and let E be an odd k-instanton on Q. The decomposition of $\mathrm{D}^{\mathrm{b}}(Q)$ allows us to write E as cohomology of the complex $\mathscr{C}_E^{\cdot}$ whose terms are given by (1.5), once we compute the cohomology table of E with respect to the collection (2.3). By the previous lemma, this gives that E is the cohomology of a monad of the form:

$$\mathrm{H}^1(Q, E \otimes \mathscr{T}_{\mathbb{P}^4}(-2)|_Q) \otimes \mathscr{O}_Q(-1) \to \mathrm{H}^1(Q, E \otimes \mathscr{S}) \otimes \mathscr{S} \to \mathrm{H}^1(Q, E) \otimes \mathscr{O}_Q.$$

We set:

$$I = \mathrm{H}^1(Q, E), \qquad W = \mathrm{H}^2(Q, E \otimes \mathscr{S})^*.$$

We have computed in the previous lemma $\dim(I) = k - 1$, $\dim(W) = k$, and the proof of the previous lemma gives $\mathrm{H}^1(Q, E \otimes \mathscr{T}_{\mathbb{P}^4}(-2)|_Q) \simeq I^*$. We can thus rewrite the above resolution as:

$$(2.7) \qquad\qquad 0 \to I^* \otimes \mathscr{O}_Q(-1) \xrightarrow{A'} W^* \otimes \mathscr{S} \xrightarrow{A} I \otimes \mathscr{O}_Q \to 0.$$

Now, since E is locally free we have a natural skew-symmetric duality $\kappa : E \to E^*(-1)$. One can easily prove that κ must lift to a skew-symmetric isomorphism between the resolution (2.7) and:

$$0 \to I^* \otimes \mathscr{O}_Q(-1) \xrightarrow{A^{\mathrm{t}}} W \otimes \mathscr{S}^*(-1) \xrightarrow{(A')^{\mathrm{t}}} I \otimes \mathscr{O}_Q \to 0.$$

Let $\tilde{\kappa}$ be the isomorphism from the middle term of the resolution above to the middle term of our original resolution. Then $\tilde{\kappa}$ is skew-symmetric and lies in:

$$\mathrm{H}^0\!\left(Q, \bigwedge^2 (W^* \otimes \mathscr{S}^*) \otimes \mathscr{O}_Q(-1)\right) \simeq \begin{cases} \mathrm{H}^0(Q, \mathrm{S}^2\, W^* \otimes \mathscr{O}_Q) \\ \qquad\quad \oplus \\ \mathrm{H}^0(Q, \wedge^2 W^* \otimes \mathrm{S}^2\, \mathscr{S}^*(-1)). \end{cases}$$

The second summand in the above decomposition is zero. On the other hand, we may identify the first one with $\mathrm{S}^2\, W^*$. The element D of $\mathrm{S}^2\, W^*$ induced by $\tilde{\kappa}$ thus gives a symmetric isomorphism $W \to W^*$, and we have $A' = D\, A^{\mathrm{t}}$.

For the converse implication, let E be the cohomology sheaf of such a monad. A straightforward computation shows that E has the desired Chern classes. Further, it is easy to see that E satisfies $\mathrm{H}^0(Q, E) = 0$ for $\mathrm{H}^1(Q, \mathscr{O}_Q(-1)) = 0$ and $\mathrm{H}^0(Q, \mathscr{S}) = 0$, so E is stable. Finally, twisting the monad by $\mathscr{O}_Q(-1)$ we immediately see that E satisfies the cohomology vanishing required to lie in $\mathrm{MI}_Q(k)$. $\qquad\square$

Note that, for $k = 1$, the above lemma says that $\mathrm{MI}_Q(1)$ consists only of the sheaf $\mathscr{S}$. Here we describe the moduli space $\mathrm{MI}_Q(k)$ by rephrasing the non-degeneracy condition of the map A in terms of an invariant resembling the hyperdeterminant.

Proof of Theorem C, first part. We prove here all statements of Theorem C except the part related to 't Hooft bundles.

Let us first define the form $\mathscr{D}$. Recall that there is a natural identification of U and U^*. Consider an element A of $I \otimes W \otimes U$ as a linear map:

$$\phi_A : W^* \otimes U \to I.$$

Taking the symmetric square of ϕ_A we obtain a linear map:

$$(2.8) \qquad \mathrm{S}^2(\phi_A) : \wedge^2 W^* \otimes \wedge^2 U \to \mathrm{S}^2 I,$$

and recall that $\wedge^2 U$ contains a unique Spin(5)-invariant subspace of rank 1, generated by a 2-form ω. We consider thus $\langle \omega \rangle \subset \wedge^2 U$, and the restriction of (2.8) to $\wedge^2 W^* \otimes \langle \omega \rangle \to \mathrm{S}^2 I$. This gives a square matrix M_A of order $\binom{k+1}{2}$. We define:

$$\mathscr{D}(A) = \det(M_A).$$

Clearly, $\mathscr{D}$ is $G_k \times \mathrm{Spin}(5)$-invariant by definition.

In view of the previous lemma, the next thing to do is to show that $\mathscr{Q}_k^{\circ} = \{A \in \mathscr{Q}_k \mid \mathscr{D}(A) \neq 0\}$, i.e. that $A \in \mathscr{Q}_k$ is surjective as a map $W^* \otimes \mathscr{S} \to I \otimes \mathscr{O}_Q$ if and only if $A \in \mathscr{Q}_k$ satisfies $\mathscr{D}(A) \neq 0$.

Let us now show that, that the first condition implies the second. So, assume that $A \in \mathscr{Q}_k$ corresponds to a surjective map $W^* \otimes \mathscr{S} \to I \otimes \mathscr{O}_Q$, let E be the odd instanton bundle associated with A, and let us check that $\mathscr{D}(A)$ is non-zero. So let E be defined by the monad (2.6), and set $K = \ker(A)$. It is easy to check the vanishing of $\mathrm{H}^j(Q, K(-1))$ for all j.

Considering the symmetric square of (2.6) we obtain the exact sequences:

$$(2.9) \qquad 0 \to \mathrm{S}^2 I^*(-2) \to K \otimes I^*(-1) \to \wedge^2 K \to \wedge^2 E \to 0,$$

$$(2.10) \qquad 0 \to \wedge^2 K \to \begin{matrix} \wedge^2 W^* \otimes \mathrm{S}^2 \mathscr{S} \\ \oplus \\ \mathrm{S}^2 W^* \otimes \wedge^2 \mathscr{S} \end{matrix} \to W^* \otimes I \otimes \mathscr{S} \to \mathrm{S}^2 I \to 0.$$

Since all the cohomology groups of the first, second, and fourth term in (2.9) vanish, so do those of $\wedge^2 K$. Plugging into (2.10), since $\mathrm{H}^j(Q, \mathscr{S}) = 0$ for all j, and since $\wedge^2 \mathscr{S} \simeq \mathscr{O}_Q(-1)$ we obtain an isomorphism:

$$(2.11) \qquad \wedge^2 W^* \otimes \mathrm{H}^1(Q, \mathrm{S}^2 \mathscr{S}) \simeq \mathrm{S}^2 I.$$

Note that the 1-dimensional vector space $\mathrm{H}^1(Q, \mathrm{S}^2 \mathscr{S})$ is isomorphic to $\mathrm{Ext}^1_Q(\mathscr{S}^*, \mathscr{S})$, and this extension corresponds to the universal exact sequence (2.1). Recall that

$$\wedge^2 U \simeq V \oplus \langle \omega \rangle,$$

as Spin(5)-modules, and $\mathrm{H}^0(Q, \mathscr{O}_Q(1)) \simeq V$. Then, taking cohomology in the symmetric square of (2.1) we see that $\mathrm{H}^1(Q, \mathrm{S}^2 \mathscr{S})$ equals the kernel of $\wedge^2 U \to V$. Therefore, taking cohomology in (2.10) and restricting to $\wedge^2 W^* \otimes \mathrm{H}^1(Q, \mathrm{S}^2 \mathscr{S})$ (i.e. considering the isomorphism of (2.11)) amounts to looking at the restriction to $\wedge^2 W^* \otimes \langle \omega \rangle$ of $\mathrm{S}^2(\phi_A)$ (i.e. the map whose determinant is $\mathscr{D}(A)$). So $\mathscr{D}(A) \neq 0$.

Conversely, let us now prove that, if A lies in $\mathscr{Q}_k$ and A is not surjective, then $\mathscr{D}(A) = 0$. This time we prefer to work dually and consider the map A^{t} and its lift $\phi_A^{\mathrm{t}} : I^* \to W \otimes U$. A point $[v] \in Q$ is represented by a vector $v \in V$ and we have a projection $\pi_v : U \to \mathscr{S}_v^*$. The assumption that A fails to be surjective (i.e., A^{t} is not injective) means that we can find a vector $y \in I^*$ such that $A_v^{\mathrm{t}}(y) = 0$ at some point $[v] \in Q$. Equivalently, $\mathrm{id}_W \otimes \pi_v\, \phi_A^{\mathrm{t}}(y) = 0$.

We consider the map g defined as the composition:

$$\mathrm{S}^2\, I^* \xrightarrow{\mathrm{S}^2(\phi_A^{\mathrm{t}})} \wedge^2 W \otimes U \otimes U \xrightarrow{\mathrm{id}_{\wedge^2 W \otimes U}\, \otimes\, \pi_v} \wedge^2 W \otimes U \otimes \mathscr{S}_v^*.$$

Under our degeneracy assumption, we have $g(y^2) = 0$. We would like to check that $\mathrm{S}^2(\phi_A^{\mathrm{t}})(y^2)$ goes to zero under the projection Π of $\wedge^2 W \otimes U \otimes U$ onto $\wedge^2 W \otimes \langle \omega \rangle$ (recall that we have an invariant splitting $\wedge^2 U \simeq V \oplus \langle \omega \rangle$), for in this case we clearly have $\mathscr{D}(A) = 0$. Note that we have the factorization:

$$g = \mathrm{id}_{\wedge^2 W} \otimes j_v\, \Pi\, \mathrm{S}^2(\phi_A^{\mathrm{t}}),$$

where $j_v : \langle \omega \rangle \to U \otimes \mathscr{S}_v^*$ is defined by the symmetric square of (2.1).

Hence we are done if we prove that j_v is injective. But the map j_v is the evaluation at $[v] \in Q$ of the natural map $\mathscr{O}_Q \to \mathrm{H}^0(Q, \mathscr{S}^*)^* \otimes \mathscr{S}^*$, which is a fibrewise injective map of vector bundles.

We have thus proved that $\mathscr{Q}_k^\circ = \{A \in \mathscr{Q}_k \mid \mathscr{D}(A) \neq 0\}$. One then shows that the map $\{A \in \mathscr{Q}_k \mid \mathscr{D}(A) \neq 0\} \to \mathrm{MI}_Q(k)$ that associates with A the cohomology of the monad given by A is an affine geometric quotient, in the same way as in [CO03, Section 2]. $\qquad\square$

This proves the first part of Theorem C. In the next section we look at 't Hooft bundles on Q, and give the proof of the second part of Theorem C.

2.3. Unobstructedness of odd 't Hooft bundles.
By analogy with the case of $\mathbb{P}^3$, we speak of (odd) 't Hooft bundles for (odd) instantons associated with a configuration of skew lines contained in Q.

Definition 2. Let $\mathrm{L} = L_1 \cup \cdots \cup L_k$ be the union of $k \geq 2$ disjoint lines in Q. Then by Hartshorne-Serre's construction we have a rank-2 bundle E_{L} fitting into:

$$(2.12) \qquad\qquad 0 \to \mathscr{O}_Q(-1) \to E_{\mathrm{L}} \to \mathscr{I}_{\mathrm{L},Q} \to 0.$$

The bundle E_{L} is said to be the 't Hooft (odd) bundle associated with L. If L is contained in (one ruling of) a smooth hyperplane section of Q, then E_{L} is said to be a special 't Hooft bundle.

The following proposition shows that $\mathrm{MI}_Q(k)$ is smooth along the subvariety of 't Hooft bundles, thus completing the proof of Theorem C.

Proposition 2.3. *Let $\mathrm{L} \subset Q$ be a union of k disjoint lines, and let $E = E_{\mathrm{L}}$ be defined as above. Then E lies in $\mathrm{MI}_Q(k)$, and satisfies $\mathrm{Ext}_Q^2(E, E) = 0$.*

Proof. Set $E = E_{\mathrm{L}}$. Let us first note that $c_1(E) = -1$ (so $E^*(-1) \simeq E$) and $c_2(E) = k$. We observe that there are isomorphisms:

$$(2.13) \qquad\qquad E^* \otimes \mathscr{O}_{\mathrm{L}} \simeq \mathscr{N}_{\mathrm{L},Q} \simeq \bigoplus_{j=1,\ldots,k} \mathscr{O}_{L_j} \oplus \mathscr{O}_{L_j}(1).$$

To prove that E satisfies the cohomology vanishing (2.5), it suffices to take cohomology of (2.12), twisted by $\mathscr{O}_Q(-1)$, and to observe that $\mathrm{H}^i(Q, \mathscr{O}_{\mathrm{L}}(-1)) = 0$ for all i. Moreover, stability of E is easily checked, since $\mathrm{H}^0(Q, E_{\mathrm{L}}) = 0$ follows immediately from (2.12).

Let us now prove $\mathrm{Ext}_Q^2(E, E) = 0$. Applying the functor $\mathrm{Hom}_Q(E, -)$ to the exact sequence (2.12), we obtain:

$$\mathrm{H}^2(Q, E) \to \mathrm{Ext}_Q^2(E, E) \to \mathrm{H}^2(Q, E^* \otimes \mathscr{I}_{\mathrm{L},Q}).$$

Since E is an odd instanton bundle, we have seen in Lemma 2.1 that $\mathrm{H}^2(Q, E) = 0$. So we only need to check $\mathrm{H}^2(Q, E^* \otimes \mathscr{I}_{\mathrm{L},Q}) = 0$. To do this, we take cohomology of the exact sequence:

$$0 \to E^* \otimes \mathscr{I}_{\mathrm{L},Q} \to E^* \to E^* \otimes \mathscr{O}_{\mathrm{L}} \to 0.$$

Using (2.13) we see that $\mathrm{H}^1(Q, E^* \otimes \mathscr{O}_{\mathrm{L}}) = 0$ and Lemma 1.1 gives $\mathrm{H}^2(Q, E^*) = 0$. This implies the desired vanishing. □

2.4. **Jumping lines of an odd instanton over a quadric threefold.** Let E be an odd instanton on Q, and L a line contained in Q.

Definition 3. We say that L is *jumping* for E if:

$$\mathrm{H}^1(L, E|_L) \neq 0.$$

Note that [CF12, Proposition 5.3] ensures that not all lines contained in Q are jumping for E. We say that E has *generic splitting* if the set of jumping lines of E has codimension 2 in $\mathbb{P}(U)$. We denote by $\mathscr{C}_E$ the curve of jumping lines of an instanton E with generic splitting, and by i its embedding in $\mathbb{P}(U)$.

Proposition 2.4. *Let E be a k-instanton on Q with generic splitting. Then $\mathscr{C} = \mathscr{C}_E$ is equipped with a torsion-free sheaf $\mathscr{F}$ fitting into:*

$$0 \to \mathscr{O}_{\mathbb{P}(U)}(-k) \to W^* \otimes \mathscr{O}_{\mathbb{P}(U)}(-1) \xrightarrow{B_E} I \otimes \mathscr{O}_{\mathbb{P}(U)} \to i_*\mathscr{F} \to 0.$$

The instanton E can be recovered from $(\mathscr{C}, \mathscr{F})$.

Proof. Recall that we described in Section 2.1.1 the family of lines in Q as $\mathbb{P}(U)$, and the universal line $\mathbb{L}$ over Q as $\mathbb{P}(\mathscr{S})$. Denote by p and q the projections from $\mathbb{L}$ to Q and $\mathbb{P}(U)$. A line L is jumping for E if and only if the point of $\mathbb{P}(U)$ corresponding to L lies in the support of $\mathbf{R}^1 q_* p^*(E)$.

Applying $q_* p^*$ to the terms of the monad of E with the help of (2.2), we get $q_* p^*(\mathscr{O}_Q) \simeq \mathscr{O}_{\mathbb{P}(U)}$, $q_* p^*(\mathscr{O}_Q(-1)) = 0$, and $q_* p^*(\mathscr{S}) \simeq \mathscr{O}_{\mathbb{P}(U)}(-1)$, where this last isomorphism follows easily from $\mathrm{H}^i(Q, \mathscr{S} \otimes \mathscr{S}) = 0$ for $i \neq 0$ and $\mathrm{H}^1(Q, \mathscr{S} \otimes \mathscr{S}) \simeq \mathbb{C}$. All higher direct images of the terms of the monad vanish.

Therefore, we get the desired map B_E as $q_* p^*(A)$, and $\mathbf{R}^1 q_* p^*(E) \simeq \mathrm{coker}(B_E)$. Since E has generic splitting, the support $\mathscr{C}$ of $\mathrm{coker}(B_E)$ is determinantal a curve in $\mathbb{P}(U)$, so that $\mathrm{coker}(B_E)$ is the extension by zero to $\mathbb{P}(U)$ of a Cohen-Macaulay sheaf $\mathscr{F}$ on $\mathscr{C}$ and $\ker(B_E) \simeq \mathscr{O}_{\mathbb{P}(U)}(-k)$. We get thus the desired exact sequence.

Finally, given $(\mathscr{F}, \mathscr{C})$, we obtain the displayed exact sequence by sheafifying the minimal graded free resolution of the graded $\mathbb{C}[x_0, \ldots, x_3]$-module associated with $i_*\mathscr{F}$, so that B_E is determined by $\mathscr{F}$ up to conjugacy. But $B_E \in \mathrm{Hom}_{\mathbb{P}(U)}(W^* \otimes \mathscr{O}_{\mathbb{P}(U)}(-1), I \otimes \mathscr{O}_{\mathbb{P}(U)}) \simeq W \otimes I \otimes U$ is nothing but A, so E is reconstructed from $(\mathscr{F}, \mathscr{C})$. □

3. Instanton bundles on Del Pezzo threefolds

Here we will look at instanton bundles on Fano threefolds of index 2, also called Del Pezzo threefolds, still in the assumption of Picard number one. According to Iskovskikh's classification, see [Isk77, Isk78], consult also [IP99], there are 5 deformation classes of these threefolds, characterized by the degree $d_X = H_X^3$, that ranges from 1 to 5.

Recall that a k-instanton bundle E in this case is a stable rank-2 bundle on X with:

(3.1) $$c_1(E) = 0, \qquad c_2(E) = k, \qquad \mathrm{H}^1(X, E(-1)) = 0.$$

The Hilbert polynomial of E is:

$$\chi(E(t)) = \frac{t+1}{3}(dt^2 + 2dt - 3k + 6).$$

3.1. **Construction of instanton bundles on Del Pezzo threefolds.** Some properties of the moduli spaces $\mathrm{M}_X(2,0,k)$ and $\mathrm{MI}_X(k)$ have been already investigated. In [MT01], it is proved that $\mathrm{MI}_X(2)$ is étale over an open subset of the intermediate Jacobian $J(X)$ of X via the Abel-Jacobi mapping when $d_X = 3$, i.e. when X is a cubic threefold, and the degree of this cover is one according to [IM00a]. In fact $\mathrm{MI}_X(2,0,2)$ is a blowup of $J(X)$ along $\mathscr{H}_0^1(X)$ by [Dru00]. Via the same map, by [MT03], $\mathrm{MI}_X(3)$ parametrizes the Theta divisor of $J(X)$ when $d = 2$.

The aim of this section, however, is to construct a well-behaved component of $\mathrm{MI}_X(k)$ for all X and for all $k \geq 2$. This, together with the construction of Proposition 2.3 (for $i_X = 3$) and [BF08, Theorem 3.9] (for $i_X = 2$) will achieve the proof of Theorem 3.1.

Theorem 3.1. *Let X be a Fano threefold of index 2 and $k \geq 2$ an integer. Then there exists a $(4k-3)$-dimensional, generically smooth component of $\mathrm{MI}_X(k)$.*

Proof. Let X be a smooth Fano threefold of index 2 and degree $d = d_X$, so that H_X maps X to $\mathbb{P}^{d+1}$ (with a point of indeterminacy if $d = 1$). We work by induction on $k \geq 2$, following an idea developed in [BF08]. We would like to prove, for all $k \geq 2$, that there exists a vector bundle E satisfying (3.1), and a line $L \subset X$ with:

$$(3.2) \qquad\qquad \mathrm{Ext}_X^2(E,E) = 0, \qquad E|_L \simeq \mathscr{O}_L^2.$$

Step 1. *Base of the induction, $k = 2$.* This case has been studied in several papers, see for instance [AC00, MT03, MT01, Fae05]. We sketch here a uniform argument. Let S be a general hyperplane section of X, so that S is a Del Pezzo surface of degree d, given as blowup of $\mathbb{P}^2$ in $9 - d$ points. Let h be the pullback to S of the class of a line in $\mathbb{P}^2$, and denote by e_i's the exceptional divisors, hence $(H_X)|_S = 3h - e_1 - \cdots - e_{9-d}$.

Take a general curve C of class $3h - e_1 - \cdots - e_{7-d}$, so that C is a smooth irreducible curve of genus 1 and degree $d + 2$. Note that the normal bundle $\mathscr{N}_{C,X}$ fits into:

$$0 \to \mathscr{O}_C(C) \to \mathscr{N}_{C,X} \to \mathscr{O}_C(H_C) \to 0,$$

where we set $H_C = (H_X)|_C$. Clearly, we have $\mathrm{h}^0(C, \mathscr{O}_C(H_C)) = d + 2$ and $\mathrm{H}^1(C, \mathscr{O}_C(H_C)) = 0$, since C is an elliptic curve of degree $d + 2$. Also, working in the Del Pezzo surface S, since C is a smooth irreducible elliptic curve with $C^2 = 9 - 7 + d = d + 2$, we have $\mathrm{h}^0(C, \mathscr{O}_C(C)) = d + 2$ and $\mathrm{H}^1(C, \mathscr{O}_C(C)) = 0$. By the above exact sequence we get $\mathrm{h}^0(C, \mathscr{N}_{C,X}) = 2d + 4$ and $\mathrm{H}^1(C, \mathscr{N}_{C,X}) = 0$. So the Hilbert scheme $\mathscr{H}_1^{d+2}(X)$ is smooth of dimension $2d + 4$ at the point corresponding to C. Let us prove that a general deformation of C in $\mathscr{H}_1^{d+2}(X)$ is non-degenerate.

Assuming the contrary, we consider the incidence of pairs (D, H) in $\mathscr{H}_1^{d+2}(X) \times \check{\mathbb{P}}^{d+1}$ such that D lies in H, and we may suppose that the projection onto (an open neighborhood of the point given by C) of $\mathscr{H}_1^{d+2}(X)$ is dominant. Note that the projection onto $\check{\mathbb{P}}^{d+1}$ is dominant by construction. Observe now that the fibre over C is a single point corresponding to the hyperplane section S, so the same happens to the fibre over a general D specializing to C. On the other hand, the general fibre over a point of $\check{\mathbb{P}}^{d+1}$ corresponding to a hyperplane $H \subset X$ can be identified with $\mathbb{P}(\mathrm{H}^0(H, \mathscr{O}_H(C))) = \mathbb{P}^{d+2}$. The dimension count leads thus to argue that $\mathscr{H}_1^{d+2}(X)$ should be of dimension $2d + 3$, locally around the point corresponding to C. This is a contradiction, so we have proved that a general deformation D of C lies in no hyperplane.

Now, the correspondence of Hartshorne-Serre associates with D a rank-2 vector bundle E fitting into:

$$(3.3) \qquad\qquad 0 \to \mathscr{O}_X(-1) \to E \to \mathscr{I}_{D,X}(1) \to 0,$$

$$0 \to \mathscr{I}_{D,X}(1) \to \mathscr{O}_X(1) \to \mathscr{O}_D(1) \to 0.$$

Computing Chern classes we get $c_1(E) = 0$, $c_2(E) = 2$. Taking cohomology of the above sequences, and of the same sequences twisted by $\mathscr{O}_X(-1)$, immediately says that E

is stable and satisfies (3.1). To compute $\mathrm{Ext}_X^2(E,E)$, note that this group is isomorphic to $\mathrm{H}^2(X, E \otimes E)$ and tensor the sequences in (3.3) by E. We have shown $\mathrm{H}^j(X, E(-1)) = 0$ for all j, and it is easy to show that $\mathrm{H}^2(X, E(1)) = 0$. Therefore, to prove $\mathrm{H}^2(X, E \otimes E) = 0$, it suffices to show $\mathrm{H}^1(D, E(1)|_D) = 0$. But we have $E(1)|_D \simeq \mathscr{N}_{D,X}$, and since D is a general deformation of C and $\mathrm{H}^1(C, \mathscr{N}_{C,X}) = 0$, we also have $\mathrm{H}^1(D, \mathscr{N}_{D,X}) = 0$. We have thus proved $\mathrm{Ext}_X^2(E,E) = 0$.

It remains to find a line $L \subset X$ with $E|_L \simeq \mathscr{O}_L^2$. We choose L in S so that $L \cap C$ is a single point x. Then we have $\mathscr{I}_{C,X}(1) \otimes \mathscr{O}_L \simeq \mathscr{O}_x \oplus \mathscr{O}_L$. Therefore, tensoring the first exact sequence of (3.3) by $\mathscr{O}_L$, we get a surjection of $E|_L$ to $\mathscr{O}_x \oplus \mathscr{O}_L$. Therefore $E|_L$ cannot be $\mathscr{O}_L(t) \oplus \mathscr{O}_L(-t)$ for any $t > 0$. Hence $E|_L$ is trivial. Note that this is an open condition on the variety of lines contained in X, so it takes place for a general line.

Let us now take care of the induction step. This is very similar to the argument used in [BF08, Theorem 3.9]. We give a proof only accounting for the differences with respect to that paper.

Step 2 (*Induction step: defining a non-reflexive sheaf with increasing c_2*)**.** We take a k-instanton E satisfying (3.2) for a general line $L \subset X$. We want to construct a $(k+1)$-instanton F again satisfying (3.2) for some line $L \subset X$, a deformation of a sheaf G with $c_1(G) = 0$, $c_2(G) = k+1$. To do this, we choose a projection $\pi : E \to \mathscr{O}_L$ and we define $G = \ker(\pi)$, so we have:

$$(3.4) \qquad 0 \to G \to E \to \mathscr{O}_L \to 0.$$

It is easy to prove that G is a stable sheaf of rank 2 with $c_1(G) = 0$, $c_2(G) = k+1$ and $c_3(G) = 0$, and using (3.4) we see also $\mathrm{H}^i(X, G(-1)) = 0$ for all i. Further, applying $\mathrm{Hom}_X(-, G)$ to (3.4), it is not hard to prove $\mathrm{Ext}_X^2(G, G) = 0$. Then $\mathrm{M}_X(2, 0, k+1)$ is smooth at the point corresponding to G, and by a Hirzebruch-Riemann-Roch computation it has dimension $4(k+1) - 3 = 4k + 1$. Moreover, tensoring (3.4) by $\mathscr{O}_L$ and using $\mathscr{T}or_1^{\mathscr{O}_X}(\mathscr{O}_L, \mathscr{O}_L) \simeq \mathscr{N}_{L,X}^* \simeq \mathscr{O}_L^2$ for a general choice of L, one sees that $\mathrm{H}^0(X, G \otimes \mathscr{O}_L(-1)) = 0$.

Step 3 (*Induction step: deforming to a locally free sheaf*)**.** We let now F be a general deformation of G in $\mathrm{M}_X(2, 0, k+1)$. By semicontinuity of Ext sheaves (see [BPS80]), we get that F will satisfy (3.1), $\mathrm{Ext}_X^2(F, F) = 0$, and $\mathrm{H}^0(X, F \otimes \mathscr{O}_L(-1)) = 0$. So it only remains to prove that F is locally free, for then F is a $(k+1)$-instanton satisfying (3.2).

To achieve this, one first notes that, for any line $L' \subset X$, in a neighborhood in $\mathscr{H}_0^1(X)$ of L, and for E' in a neighborhood in $\mathrm{M}_X(2, 0, k)$ of E, the sheaf F cannot fit into an exact sequence of the form:

$$(3.5) \qquad 0 \to F \to E' \xrightarrow{\pi'} \mathscr{O}_{L'} \to 0.$$

Indeed, the sheaves fitting into such sequence form a family of dimension $4k-3+2+1 = 4k$, indeed we have $4k - 3$ choices for E', a 2-dimensional family of lines contained in X for the choice of L' and a $\mathbb{P}^1 = \mathbb{P}(\mathrm{H}^0(L', E'|_{L'}))$ for the choice of π'. But F moves in a smooth (open) part of $\mathrm{M}_X(2, 0, k+1)$ of dimension $4k + 1$.

Then, one proves that if F was not locally free, it would have to fit into (3.5), thus finishing the proof. To get this, one takes the double dual:

$$(3.6) \qquad 0 \to F \to F^{**} \to T \to 0,$$

where T is a coherent torsion sheaf having dimension ≤ 1. What we have to show is now that T takes the form $\mathscr{O}_{L'}$, for some line L' contained in X, and that F^{**} is a deformation of E in $\mathrm{M}_X(2, 0, k)$.

Let us first prove $T \simeq \mathscr{O}_{L'}$ for some line $L' \subset X$. To achieve this, it suffices to show the three conditions $\mathrm{H}^0(X, T(-1)) = 0$, $c_2(T) = -1$ and $c_3(T) = 0$. Indeed, $c_2(T) = -1$

means that T is a sheaf of generic rank 1 on a curve of degree 1, which is thus a line L' contained in X. The first condition ensures that T is purely 1-dimensional, i.e., T is torsion-free on L' and the support of T has no isolated points, so that $T \simeq \mathscr{O}_{L'}(t)$ for some t. The condition $c_3(T) = 0$ then amount to $t = 0$. We already have $c_2(T) \leq -1$ since T is supported in codimension 2.

Now, we first see that $\mathrm{H}^0(X, F^{**}(-1)) = 0$. Indeed, a non-zero global section of $F^{**}(-1)$ would induce, via (3.6), a subsheaf K of F having $c_1(K) = 1$, which contradicts F being stable. Then, from $\mathrm{H}^1(X, F(-1)) = 0$ we get $\mathrm{H}^0(T(-1)) = 0$.

Next comes the semicontinuity argument. Take cohomology of (3.6) twisted by $\mathscr{O}_X(t)$, and note that F^{**} is reflexive, so $\mathrm{H}^1(X, F^{**}(t)) = 0$ for $t \ll 0$, by [Har80, Remark 2.5.1] (this remark is formulated for $\mathbb{P}^3$, but it holds in fact for any smooth projective threefold). We get $\mathrm{h}^1(X, T(t)) \leq \mathrm{h}^2(X, F(t))$ for $t \ll 0$. Recall that $c = c_3(F^{**}) \geq 0$ since F^{**} is reflexive, and note that c and $c_2(T)$ are invariant under twist by $\mathscr{O}_X(t)$, so $c_3(T(t)) = -2tc_2(T) + c$. Using Hirzebruch-Riemann-Roch we get:

$$\mathrm{h}^1(X, T(t)) = -\chi(T(t)) = c_2(T)(t+1) - c/2, \qquad \text{for } t \ll 0.$$

On the other hand, by semicontinuity we have $\mathrm{h}^2(X, F(t)) \leq \mathrm{h}^2(X, G(t)) = \mathrm{h}^1(X, \mathscr{O}_L(t)) = -(t+1)$ (by (3.4)) for $t \ll 0$. Summing up, we have:

$$-c_2(T)(t+1) + c/2 \leq -(t+1), \qquad \text{for } t \ll 0.$$

This clearly implies $c_2(T) \geq -1$, hence $c_2(T) = -1$ so by the inequality above (and $c \geq 0$) we deduce $c = 0$ hence also $c_3(T) = 0$. This finishes the proof that $T \simeq \mathscr{O}_{L'}$ for some line $L' \subset X$.

To conclude, we have to show that F^{**} is a deformation of E in $\mathrm{M}_X(2, 0, k)$. But $c_2(T) = -1$ and $c_3(T) = 0$, so $c_2(F^{**}) = k$ and $c_3(F^{**}) = 0$, which implies that F^{**} has the same Chern classes as E. Therefore, F^{**} is clearly a flat deformation of E, and as such is also a semistable sheaf by [Mar76], so F^{**} lies in a neighborhood of E in $\mathrm{M}_X(2, 0, k)$.

$\square$

3.2. **Instanton bundles on Fano threefolds of degree 5.** Here, we focus on the case when X satisfies $\mathrm{H}^3(X) = 0$, in other words to the case when the derived category of X is finitely generated, and study instanton bundles in a monad-theoretic fashion.

In this case, the threefold X is obtained fixing a 5-dimensional vector space U, and cutting $\mathbf{G}(2, U^*) \subset \mathbb{P}^9$ with a $\mathbb{P}^6 \subset \mathbb{P}^9$, i.e.

$$X = \mathbf{G}(2, U^*) \cap \mathbb{P}^6 \subset \mathbb{P}^9 = \mathbb{P}(\wedge^2 U),$$

where the $\mathbb{P}^6$ is chosen so that X is smooth of dimension 3. This threefold is usually denoted as V_5, as its degree $d_X = H_X^3$ is 5. The choice of $\mathbb{P}^6 = \mathbb{P}(V)$ in $\mathbb{P}^9 = \mathbb{P}(\wedge^2 U)$ corresponds to the choice of a 3-dimensional subspace B of $\wedge^2 U$, so that $V = \wedge^2 U/B = \mathrm{H}^0(X, \mathscr{O}_X(1))$. In other words, a 2-dimensional vector subspace Λ of U^* is an element of X if the composition $\sigma_\Lambda : \Lambda \otimes \Lambda \to U^* \otimes U^* \to \wedge^2 U^* \to B^*$ is zero.

Let us now describe the moduli space $\mathrm{MI}_X(k)$ of instanton bundles in this case in terms of nets of quadrics (in fact nets of conics). Given an integer $k \geq 2$, we fix a vector space I of dimension k, and we consider the space $I \otimes U$, and the space $\bigwedge^2(I \otimes U)$ of skew-symmetric vectors on it. An element ω of this space can be regarded as a skew-symmetric map $I^* \otimes U^* \to I \otimes U$, so for all integers j, $\bigwedge^2(I \otimes U)$ contains the locally closed subvarieties of maps of given (necessarily even) rank:

$$R_j = \{\omega \in \bigwedge^2 (I \otimes U) \mid \mathrm{rk}(\omega) = 2j\}.$$

We consider the space of nets of quadrics $\mathrm{S}^2 I \otimes B$ as a subspace of $\bigwedge^2(I \otimes U) = \mathrm{S}^2 I \otimes \wedge^2 B \oplus \wedge^2 I \otimes \mathrm{S}^2 U$, via the embedding $B \hookrightarrow \wedge^2 U$. We have thus the varieties of

nets of quadrics of fixed rank:

$$M_j = R_j \cap S^2\, I \otimes B.$$

We will also need to consider the space Δ of degenerate vectors ω, namely:

$$\Delta = \{\omega \in \bigwedge^2 (I \otimes U) \mid \exists \Lambda = \mathbb{C}^2 \subset U^* \text{ with } \sigma_\Lambda = 0 \text{ and } \omega|_{\Lambda \otimes I^*} \text{ is not injective}\}.$$

The group $\mathrm{GL}(k) = \mathrm{GL}(I)$ acts on I, hence on the nets of quadrics $S^2\, I \otimes B$. Also, $\mathrm{GL}(k)$ acts on M_j, for all j.

Theorem D. *The variety $M_{2k+1} \setminus \Delta$ is a $\mathrm{GL}(k)/\{\pm 1\}$-principal bundle over $\mathrm{MI}_X(k)$ and the associated map $M_{2k+1} \setminus \Delta \to \mathrm{MI}_X(k)$ is a geometric quotient for the action of $\mathrm{GL}(k)$.*

3.2.1. *Derived category of Fano threefolds of degree 5.* We denote by $\mathscr{U}$ the rank 2 universal sub-bundle on X, defined by restriction from $\mathbf{G}(2, U^*)$. We have $\mathrm{Hom}_X(\mathscr{U}, \mathscr{O}_X) \simeq \mathrm{H}^0(X, \mathscr{U}^*) \simeq U$, and $c_1(\mathscr{U}) = -1$, $c_2(\mathscr{U}) = 2$, $\mathscr{U}^*(-1) \simeq \mathscr{U}$. We have a canonical exact sequence:

$$(3.7) \qquad 0 \to \mathscr{U} \to U^* \otimes \mathscr{O}_X \to \mathscr{Q} \to 0,$$

where $\mathscr{Q}$ is the universal rank-3 quotient bundle. According to [Orl91] (see also [Fae05]), a decomposition of the category $\mathrm{D}^{\mathrm{b}}(X)$ in this case has the form:

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathscr{O}_X(-1), \mathscr{U}, \mathscr{Q}^*, \mathscr{O}_X \rangle,$$

and its dual collection is:

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathscr{O}_X(-1), \mathscr{Q}(-1), \mathscr{U}, \mathscr{O}_X \rangle.$$

We can use mutations to obtain an exceptional collection having the form that we need. Right mutating $\mathscr{Q}^*$ through $\mathscr{O}_X$, we get $\mathscr{U}^*$ and this gives the decomposition (1) satisfying (2), namely:

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathscr{O}_X(-1), \mathscr{U}, \mathscr{O}_X, \mathscr{U}^* \rangle,$$

and the dual collection:

$$(3.8) \qquad \mathrm{D}^{\mathrm{b}}(X) = \langle \mathscr{O}_X(-1), \mathscr{Q}(-1), \mathscr{R}, \mathscr{U} \rangle,$$

where $\mathscr{R}$ is the kernel of the natural evaluation $U \otimes \mathscr{U} \to \mathscr{O}_X$. The following vanishing results are well-known:

$$\mathrm{H}^i(X, \mathscr{U}(t)) = 0, \qquad \text{for all } t \text{ if } i = 1, 2, \text{ and for } t \le 0 \text{ if } i = 0.$$

3.2.2. *Monads for instanton bundles on Del Pezzo threefolds of degree 5.* We will write the monad associated with an instanton bundle on a Del Pezzo threefold X of degree 5. Fix an integer $k \ge 2$, a vector space I of dimension k, a vector space W of dimension $4k + 2$ and a skew-symmetric isomorphism $D : W \to W^*$. We look at an element A of $I \otimes W \otimes U$ as a map:

$$A : W^* \otimes \mathscr{O}_X \to I \otimes \mathscr{U}^*,$$

and we can consider:

$$D\, A^{\mathrm{t}} : I^* \otimes \mathscr{U} \to W^* \otimes \mathscr{O}_X.$$

Here the variety $\mathscr{Q}_k = \mathscr{Q}_{X,k}$ is given by $\mathscr{Q}_k = \{A \in I \otimes W \otimes U \mid ADA^{\mathrm{t}} = 0\}$.

Proposition 3.2. *Let E be a sheaf in $\mathrm{MI}_X(k)$ with $k \ge 2$. Then E is the cohomology of a monad:*

$$I^* \otimes \mathscr{U} \xrightarrow{D\, A^{\mathrm{t}}} W^* \otimes \mathscr{O}_X \xrightarrow{A} I \otimes \mathscr{U}^*,$$

with A, I, W, D as above. Conversely, the cohomology of such a monad is a k-instanton bundle on X.

Proof. We have to write the cohomology table (1.4) for a k-instanton E with respect to the collection (3.8), in order to write the complex (1.5) whose cohomology is E. We first tensor (3.7) by $E(-1)$, and use the instanton condition (3.1) together with Lemma 1.1 to obtain natural isomorphisms:

$$\mathrm{H}^{i-1}(X, E \otimes \mathscr{Q}(-1)) \simeq \mathrm{H}^i(X, E \otimes \mathscr{U}(-1)) \simeq \mathrm{H}^{3-i}(X, E \otimes \mathscr{U})^*,$$

where the second isomorphism uses Serre duality and the canonical isomorphisms $E \simeq E^*$, $\mathscr{U}^*(-1) \simeq \mathscr{U}$, $\omega_X \simeq \mathscr{O}_X(-2)$. Now, by stability of E, $\mathscr{Q}$ and $\mathscr{U}$ (proved in [Fae05]), we get that the above groups vanish for $i = 0, 1, 3$. We are only left with the space $\mathrm{H}^1(X, E \otimes \mathscr{Q}(-1))$, which we call I^*, and whose dimension can be computed by Hirzebruch-Riemann-Roch and equals k. This takes care of the second column of (1.4), and also of the fourth, where the only non-vanishing group is $\mathrm{H}^1(X, E \otimes \mathscr{U}) \simeq I$.

We have to compute the cohomology groups $\mathrm{H}^i(X, E \otimes \mathscr{R})$. By Lemma 1.1 we have $\mathrm{H}^{i-1}(X, E) = \mathrm{H}^i(X, E(-2)) = 0$ for $i \neq 2$. We have the defining exact sequence:

$$(3.9) \qquad\qquad 0 \to \mathscr{R} \to U \otimes \mathscr{U} \to \mathscr{O}_X \to 0,$$

and the resolution of $\mathscr{R}$, which is a part of a helix on X (see again [Fae05]):

$$(3.10) \qquad\qquad 0 \to \mathscr{O}_X(-2) \to V^* \otimes \mathscr{O}_X(-1) \to U^* \otimes \mathscr{Q}(-1) \to \mathscr{R} \to 0.$$

Twisting (3.9) by E and using the vanishing we already proved, we get $\mathrm{H}^i(X, E \otimes \mathscr{R}) = 0$ for $i = 0, 3$. Twisting (3.10) by E and again using the vanishing we already have, we also get $\mathrm{H}^2(X, E \otimes \mathscr{R}) = 0$. Thus we are only left with $\mathrm{H}^1(X, E \otimes \mathscr{R})$, we call this space W^* and we compute its dimension $4k + 2$ by Hirzebruch-Riemann-Roch.

We have thus proved that a k-instanton E is the cohomology of a monad of the form (1.9). We still have to check that the monad is self-dual. But this is rather clear, indeed the skew-symmetric duality $\kappa : E \to E^*$ lifts to an isomorphism between our resolution and its transpose, and induces a skew-symmetric duality $D : W \to W^*$ in the middle term.

Finally, to check the converse implication, we first note that the Chern classes of the cohomology sheaf E of a monad of this form are indeed those of a k-instanton on X. In order to see that E is stable, we need to check that $\mathrm{H}^0(X, E) = 0$. Note that this space is isomorphic to $\mathrm{Hom}_X(E, \mathscr{O}_X)$ because A is surjective so that E is locally free. Then, a non-zero global section of E gives a non-zero map $s : E \to \mathscr{O}_X$, which therefore lifts to a map $W^* \otimes \mathscr{O}_X \to \mathscr{O}_X$. In particular s is surjective. Note that $\ker(s)$ is reflexive of rank 1 with vanishing c_1, so $\ker(s) \simeq \mathscr{O}_X$. Hence E is an extension of $\mathscr{O}_X$ by $\mathscr{O}_X$, i.e. $E \simeq \mathscr{O}_X^2$, which contradicts our Chern class computation.

Moreover, twisting the monad by $\mathscr{O}_X(-1)$, and using the cohomology vanishing for $\mathscr{U}$, we immediately see that E satisfies the instanton condition (3.1). $\qquad\qquad \square$

Proof of Theorem D. Let us consider an element ω of $\bigwedge^2(I \otimes U)$ as a skew-symmetric map $I^* \otimes U^* \to I \otimes U$. This induces a commutative diagram:

$$(3.11) \qquad\qquad \begin{array}{ccc} I^* \otimes U^* & \xrightarrow{\;\omega\;} & I \otimes U \\ \Big\downarrow{\scriptstyle A_\omega^{\mathrm{t}}} & & {\scriptstyle A_\omega}\Big\uparrow \\ W & \xrightarrow{\;D\;} & W^*, \end{array}$$

where D is the skew-symmetric duality induced by ω on its image W, and A_ω is the restriction of ω to its image. Since ω lies in R_{2k+1} we have $\dim(W) = 4k + 2$.

We can now consider A_ω as a map:

$$A_\omega : W^* \otimes \mathscr{O}_X \to I \otimes \mathscr{U}^*,$$

and we would like to check that A_ω satisfies $A_\omega \, D \, A_\omega^{\mathrm{t}} = 0$. This composition is a skew-symmetric map $I^* \otimes \mathscr{U} \to I \otimes \mathscr{U}^*$, that corresponds to a skew-symmetric map $I^* \otimes U^* \to I \otimes U$, which is nothing but ω itself.

But the space of skew-symmetric maps $I^* \otimes \mathscr{U} \to I \otimes \mathscr{U}^*$ is:

$$\mathrm{H}^0(X, \bigwedge^2 (I \otimes \mathscr{U}^*)) \simeq \wedge^2 I \otimes \mathrm{H}^0(X, \mathrm{S}^2 \mathscr{U}^*) \oplus \mathrm{S}^2 I \otimes \mathrm{H}^0(X, \wedge^2 \mathscr{U}^*)$$
$$\simeq \wedge^2 I \otimes \mathrm{S}^2 U \oplus \mathrm{S}^2 I \otimes V.$$

So, $A_\omega \, D \, A_\omega^{\mathrm{t}}$ as an element of $\bigwedge^2(I \otimes U)$ is obtained by taking $\omega \in \bigwedge^2(I \otimes U)$ and projecting it on the summand $\wedge^2 I \otimes \mathrm{S}^2 U \oplus \mathrm{S}^2 I \otimes V$. But ω lies in $\mathrm{S}^2 I \otimes B$, and $V = \wedge^2 U / B$, so $A_\omega \, D \, A_\omega^{\mathrm{t}} = 0$. Further, the fact that A is a surjective map of sheaves amounts to the fact that A^{t} is an injective map of vector bundles, which is equivalent to the non-degeneracy condition for ω to lie away from Δ. So, A_ω defines a k-instanton bundle according to Proposition 3.2. This defines our map $M_{2k+1} \setminus \Delta \to \mathrm{MI}_X(k)$, that we denote by Ψ, and we let E_ω be the k-instanton bundle associated with ω.

To continue the proof, we show that Ψ is surjective. This is essentially Proposition 3.2. Indeed, any k-instanton E is the cohomology of a monad given by a map A and a duality D, and the pair (A, D) gives an element ω in view of a diagram of the form (3.11). The vector ω obtained in this way lies in R_{2k+1} because $\dim(W) = 4k + 2$, and away from Δ, because A is surjective. Finally, we have checked that the condition $A_\omega \, D \, A_\omega^{\mathrm{t}} = 0$ amounts to ask that ω lies in $\mathrm{S}^2 I \otimes B \subset \bigwedge^2(I \otimes U)$.

In order to analyze the fibres of Ψ, we note that E_ω is equipped with a distinguished skew-symmetric duality κ_ω, arising from the skew-symmetry of the defining monad. Moreover, it is easy to see that, given ω_1, ω_2, there is an isomorphism $(E_{\omega_1}, \kappa_{\omega_1}) \to (E_{\omega_2}, \kappa_{\omega_2})$ if and only if ω_1 and ω_2 are in the same $\mathrm{GL}(k) = \mathrm{GL}(I)$-orbit. Since E is simple, we have $\omega_i = \pm \mathrm{id}_E$, so the stabilizer of ω under the $\mathrm{GL}(k)$-action is the set of automorphisms of $(E_\omega, \kappa_\omega)$, i.e. ± 1. Then the map Ψ has fibers $\mathrm{GL}(k)/\{\pm 1\}$. Moreover, according to [Kuz12, Proposition 4.9], a net of quadric ω corresponding to an instanton bundle is semistable in the sense of GIT for the action of $\mathrm{GL}(k)$. The fact that the quotient is geometric is now clear. $\qquad\square$

Remark 3.3. I do not know if the moduli space $\mathrm{MI}_X(k)$ affine, if X is a Del Pezzo threefold of degree 5. I do not know if it is smooth and/or irreducible for $k \geq 3$ (for $k = 2$ it is, as proved in [Fae05]). I do not know if there are instantons with an SL_2-structure (cf. [Fae07b] for the case of $\mathbb{P}^3$).

3.3. **Instanton bundles on Fano threefolds of degree 4.** Here we let X be a Fano threefold X of Picard number 1, index 2 and degree 4, so that, if H_X is the (very) ample line bundle generating $\mathrm{Pic}(X)$, we have $K_X = -2H_X$. It is well-known that the Del Pezzo threefold X is the complete intersection of two quadrics in $\mathbb{P}^5$. We will first recall the structure of the derived category of X and then state and prove a theorem on the moduli space of instanton bundles on X, where monads are replaced by vector bundles on a curve of genus 2.

3.3.1. *Derived category of Fano threefolds of degree 4.* Given the variety X, we consider the moduli space $\mathrm{M}_X(2, 1, 2)$, which is isomorphic to a smooth curve Γ of genus 2. This curve is obtained as a double cover of the projective line representing the pencil of quadrics vanishing on X, ramified along the 6 points corresponding to degenerate quadrics. The moduli space is fine and we denote by $\mathscr{E}$ a universal sheaf on $X \times \Gamma$, determined up to a twist by a line bundle on Γ. We denote by $\mathscr{E}_y$ the sheaf in $\mathrm{M}_X(2, 1, 2)$ corresponding to y. We have:

$$c_1(\mathscr{E}) = H_X + N, \qquad c_2(\mathscr{E}) = 2L_X + H_X M + \eta,$$

where L_X is the class of a line in X, M and N are divisor classes on Γ of degree respectively m and $2m - 1$ (here m is an integer that we may choose arbitrarily), and η lies in $\mathrm{H}^{1,2}(X) \otimes \mathrm{H}^{0,1}(\Gamma)$ and satisfies $\eta^2 = 4$.

It is not difficult to show that, if $\mathscr{E}_y$ is a sheaf in $\mathrm{M}_X(2,1,2)$, then $\mathscr{E}_y$ is a stable, locally free, globally generated sheaf, and that we have:

$$(3.12) \qquad\qquad 0 \to \mathscr{E}_{y'}(-1) \to \mathrm{H}^0(X, \mathscr{E}_y) \otimes \mathscr{O}_X \to \mathscr{E}_y \to 0,$$

where y' is conjugate to y in the $2:1$ cover $\Gamma \to \mathbb{P}^1$.

We denote by p and q the projections of $X \times \Gamma$ onto X and Γ, and we consider the functors:

$$\boldsymbol{\Phi} : \mathrm{D}^{\mathrm{b}}(\Gamma) \to \mathrm{D}^{\mathrm{b}}(X), \qquad\qquad \boldsymbol{\Phi}(a) = \mathbf{R}p_*(q^*(a) \otimes \mathscr{E}),$$

$$\boldsymbol{\Phi}^! : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(\Gamma), \qquad\qquad \boldsymbol{\Phi}^!(b) = \mathbf{R}q_*(p^*(b) \otimes \mathscr{E}^* \otimes q^*(\omega_\Gamma))[1],$$

$$\boldsymbol{\Phi}^* : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(\Gamma), \qquad\qquad \boldsymbol{\Phi}^*(b) = \mathbf{R}q_*(p^*(b) \otimes \mathscr{E}^* \otimes p^*(\omega_X))[3].$$

We recall that $\boldsymbol{\Phi}$ is fully faithful, $\boldsymbol{\Phi}^*$ is left adjoint to $\boldsymbol{\Phi}$, and $\boldsymbol{\Phi}^!$ is right adjoint to $\boldsymbol{\Phi}$. If follows from [BO95] that we have the following semiorthogonal decomposition:

$$\mathrm{D}^{\mathrm{b}}(X) \simeq \langle \mathscr{O}_X(-1), \mathscr{O}_X, \boldsymbol{\Phi}(\mathrm{D}^{\mathrm{b}}(\Gamma)) \rangle.$$

An easy computation shows that $\mathscr{V} = (\boldsymbol{\Phi}^*(\mathscr{O}_X))^*$ is a vector bundle of rank 4 and degree $4(1 - m)$ on Γ.

3.3.2. *Instanton bundles on Del Pezzo threefolds of degree 4 and Brill-Noether loci on curves of genus 2.* The goal of this section will be to prove the following result, relating k-instanton bundles on X to simple vector bundles of higher rank on Γ, sitting in a specific Brill-Noether locus. This locus is given by bundles $\mathscr{F}$ on Γ having rank k and degree $k(2 - m)$ that have at least $k - 2$ independent global sections when twisted by $\mathscr{V} = (\boldsymbol{\Phi}^*(\mathscr{O}_X))^*$.

Theorem E. *The map $E \to \boldsymbol{\Phi}^!(E)$ gives:*

i) an open immersion of $\mathrm{MI}_X(2)$ into the moduli space of semistable bundles of rank 2 on Γ, so that $\mathrm{MI}_X(2)$ is a smooth irreducible fivefold;

ii) for $k \geq 3$, an open immersion of $\mathrm{MI}_X(k)$ into the locus $\mathrm{W}(k)$ defined as the isomorphisms classes of:

$$\left\{ \begin{array}{c} \text{simple bundles } \mathscr{F} \text{ on } \Gamma \\ \text{of rank } k \text{ and degree } k(2-m) \end{array} \middle| \mathrm{h}^0(\Gamma, \mathscr{V} \otimes \mathscr{F}) = k - 2 \right\}$$

The main tool will be the next proposition, that provides the analogue of the notion on monad for a classical instanton.

Proposition 3.4. *Let E be a sheaf in $\mathrm{MI}_X(k)$, and set $\mathscr{F} = \boldsymbol{\Phi}^!(E)$. Then:*

i) $\mathscr{F}$ is a simple vector bundle of rank k and degree $k(2 - m)$ on Γ;

ii) there is a functorial exact sequence:

$$(3.13) \qquad\qquad 0 \to \mathscr{O}_X^{k-2} \to \boldsymbol{\Phi}(\mathscr{F}) \to E \to 0,$$

where $\boldsymbol{\Phi}(\mathscr{F})$ is a simple vector bundle of rank k on X;

iii) we have:

$$\mathrm{h}^0(\Gamma, \mathscr{V} \otimes \mathscr{F}) = k - 2.$$

Proof. Since the space $\mathrm{MI}_X(k)$ is not empty we must have $k \geq 2$, see Lemma 1.2. Let y be a point of Γ. We first prove:

$$(3.14) \qquad\qquad \mathrm{H}^i(X, E \otimes \mathscr{E}_y^*) = 0, \qquad \text{for } i \neq 1.$$

Notice that (3.14) holds for $i = 0$ and $i = 3$, as is easily proved using stability of E and $\mathscr{E}_y$ and Serre duality. Then we only need to prove that (3.14) holds for $i = 2$. We use

$\mathscr{E}_y^* \simeq \mathscr{E}_y(-1)$, we twist (3.12) by $E(-1)$ and we use $\mathrm{H}^i(X, E(-1))$ for all i (see Lemma 1.1). We get that (3.14) holds for $i = 2$ if $\mathrm{H}^3(X, E \otimes \mathscr{E}_{y'}(-2)) = 0$. But this vanishing is clear from the stability of E and $\mathscr{E}_{y'}$ and from Serre duality.

We have thus proved (3.14), and we note that $\mathscr{F}$ is then a vector bundle on Γ, whose rank equals $\mathrm{h}^1(X, E \otimes \mathscr{E}_y^*) = k$, which can be computed by Hirzebruch-Riemann-Roch. The degree of $\mathscr{F}$ can by computed by Grothendieck-Riemann-Roch. This finishes the proof of (i), except the statement that $\mathscr{F}$ is simple.

To show (ii), we note that E lies in $^\perp\langle \mathscr{O}_X(-1)\rangle$ by Lemma 1.1. By the same lemma, and by stability of E, we have $\mathrm{H}^i(X, E) = 0$ for $i \neq 1$ and by Hirzebruch-Riemann-Roch we have $\mathrm{h}^1(X, E) = k - 2$. Note also that $\mathrm{Ext}^1_X(E, \mathscr{O}_X) \simeq \mathrm{H}^1(X, E)$. Then we have a functorial exact sequence:

$$0 \to \mathscr{O}_X^{k-2} \to \tilde{E} \to E \to 0,$$

for some universal extension represented by a vector bundle $\tilde{E}$. Then, $\tilde{E}$ belongs to $^\perp\langle \mathscr{O}_X(-1), \mathscr{O}_X \rangle = \boldsymbol{\Phi}(\mathrm{D}^\mathrm{b}(\Gamma))$, so $\tilde{E} \simeq \boldsymbol{\Phi}(a)$ for some object a in $\mathrm{D}^\mathrm{b}(\Gamma)$. But applying $\boldsymbol{\Phi}^!$ to the above sequence one gets $\mathscr{F} \simeq \boldsymbol{\Phi}^!(\boldsymbol{\Phi}(a))$ so $\mathscr{F} \simeq a$, and we get the resolution (3.13).

Note that, to finish the proof of (i) and (ii), it will be enough to show that $\mathscr{F}$ is simple, because $\boldsymbol{\Phi}$ is fully faithful, so that $\boldsymbol{\Phi}(\mathscr{F})$ will be simple as well. In order to achieve this, we note that:

$$\mathrm{Hom}_\Gamma(\mathscr{F}, \mathscr{F}) \simeq \mathrm{Hom}_X(\boldsymbol{\Phi}(\mathscr{F}), E).$$

Then, applying $\mathrm{Hom}_X(-, E)$ to (3.13) we get that this group has dimension 1 since E is stable (hence simple) and $\mathrm{H}^0(X, E) = 0$.

It remains to show (iii). Note that $\mathrm{h}^0(\Gamma, \mathscr{V} \otimes \mathscr{F})$ equals the dimension of:

$$\mathrm{Hom}_\Gamma(\mathscr{V}^*, \mathscr{F}) = \mathrm{Hom}_\Gamma(\boldsymbol{\Phi}^*(\mathscr{O}_X), \mathscr{F}) \simeq \mathrm{Hom}_X(\mathscr{O}_X, \boldsymbol{\Phi}(\mathscr{F})).$$

Note also that, applying $\mathrm{Hom}_X(\mathscr{O}_X, -)$ to (3.13) we get $\mathrm{h}^0(X, \boldsymbol{\Phi}(\mathscr{F})) = k - 2$ since $\mathrm{H}^0(X, E) = 0$. This gives (iii). $\qquad\square$

Proof of Theorem E. We first note that the map $E \to \boldsymbol{\Phi}^!(E)$ is injective. Indeed, after setting $\mathscr{F} = \boldsymbol{\Phi}^!(E)$, we recall from the proof of Proposition 3.4 that $\mathrm{H}^0(X, \boldsymbol{\Phi}(\mathscr{F})) \simeq \mathrm{H}^0(\Gamma, \mathscr{F})$ and that this space has dimension $k-2$. Then, the map $\mathscr{O}_X^{k-2} \to \boldsymbol{\Phi}(\mathscr{F})$ appearing in (3.13) is naturally determined from $\boldsymbol{\Phi}(\mathscr{F})$ as the natural evaluation of sections, so that we can recover its cokernel E only from $\boldsymbol{\Phi}(\mathscr{F})$, i.e. from $\mathscr{F}$.

Then, we note that the differential of this map at the point of $\mathrm{MI}_X(k)$ given by E identifies the tangent space of $\mathrm{MI}_X(k)$ and the space containing obstructions to $\mathrm{MI}_X(k)$ at E with those of $\mathrm{W}(k)$ at $\mathscr{F}$. Indeed, we first rewrite (3.13) functorially as:

$$0 \to \mathrm{H}^1(X, E)^* \otimes \mathscr{O}_X \to \boldsymbol{\Phi}(\mathscr{F}) \to E \to 0,$$

and we deduce the natural isomorphisms:

$$\mathrm{H}^1(X, E)^* \simeq \mathrm{H}^0(X, \boldsymbol{\Phi}(\mathscr{F})) \simeq \mathrm{H}^0(\Gamma, \mathscr{V} \otimes \mathscr{F}),$$
$$\mathrm{H}^1(X, E) \simeq \mathrm{H}^1(X, \boldsymbol{\Phi}(\mathscr{F})) \simeq \mathrm{H}^1(\Gamma, \mathscr{V} \otimes \mathscr{F}).$$

Using these isomorphisms, the adjoint pair $(\boldsymbol{\Phi}, \boldsymbol{\Phi}^!)$, and applying $\mathrm{Hom}_X(-, E)$ to the exact sequence above we get a commutative diagram:

$$\mathrm{Ext}^1_X(E, E) \to \mathrm{Ext}^1_X(\boldsymbol{\Phi}(\mathscr{F}), E) \longrightarrow \mathrm{H}^1(X, E) \otimes \mathrm{H}^1(X, E) \longrightarrow \mathrm{Ext}^2_X(E, E)$$
$$\parallel \qquad\qquad\qquad\qquad\qquad \parallel$$
$$\mathrm{Ext}^1_\Gamma(\mathscr{F}, \mathscr{F}) \longrightarrow \mathrm{H}^0(\Gamma, \mathscr{V} \otimes \mathscr{F})^* \otimes \mathrm{H}^1(\Gamma, \mathscr{V} \otimes \mathscr{F})$$

This identifies $\mathrm{Ext}^1_X(E, E)$ and $\mathrm{Ext}^2_X(E, E)$ respectively with the kernel and the cokernel of the bottom map in this diagram, the dual Petri map. In turn, by standard Brill-Noether

theory, these are the tangent space at $\mathscr{F}$ of $\mathrm{W}(k)$ and the space of obstructions at $\mathscr{F}$ of $\mathrm{W}(k)$. We have thus proved part (ii) of the theorem.

In order to prove part (i), we only need to show that $\mathscr{F}$ is semistable, and to recall the well-known fact that the moduli space of semistable bundles of rank 2 on Γ is a smooth irreducible variety of dimension 5. For the semistability of $\mathscr{F}$, simply note that, by Proposition 3.4, we have $\mathrm{H}^i(\Gamma, \mathscr{V} \otimes \mathscr{F}) = 0$ for $i = 0, 1$. This is a well-known sufficient condition for semistability. $\square$

4. Odd instanton bundles on prime Fano threefolds

Here we study the case of prime Fano threefolds, namely the case when the Fano threefold X has $i_X = 1$. In this case, the genus g of X is defined as the genus of the canonical curve obtained by taking the intersection of two general hyperplane sections of X, and one has $2g - 2 = H_X^3$. According to Iskovskikh's classification [Isk77, Isk78] there are 10 deformation classes of these threefolds, one for each genus $g \in \{2, 3, \ldots, 12\} \setminus \{11\}$.

An (odd) k-instanton on X is a rank-2 stable bundle E with $c_1(E) = -1$ and $c_2(E) = k$. The Hilbert polynomial of E is:

$$\chi(E(t)) = \frac{t}{3}((2g - 2)t^2 - 3k + g + 11).$$

In [BF08], we proved an existence result, when X is ordinary and not hyperelliptic (see the definition in the introduction, and see [BF08] for a short review of these notions). The claim is that, at least under these assumptions, and setting $m_g = \lceil g/2 \rceil + 1$, there is a $(2k - g - 2)$-dimensional, generically smooth irreducible component of $\mathrm{MI}_X(k)$, as soon as $k \geq m_g$, while $\mathrm{M}_X(2, -1, k)$ is empty for $k < m_g$.

Here, we will focus on the case $\mathrm{H}^3(X) = 0$ (recall that some cases with $\mathrm{H}^3(X) \neq 0$ were studied in [BF08] and [BF12]). This vanishing takes place if and only if $g = 12$. So, we let $k \geq 7$, and there exists a generically smooth $(2k - 14)$-dimensional irreducible component of $\mathrm{MI}_X(k)$. In this case, X is not hyperelliptic, and X is ordinary unless X is the Mukai-Umemura threefold, [MU83, Pro90].

The threefold X naturally sits in $\mathbf{G}(3, 7)$ and in $\mathbf{G}(2, 8)$, and we denote by $\mathscr{U}$ and $\mathscr{E}$ the universal sub-bundles of rank 3 and of rank 2, restricted to X. We set $U = \mathrm{Hom}_X(\mathscr{E}, \mathscr{U})$ (this vector space U has dimension 4). It turns out that $\mathrm{D}^b(X)$ admits a decomposition of the form (1) satisfying (2), with $\mathscr{E}_2 = \mathscr{E}$ and $\mathscr{E}_3 = \mathscr{U}$. In fact $\mathscr{E}$ is the only element in $\mathrm{M}_X(2, -1, 7) = \mathrm{MI}_X(7)$. The embedding of X in $\mathbf{G}(2, 8)$ comes from the description of X as twisted cubics (i.e., Cohen-Macaulay curves of degree 3 and arithmetic genus 0) in $\mathbb{P}(U^*)$ that annihilate a net of quadrics $\mathrm{S}^2 U^* \to B = \mathbb{C}^3$. The transpose of this map is the structural embedding $B^* \hookrightarrow \mathrm{S}^2 U$. In this sense, a point x of X corresponds to a twisted cubic T_x in $\mathbb{P}(U^*)$ equipped with the resolution:

$$0 \to \mathscr{O}_{\mathbb{P}(U^*)}(-4) \to \mathscr{U}^*(-1)_x \otimes \mathscr{O}_{\mathbb{P}(U^*)}(-2) \xrightarrow{\psi_x} \mathscr{E}_x \otimes \mathscr{O}_{\mathbb{P}(U^*)}(-1) \to \omega_{T_x} \to 0.$$

Let us now describe $\mathrm{MI}_X(k+7)$ for $k \geq 1$ in terms of nets of quadrics (in fact, of conics). Take a vector space I of dimension k, and consider the spaces $I \otimes U$ and the symmetric square $\mathrm{S}^2(I \otimes U)$. An element α of $\mathrm{S}^2(I \otimes U)$ is a symmetric map $I^* \otimes U^* \to I \otimes U$, so for any integer j, $\mathrm{S}^2(I \otimes U)$ contains the locally closed subvariety:

$$R_j = \{\alpha \in \mathrm{S}^2(I \otimes U) \mid \mathrm{rk}(\alpha) = j\}.$$

The space of nets of quadrics $\mathrm{S}^2 I \otimes B^*$ is a subspace of $\mathrm{S}^2(I \otimes U) = \mathrm{S}^2 I \otimes \mathrm{S}^2 U \oplus \wedge^2 I \otimes \wedge^2 U$ via the structural embedding $B^* \hookrightarrow \mathrm{S}^2 U$. The varieties of nets of quadrics of fixed rank are:

$$M_j = R_j \cap (\mathrm{S}^2 I \otimes B^*).$$

Using the map ψ_x, we associate with a point $x \in X$ and a point $0 \neq u \in U$ a linear map $\psi_x(u) : \mathscr{U}^*(-1)_x \to \mathscr{E}_x$. Therefore, for any $x \in X$ and $\alpha \in \mathrm{S}^2(I \otimes U)$, we get a linear map:
$$\Psi_x(\alpha) : I^* \otimes \mathscr{U}^*(-1)_x \to I \otimes U \otimes \mathscr{E}_x.$$
The variety Δ of degenerate elements of M_j is thus:
$$\Delta = \{\alpha \in M_j \mid \exists T_x \text{ twisted cubic that annihilates } B \text{ and } \Psi_x(\alpha) \text{ is not injective}\}.$$
The group $\mathrm{GL}(k) = \mathrm{GL}(I)$ acts on I, on M_j, and on $M_j \setminus \Delta$.

Theorem F. *The variety $M_{3k+1} \setminus \Delta$ is a $\mathrm{GL}(k)/\{\pm 1\}$-principal bundle over $\mathrm{MI}_X(k+7)$, and the associated map $M_{3k+1} \setminus \Delta \to \mathrm{MI}_X(k+7)$ is a geometric quotient for the action of $\mathrm{GL}(k)$.*

We will give some more results on $\mathrm{MI}_X(k)$, including a monad-theoretic parametrization, a complete description for low k, and an analysis in terms of jumping conics.

4.1. **Geometry of prime Fano threefolds of genus 12.** Here we briefly recall some of the geometry of a prime Fano threefold X of genus 12, the main references being [Muk92, Sch01, Fae07a].

4.1.1. *Constructions of threefolds of genus 12 and related bundles.* To give the first construction of our threefold, we have to fix two vector spaces $V = \mathbb{C}^7$ and $B = \mathbb{C}^3$ and a net of alternating forms $\sigma : B \to \wedge^2 V^*$. Then X is obtained as the locus in $\mathbf{G}(3, V)$:
$$X = \{\Lambda \subset V \mid \sigma(b)(u \wedge v) = 0 \text{ for any } u, v \in \Lambda, \text{ for any } b \in B\},$$
where we assume that σ is general enough so that X is smooth of the expected codimension 9. The threefold X is said to be of type V_{22}, for its degree H_X^3 is 22 (this is the maximal possible degree of a smooth prime Fano threefold). Since X sits in $\mathbf{G}(3, 7)$, we have a universal sub-bundle (of rank 3) and quotient bundle (of rank 4), and we denote their restrictions to X respectively by $\mathscr{U}$ and $\mathscr{Q}$. The universal exact sequence reads:
$$(4.1) \qquad\qquad 0 \to \mathscr{U} \to V \otimes \mathscr{O}_X \to \mathscr{Q} \to 0,$$
We have $\mathrm{Hom}_X(\mathscr{U}, \mathscr{Q}^*) \simeq B$, and an exact sequence of vector bundles:
$$0 \to \mathscr{K} \to B \otimes \mathscr{U} \to \mathscr{Q}^* \to 0,$$
where $\mathscr{K}$ is defined by the sequence, and turns out to be an exceptional bundle of rank 5.

As already mentioned, the second description of X is as the subvariety of Hilbert scheme of twisted cubics in $\mathbb{P}^3 = \mathbb{P}(U^*)$ consisting of cubics T whose ideal is annihilated by a fixed net of quadrics, again parametrized by B, i.e. all quadrics in the ideal of T should be in the kernel of $\mathrm{S}^2 U^* \to B$.

This gives rise to a universal rank-3 bundle (the 3 generators of the ideal of a twisted cubic), which turns out to be isomorphic to $\mathscr{U}$, and to a universal rank-2 bundle (the 2 syzygies among the 3 generators) which we denote by $\mathscr{E}$. We have $\mathrm{Hom}_X(\mathscr{E}, \mathscr{U}) \simeq U$. By this description, over $\mathbb{P}(B)$ we have a determinantal quartic curve given by the degenerate quadrics in the net. It is called the apolar quartic of X. It turns out that $\mathscr{E}^*$ embeds X in $\mathbf{G}(2, 8)$, and we have a universal exact sequence:
$$(4.2) \qquad\qquad 0 \to \mathscr{E} \to \mathrm{H}^0(X, \mathscr{E}^*)^* \otimes \mathscr{O}_X \to \mathscr{F} \to 0,$$
for some universal quotient bundle $\mathscr{F}$ of rank 6.

Let us summarize the cohomology vanishing satisfied by the bundles under consideration:
$$(4.3) \qquad \mathrm{H}^i(X, \mathscr{E}(t)) = \mathrm{H}^i(X, \mathscr{U}(t)) = 0, \qquad\qquad \text{for all } t \text{ if } i = 1, 2,$$
$$\mathrm{H}^0(X, \mathscr{E}) = \mathrm{H}^0(X, \mathscr{U}) = \mathrm{H}^0(X, \mathscr{U}^*(-1)) = 0.$$

4.1.2. *Derived category of prime Fano threefolds of genus 12.* According to [Fae07a], the category $\mathrm{D}^b(X)$ admits a pair of dual semiorthogonal decompositions:

$$(4.4) \qquad\qquad \mathrm{D}^b(X) = \langle \mathscr{E}, \mathscr{K}, \mathscr{U}, \mathscr{O}_X \rangle,$$

$$(4.5) \qquad\qquad \mathrm{D}^b(X) = \langle \mathscr{E}, \mathscr{U}, \mathscr{Q}^*, \mathscr{O}_X \rangle.$$

Mutating the exceptional collections above, we get a semiorthogonal decomposition satisfying (2):

$$\mathrm{D}^b(X) = \langle \mathscr{O}_X(-1), \mathscr{U}^*(-1), \mathscr{E}, \mathscr{U} \rangle,$$

and the dual decomposition:

$$(4.6) \qquad\qquad \mathrm{D}^b(X) = \langle \mathscr{O}_X, \mathscr{Q}, \mathscr{R}^*, \mathscr{U}^* \rangle,$$

where $\mathscr{R}$ is defined as right mutation of $\mathscr{E}$ with respect to $\mathscr{U}$:

$$(4.7) \qquad\qquad 0 \to \mathscr{E} \to U^* \otimes \mathscr{U} \to \mathscr{R} \to 0.$$

4.1.3. *Conics contained in a prime Fano threefolds of genus 12.* One way to study odd instantons on a smooth prime Fano threefold of genus is to look at their scheme of jumping conics. We will develop this in the case of threefolds of genus 12.

In order to do so, we first take a look at the Hilbert scheme $\mathscr{H}_0^2(X)$ of conics contained in X. It turns out that $\mathscr{H}_0^2(X)$ is identified with the projective plane $\mathbb{P}^2 = \mathbb{P}(B^*)$, cf [IP99, 5.2.15]. However we will use the following description.

Lemma 4.1. *There is a natural isomorphism $B \simeq \mathrm{Hom}_X(\mathscr{U}, \mathscr{Q}^*)$, and:*

i) any element $b \in \mathbb{P}(B^*)$ gives a conic C_b contained in X and an exact sequence:

$$0 \to \mathscr{U} \xrightarrow{b} \mathscr{Q}^* \to \mathscr{I}_{C_b} \to 0;$$

ii) the map $b \mapsto C_b$ defines an isomorphism $\mathbb{P}(B^*) \simeq \mathscr{H}_0^2(X)$;
iii) over the product $X \times \mathbb{P}(B^*)$ there is an exact sequence:

$$(4.8) \qquad 0 \to \mathscr{O}_{X \times \mathbb{P}(B^*)}(-1, -3) \to p_1^*(\mathscr{Q}(-1)) \to \mathscr{U}^*(-1) \boxtimes \mathscr{O}_{\mathbb{P}(B^*)}(1) \to \omega_{\mathscr{C}} \to 0,$$

where p_1 and p_2 denote the projections from $X \times \mathbb{P}(B^*)$ onto the two factors, and $\mathscr{C}$ is the universal conic.

Proof. It is proved in [Fae07a, Lemma 6.1] that there is a natural isomorphism $B \simeq \mathrm{Hom}_X(\mathscr{U}, \mathscr{Q}^*)$. Given $0 \neq b \in B$, we have a map $b : \mathscr{U} \to \mathscr{Q}^*$, and using the stability of $\mathscr{U}$ and $\mathscr{Q}^*$ (proved in [Fae07a, Lemma 6.2]) we get that b is necessarily injective. Then $b^{\mathrm{t}} : \mathscr{Q} \to \mathscr{U}^*$ has generic rank 3 and $\ker(b^{\mathrm{t}})$ is reflexive of rank 1, hence invertible, i.e., $\ker(b^{\mathrm{t}}) \simeq \mathscr{O}_X(t)$.

We set $\mathscr{F}_b = \mathrm{coker}(b^{\mathrm{t}})$ and note that $\mathscr{F}_b$ is a torsion sheaf. By stability of $\mathscr{U}$ we have $t \leq 0$. But we have $c_1(\mathscr{F}_b) = t \geq 0$ so $t = 0$. Hence $c_1(\mathscr{F}_b) = 0$, and computing Chern classes we get $c_2(\mathscr{F}_b) = -2$, so that $\mathscr{F}_b$ is supported in codimension 2. Using the cohomology of $\mathscr{U}^*(-1)$, $\mathscr{Q}(-1)$ and $\mathscr{O}_X(-1)$ one sees that $\mathrm{H}^0(X, \mathscr{F}_b(-1)) = 0$. So, $\mathscr{F}_b$ is supported on a curve C_b without embedded points, and is locally Cohen-Macaulay (hence torsion-free) on C_b. Moreover, C_b has degree 1 or 2 and $\mathscr{F}_b$ has respectively rank 2 or 1 on C_b. We get the exact sequences:

$$0 \to \mathscr{O}_X \to \mathscr{Q} \xrightarrow{b^{\mathrm{t}}} \mathscr{U}^* \to \mathscr{F}_b \to 0,$$

$$0 \to \mathscr{U} \xrightarrow{b} \mathscr{Q}^* \to \mathscr{O}_X \to \mathscr{H}om_{C_b}(\mathscr{F}_b, \omega_{C_b}(1)) \to 0.$$

This says that $\mathscr{O}_{C_b} \simeq \mathscr{H}om_{C_b}(\mathscr{F}_b, \omega_{C_b}(1))$, and since the dual of $\mathscr{F}_b$ over C_b has rank 1, $\mathscr{F}_b$ must have rank 1 too so that C_b has degree 2. Moreover, $\mathrm{coker}(b)$ is the ideal sheaf of

C_b, and by a Hilbert polynomial computation we see that C_b has arithmetic genus 0, i.e., C_b is a conic. We have proved (i). We also get:

$$(4.9) \qquad 0 \to \mathscr{O}_X(-1) \to \mathscr{Q}(-1) \xrightarrow{b^{\mathrm{t}}} \mathscr{U}^*(-1) \to \omega_{C_b} \to 0.$$

Associating C_b with b thus defines a map $\mathbb{P}(B^*) \to \mathscr{H}_0^2(X)$, which is clearly injective. Since we already know $\mathscr{H}_0^2(X) \simeq \mathbb{P}^2$, we conclude that $\mathbb{P}(B^*) \to \mathscr{H}_0^2(X)$ is an isomorphism. However, one could construct the inverse of this map by taking the Beilinson resolution of the ideal sheaf $\mathscr{I}_C$ of a conic C according to the decomposition (4.5). Anyway we have proved (ii).

Writing the middle map of (4.9) universally with respect to $b \in \mathbb{P}(B^*)$ gives the middle map of (4.8). Fixing a point $x \in X$, the associated map $\mathscr{Q}(-1)_x \otimes \mathscr{O}_{\mathbb{P}(B^*)} \to \mathscr{U}^*(-1)_x \otimes \mathscr{O}_{\mathbb{P}(B^*)}(1)$ is a 4×3 matrix of linear forms, degenerating on a subscheme Z of length 6 in $\mathbb{P}(B^*)$, consisting of the 6 conics through x, see [Tak89]. One sees easily that the kernel of this matrix is $\mathscr{O}_{\mathbb{P}(B^*)}(-3)$ and that its cokernel is ω_Z. Summing up, we have proved (iii). □

Dualizing (4.8) and twisting by $\omega_X \boxtimes \omega_{\mathbb{P}(B^*)}$ we get:

$$(4.10) \qquad 0 \to \mathscr{U} \boxtimes \mathscr{O}_{\mathbb{P}(B^*)}(-4) \to \mathscr{Q}^* \boxtimes \mathscr{O}_{\mathbb{P}(B^*)}(-3) \to \mathscr{O}_{X \times \mathbb{P}(B^*)} \to \mathscr{O}_{\mathscr{C}} \to 0.$$

4.2. Monads for instantons on Fano threefolds of genus 12.

According to Definition 1, a coherent sheaf E is an odd k-instanton on X if E is a rank-2 bundle with:

$$c_1(E) = -1, \qquad c_2(E) = k, \qquad \mathrm{H}^1(X, E) = 0.$$

We have said that $\mathrm{MI}_X(k)$ is empty for $k \leq 6$, and that it contains a unique rank-2 bundle $\mathscr{E}$ if $k = 7$ (we refer to [Kuz96, BF11]).

Increasing k to $k = 8$, we can control the whole moduli space $\mathrm{M}_X(2, -1, 8)$, as shown by the next proposition that puts together some ingredients, most of them already available in the literature. Indeed, the sheaves in this space that are not locally free fail to be so precisely along a line: a statement proved in [BF08, Proposition 3.5]. Intuitively, this is due to the fact that the c_2 of the double dual must go down by one because there are no rank-2 semistable bundles with $c_1 = -1$ and lower c_2.

Proposition 4.2. *The moduli space* $\mathrm{M}_X(2, -1, 8)$ *is identified with* $\mathbb{P}^2$, *and its open piece of locally free sheaves is* $\mathrm{MI}_X(8)$. *The complement of this piece is the apolar quartic of* X, *and is in bijection with the set of lines contained in* X.

Proof. Let E be a sheaf in $\mathrm{M}_X(2, -1, 8)$. By [BF08, Proposition 3.5, part (i)], we know that $\mathrm{H}^i(X, E) = 0$ for all i and $\mathrm{H}^i(X, E(1)) = 0$ for $i > 0$. This shows that any locally free sheaf in $\mathrm{M}_X(2, -1, 8)$ lies in $\mathrm{MI}_X(8)$. Further, by [BF08, Proposition 3.5, part (iii)] we know that either E is locally free, or it fits into an exact sequence:

$$(4.11) \qquad 0 \to E \to \mathscr{E} \to \mathscr{O}_L \to 0,$$

where L is a line contained in X.

We compute the complex associated with E according to the decomposition (4.4) and we calculate thus the cohomology table of E tensored with the bundles appearing in (4.5). We already settled this almost completely in the previous proposition. Indeed, assume first that E is locally free. Then, we established that $\mathrm{h}^i(X, E) = \mathrm{h}^i(X, E \otimes \mathscr{E}) = 0$ for all i (indeed the only relevant value was $\mathrm{h}^1(X, E \otimes \mathscr{E}) = k - 8 = 0$). Further, we have $\mathrm{h}^1(X, E \otimes \mathscr{U}) = \mathrm{h}^1(X, E \otimes \mathscr{Q}^*) = 1$, and all remaining cohomology groups are zero, so we get a resolution of E:

$$0 \to E \to \mathscr{K} \xrightarrow{f} \mathscr{U} \to 0,$$

and f lies in $\mathrm{Hom}_X(\mathscr{K}, \mathscr{Q}) = B$, so we have a map $\mathrm{MI}_X(8) \to \mathbb{P}^2$ given by $E \mapsto [f]$. We proved in [AF06] that a morphism $g : \mathscr{K} \to \mathscr{U}$ fails to be surjective if and only if $[g]$ lies in the apolar quartic of X, and that in this case we have an exact sequence:

$$(4.12) \qquad\qquad 0 \to \mathscr{E} \to \mathscr{K} \xrightarrow{g} \mathscr{U} \to \mathscr{O}_L(-1) \to 0.$$

On the other hand, if g is surjective then its kernel clearly lies in $\mathrm{MI}_X(8)$, and $[f]$ and E determine each other this way. This proves that $\mathrm{MI}_X(8)$ is identified with the complement of the apolar quartic in $\mathbb{P}^2$. Further, this quartic is in bijection with the set of lines contained in X, for any line $L \subset X$ determines canonically a long exact sequence as above.

To complete the proof, we only need to look at the class of sheaves E which are not locally free and therefore fit into (4.11). The sheaf E is clearly determined by L, and still determines a map $f : \mathscr{K} \to \mathscr{U}$ that this time will belong to the apolar quartic. Conversely, given $g : \mathscr{K} \to \mathscr{U}$ in the apolar quartic, dualizing (4.12) we see that E is obtained as the cokernel:

$$0 \to \mathscr{U}^*(-1) \xrightarrow{g^{\mathrm{t}}} \mathscr{K}^*(-1) \to E \to 0.$$

This gives the inverse map from the apolar quartic to the class of non-locally-free sheaves, and this clearly agrees with the construction given for locally free sheaves. The proof of the proposition is thus finished. $\qquad\qquad\square$

For higher k, we give the following monad-theoretic description of $\mathrm{MI}_X(k)$.

Proposition 4.3. *Let $k \geq 1$, and let E be a sheaf in $\mathrm{MI}_X(k+7)$. Then E is the cohomology of a monad of the form:*

$$I^* \otimes \mathscr{U}^*(-1) \xrightarrow{D\,A^{\mathrm{t}}} W^* \otimes \mathscr{E} \xrightarrow{A} I \otimes \mathscr{U},$$

where $I \simeq \mathbb{C}^k$, and $W \simeq \mathbb{C}^{3k+1}$, and $D : W^* \to W$ is a symmetric duality. Conversely, the cohomology of such a monad is a a sheaf in $\mathrm{MI}_X(k+7)$.

Proof. We first have to write the cohomology table (1.4) with respect to the collection (4.6). By Lemma 1.1, we get $\mathrm{H}^i(X, E) = 0$ for all i. Twisting (4.1) by E we get:

$$\mathrm{H}^{i-1}(X, E \otimes \mathscr{Q}) \simeq \mathrm{H}^i(X, E \otimes \mathscr{U}) \simeq \mathrm{H}^{3-i}(X, E \otimes \mathscr{U}^*)^*,$$

where the second isomorphism uses Serre duality and the fact $E^*(-1) \simeq E$. Stability of $\mathscr{U}$ and $\mathscr{Q}$ is proved in [Fae07a], and implies that the above groups vanish for $i = 0, 1, 3$. Indeed, the slope of the bundle whose group of global sections is being computed is always strictly negative. Therefore we are left with $\mathrm{H}^1(X, E \otimes \mathscr{Q})$, we denote this vector space by I^* and we compute its dimension k by Hirzebruch-Riemann-Roch. We deduce $\mathrm{H}^1(X, E \otimes \mathscr{U}^*) \simeq I$. This takes care of the second and of the fourth columns in our cohomology table.

To finish the computation of cohomology, we have to calculate $\mathrm{H}^i(X, E \otimes \mathscr{R}^*)$. To do this, we first compute $\mathrm{H}^i(X, E \otimes \mathscr{E}^*)$. We note that, by stability, $\mathrm{H}^i(X, E \otimes \mathscr{E}^*)$, vanishes for $i = 0$, indeed any morphism $\mathscr{E} \to E$ must be an isomorphism, but $c_2(\mathscr{E}) = 7 < c_2(E)$. The same group vanishes for $i = 3$, in view of Serre duality. We recall from [Fae07a] that the quotient bundle $\mathscr{F}$ is stable, and we note that this implies $\mathrm{H}^0(X, E \otimes \mathscr{F}) = 0$ and in turn $\mathrm{h}^2(X, E \otimes \mathscr{E}^*) = \mathrm{h}^1(X, E \otimes \mathscr{E}) = 0$ by using Serre duality and (4.2). We are only left with $\mathrm{H}^1(X, E \otimes \mathscr{E}^*)$ (its dimension equals $k-1$ by Hirzebruch-Riemann-Roch).

We can now proceed to compute $\mathrm{H}^i(X, E \otimes \mathscr{R}^*)$. We dualize (4.7), tensor it by E, and take cohomology. By the vanishing already obtained, we get that this group vanishes for $i = 0, 3$. Now we use the following resolution of $\mathscr{R}^*$, which is a part of a helix on X:

$$0 \to \mathscr{E} \to \mathrm{H}^0(X, \mathscr{E}^*)^* \otimes \mathscr{O}_X \to U^* \otimes \mathscr{Q}^* \to \mathscr{R}^* \to 0.$$

Tensoring it by E, taking cohomology, and using the vanishing already obtained, we get $\mathrm{H}^i(X, E \otimes \mathscr{R}^*) = 0$ for $i = 2$. This only leaves $\mathrm{H}^1(X, E \otimes \mathscr{R}^*)$. We call this space W^*, and we compute $\dim(W) = 3k + 1$ by Hirzebruch-Riemann-Roch.

To continue the proof, we have to establish the self-duality of the monad. This is provided, as usual, by the skew-symmetric duality $\kappa : E \to E^*(-1)$. We lift κ to an isomorphism between our monad and its dual, twisted by $\mathscr{O}_X(-1)$. This provides a duality of $W^* \otimes \mathscr{E}$ that lies in:

$$\mathrm{H}^0(X, \bigwedge^2 (W^* \otimes \mathscr{E}^*) \otimes \mathscr{O}_X(-1)) \simeq \left\{ \begin{array}{c} \wedge^2 W^* \otimes \mathrm{H}^0(X, \mathrm{S}^2 \mathscr{E}^*(-1)) \\ \oplus \\ \mathrm{S}^2 W^* \otimes \mathrm{H}^0(X, \wedge^2 \mathscr{E}^*(-1)). \end{array} \right.$$

In the direct sum above, the former term is zero, while the second one is naturally identified with $\mathrm{S}^2 W^*$. So we have a distinguished element of $\mathrm{S}^2 W^*$ that corresponds to the symmetric duality D, and the monad is self-dual.

To finish the proof, we must ensure that the converse implication holds, so we let E be the cohomology of a monad of the form under consideration. It is easy to see that E is a bundle of rank 2 with the appropriate Chern classes, and using (4.3) we get that E fulfills $\mathrm{H}^i(X, E) = 0$ for all i. This suffices to guarantee that E lies in $\mathrm{MI}_X(k+7)$ and we are done. $\qquad\square$

We have now proved Theorem F. This, together with Proposition 3.2 and Theorem C, achieves the proof of Theorem B.

Remark 4.4. I do not know if the moduli space $\mathrm{MI}_X(k)$ is affine, smooth, irreducible when X is a prime Fano threefold of genus 12, and $k \geq 9$. I haven't found examples of bundles with an SL_2-structure supported on the Mukai-Umemura threefold.

4.3. **Nets of quadrics for instantons on Fano threefolds of genus 12.** This subsection is devoted to the proof of Theorem F, we refer to the beginning of the section for notation.

First of all, we need to establish the natural isomorphisms:

$$(4.13) \qquad V \simeq \ker(\mathrm{S}^2 U^* \to B), \qquad \mathrm{H}^0(X, \mathrm{S}^2 \mathscr{U}^*(1)) \simeq \wedge^2 U.$$

The first one is proved in [Sch01], see also [Fae07a, Section 4]. For the second one, we use [Fae07a, Remark 8.1] (and its proof).

To begin the proof of Theorem F, we show that $\alpha \in M_{3k+1} \setminus \Delta$ gives a $(k+7)$-instanton. To do this, observe that an element $\alpha \in \mathrm{S}^2(I \otimes U)$ induces a commutative diagram:

$$\begin{array}{ccc} I^* \otimes U^* & \xrightarrow{\alpha} & I \otimes U \\ \downarrow{\scriptstyle A_\alpha^{\mathrm{t}}} & & {\scriptstyle A_\alpha}\uparrow \\ W & \xrightarrow{D} & W^*, \end{array}$$

where D is a symmetric duality induced by α on its image W. Since α lies in R_{3k+1} we have $\dim(W) = 3k+1$. Then, we have $A_\alpha : W^* \otimes \mathscr{E} \to I \otimes \mathscr{U}$, and $A_\alpha D A_\alpha^{\mathrm{t}}$ is a skew-symmetric map $I^* \otimes \mathscr{U}^*(-1) \to I \otimes \mathscr{U}$, so lies in:

$$\mathrm{H}^0(X, \bigwedge^2 (I \otimes \mathscr{U}) \otimes \mathscr{O}_X(1)) \simeq \wedge^2 I \otimes \mathrm{H}^0(X, \mathrm{S}^2 \mathscr{U}(1)) \oplus \mathrm{S}^2 I \otimes \mathrm{H}^0(X, \wedge^2 \mathscr{U}(1))$$

$$\simeq \wedge^2 I \otimes \wedge^2 U \oplus \mathrm{S}^2 I \otimes V^*,$$

where we used (4.13). Therefore, $A_\omega D A_\omega^{\mathrm{t}}$ is obtained by taking $\alpha \in \mathrm{S}^2(I \otimes U)$ and projecting it on the summand $\wedge^2 I \otimes \wedge^2 U \oplus \mathrm{S}^2 I \otimes V^*$. Since α lies in $\mathrm{S}^2 I \otimes B^*$, and $V^* = \mathrm{S}^2 U / B^*$, we get $A_\alpha D A_\alpha^{\mathrm{t}} = 0$. Also, A_α is a surjective map of sheaves if α does not lie in Δ. The conclusion that A_α gives a $(k+7)$-instanton follows from Proposition 4.3. We have thus obtained a map $M_{3k+1} \setminus \Delta \to \mathrm{MI}_X(k)$ which is invariant for $\mathrm{GL}(k)$.

Next, we want to prove that this map is surjective. But this is clear from Proposition 4.3, cf. the argument used in the proof of Theorem D.

The final step to prove Theorem F is to show that our nets of quadrics are semistable for the $\mathrm{GL}(k)$-action. This will be easier once we have an alternative monad-theoretic description of E. To exhibit it, recall from [Fae07a, Lemma 6.1] that $\mathrm{Hom}_X(\mathscr{K}, \mathscr{U}) \simeq \mathrm{Hom}_X(\mathscr{U}^*(-1), \mathscr{K}^*(-1)) \simeq B^*$. Then, for all $\alpha \in \mathrm{S}^2 I \otimes B^*$, we look at the associated map $I^* \to I \otimes B^*$ and set B_α for the following natural composition:

$$I^* \otimes \mathscr{U}^*(-1) \to I \otimes B^* \otimes \mathscr{U}^*(-1) \to I \otimes \mathscr{K}^*(-1).$$

Lemma 4.5. *Let E be the $(k+7)$-instanton on X associated with α. Then E is the cohomology of a monad:*

$$I^* \otimes \mathscr{U}^*(-1) \xrightarrow{B_\alpha} I \otimes \mathscr{K}^*(-1) \to C \otimes \mathscr{E},$$

where $C = \mathrm{coker}(W^* \to I \otimes U)$ has dimension $k - 1$.

Proof. Dualizing and twisting by $\mathscr{O}_X(-1)$ the sequence [Fae07a, (42)], we get:

$$0 \to \mathscr{K}^*(-1) \to U \otimes \mathscr{E} \to \mathscr{U} \to 0.$$

Tensoring this sequence with I and fitting it with the monad of Proposition 4.3 gives rise to the diagram:

Since $E = \ker(A)/I^* \otimes \mathscr{U}^*(-1)$, the top line of the diagram shows the desired monad. The leftmost map is B_α by construction. The dimension of C is clear. $\square$

Observe that $\alpha \in R_{3k+1}$ lies away from Δ iff B_α is fibrewise injective.

Lemma 4.6. *Let $\alpha \in \mathrm{S}^2 I \otimes B^*$ be a net of quadrics with $\mathrm{rk}(\alpha) = 3k + 1$ and $\alpha \notin \Delta$. Then α is semistable for the $\mathrm{GL}(k)$-action.*

Proof. The argument is very similar to [Kuz12, Proposition 4.9], but the computation needs to be carried our explicitly. From [Wal78], we recall that $\alpha \in \mathrm{S}^2 I \otimes B^*$ is not semistable iff there are subspaces I_1, I_2 of I^* with $\dim(I_1) + \dim(I_2) > k$ such that $B \to \mathrm{S}^2 I \to I_1 \otimes I_2$ is zero. Exactly as in [Kuz12, Proposition 4.9], we get $\dim(I_1) > \dim(I_2)^\perp$ and a natural induced diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I_1 \otimes \mathscr{U}^*(-1) & \longrightarrow & I \otimes \mathscr{U}^*(-1) & \longrightarrow & I/I_1 \otimes \mathscr{U}^*(-1) & \longrightarrow & 0 \\
 & & \downarrow b & & \downarrow B_\alpha & & \downarrow c & & \\
0 & \longrightarrow & I_2^\perp \otimes \mathscr{K}^*(-1) & \longrightarrow & I^* \otimes \mathscr{K}^*(-1) & \longrightarrow & I_2^* \otimes \mathscr{K}^*(-1) & \longrightarrow & 0
\end{array}
$$

This gives an exact sequence:

$$0 \to \ker(c) \to \mathrm{coker}(b) \xrightarrow{d} C \otimes \mathscr{E} \to \mathrm{coker}(c) \to 0.$$

Now, using the values $c_1(K) = -2$, $\mathrm{rk}(K) = 5$ (see [Fae07a, Lemma 6.1]), and setting $i_1 = \dim(I_1)$, $i_2 = \dim(I_2^\perp)$ we get $c_1(\mathrm{coker}(b)) = -3i_2 + 2i_1$ and $\mathrm{rk}(\mathrm{coker}(b)) = 5i_2 - 3i_1$. By stability of $\mathscr{E}$ and $\mathscr{U}$ (proved in [Fae07a]) we get:

$$c_1(\ker(c)) \le -\frac{2}{3} \mathrm{rk}(\ker(c)), \qquad c_1(\mathrm{Im}(d)) \le -\frac{1}{2} \mathrm{rk}(\mathrm{Im}(d)).$$

This gives:

$$c_1(\mathrm{coker}(b)) \le c_1(\ker(c)) - \frac{1}{2}\,\mathrm{rk}(\mathrm{coker}(b)) + \frac{1}{2}\,\mathrm{rk}(\ker(c)) \le$$

$$\le -\frac{2}{3}\,\mathrm{rk}(\ker(c)) - \frac{1}{2}\,\mathrm{rk}(\mathrm{coker}(b)) + \frac{1}{2}\,\mathrm{rk}(\ker(c)).$$

We obtain:

$$-6i_2 + 4i_1 \le -\frac{1}{3}\,\mathrm{rk}(\ker(c)) - 5i_2 + 3i_1,$$

which contradicts $i_1 > i_2$. $\qquad\square$

4.4. **Jumping conics of instantons on Fano threefolds of genus 12.** Here we will develop some basic considerations to describe the set of jumping conics of an odd instanton on a smooth prime Fano threefold X of genus 12.

Definition 4. Given an odd k-instanton E over X, and a conic C contained in X, we say that C is a *jumping conic* for E if:

$$\mathrm{H}^1(C, E|_C) \ne 0.$$

Looking at the universal conic $\mathscr{C}$ as a variety dominating X and $\mathbb{P}(B^*)$, and denoting by p and q the projections, we can then define the *scheme of jumping conics* as the support of the coherent sheaf:

$$\mathbf{R}^1 q_*(p^*(E) \otimes \mathscr{O}_{\mathscr{C}}).$$

We says that E has *generic splitting* if not all conics of X are jumping for E.

Lemma 4.7. *Let E be a k-instanton on X. Then the scheme of jumping conics of E is the support of the cokernel sheaf a symmetric matrix of linear forms:*

$$M_E : I^* \otimes \mathscr{O}_{\mathbb{P}(B^*)}(-4) \to I \otimes \mathscr{O}_{\mathbb{P}(B^*)}(-3).$$

Proof. In order to compute $\mathbf{R}^1 q_*(p^*(E) \otimes \mathscr{O}_{\mathscr{C}})$, we use (4.10). Since E is a $(k+7)$-instanton, we have:

$$\mathrm{H}^k(X, E) = 0, \qquad \mathrm{H}^j(X, E \otimes \mathscr{Q}^*) = \mathrm{H}^j(X, E \otimes \mathscr{U}) = 0,$$

for all k (Lemma 1.1) and for $j \ne 2$, which can be seen from the proof of Proposition 4.3 using that $\mathrm{H}^j(X, E \otimes \mathscr{Q}^*) \simeq \mathrm{H}^{3-j}(X, E^*(-1) \otimes \mathscr{Q})^* \simeq \mathrm{H}^{3-j}(X, E \otimes \mathscr{Q})^*$ and $\mathrm{H}^j(X, E \otimes \mathscr{U}) \simeq \mathrm{H}^{3-j}(X, E^*(-1) \otimes \mathscr{U}^*)^* \simeq \mathrm{H}^{3-j}(X, E \otimes \mathscr{U}^*)^*$. By the same reason, we have natural isomorphisms:

$$\mathrm{H}^2(X, E \otimes \mathscr{Q}^*) \simeq I, \qquad \mathrm{H}^2(X, E \otimes \mathscr{U}) \simeq I^*.$$

Therefore, tensoring (4.10) with $p^*(E)$ and taking direct images, we get an exact sequence:

$$0 \to q_* p^*(E \otimes \mathscr{O}_{\mathscr{C}}) \to I^* \otimes \mathscr{O}_{\mathbb{P}(B^*)}(-4) \to I \otimes \mathscr{O}_{\mathbb{P}(B^*)}(-3) \to \mathbf{R}^1 q_*(p^*(E) \otimes \mathscr{O}_{\mathscr{C}}) \to 0.$$

$\qquad\square$

Proposition 4.8. *The scheme of jumping conics of $(k+7)$-instanton E with generic splitting is a plane curve $\mathscr{C}$ of degree k, and is equipped with a coherent sheaf $\mathscr{L}$ such that $\mathscr{H}om_{\mathscr{C}}(\mathscr{L}, \mathscr{O}_{\mathscr{C}}) \simeq \mathscr{L}(k-1)$. Moreover, E is determined by $(\mathscr{C}, \mathscr{L})$.*

Proof. Let E be a $(k+7)$-instanton on X. If E has generic splitting, then there exists $b \in \mathbb{P}(B^*)$ such that the conic C_b is not jumping for E, so that the map M_E associated with E according to the previous lemma is injective at b. Therefore, $\mathrm{coker}(M_E)$ is supported on a curve $\mathscr{C}$ of degree k in $\mathbb{P}(B^*)$, defined by the equation $\det(M_E)$. Let $i : \mathscr{C} \hookrightarrow \mathbb{P}(B^*)$ be the embedding. The sheaf $\mathrm{coker}(M_E) \otimes \mathscr{O}_{\mathbb{P}(B^*)}(3)$ is supported on $\mathscr{C}$ and we have:

$$(4.14) \qquad 0 \to I^* \otimes \mathscr{O}_{\mathbb{P}(B^*)}(-1) \xrightarrow{M_E} I \otimes \mathscr{O}_{\mathbb{P}(B^*)} \to i_*(\mathscr{L}) \to 0.$$

Then $\mathscr{L}$ is a Cohen-Macaulay (hence torsion-free) sheaf on $\mathscr{C}$. Since M_E is symmetric, dualizing the above sequence we find:

$$i_*(\mathscr{L}) \simeq \mathscr{E}xt^1_{\mathbb{P}(B^*)}(i_*(\mathscr{L}), \mathscr{O}_{\mathbb{P}(B^*)}(-1)) \simeq i_*(\mathscr{H}om_{\mathscr{C}}(\mathscr{L}, \mathscr{O}_{\mathscr{C}}(k-1))),$$

where the last isomorphism follows from Grothendieck duality.

To check that E is determined by $(\mathscr{C}, \mathscr{L})$, note that the exact sequence (4.14) is the sheafification of the minimal graded free resolution of the module over $\mathbb{C}[x_0, x_1, x_2]$ associated with $i_*(\mathscr{L})$. As such, it is unique up to conjugation by $\mathrm{GL}(I)$. Therefore, from $(\mathscr{C}, \mathscr{L})$ we reconstruct M_E, i.e., the net of quadrics as en element of $\mathrm{S}^2 I \otimes B^*$, up to $\mathrm{GL}(I)$-action. This gives back E according to Theorem F. $\qquad\square$

5. Geometric quotients for moduli of instanton bundles

Here, we will carry out some considerations to check that the moduli space $\mathrm{MI}_X(k)$ is a geometric quotient, relying on the monad-theoretic presentation. In other words, we show the last statement of Theorem B.

Lemma 5.1. *The map $\mathscr{Q}^{\circ}_{X,k} \to \mathrm{MI}_X(k)$, that sends A to the cohomology sheaf $\ker(A)/\mathrm{Im}(A\,D^{\mathrm{t}})$ is a geometric quotient for the G_k-action.*

Proof. We will deduce our statement from some well-known facts concerning the action of a larger group $\hat{G}_k$ on a larger variety $\hat{\mathscr{Q}}^{\circ}_{X,k}$.

Let us first define $\hat{\mathscr{Q}}_{X,k}$. Fix vector spaces I and W as in Table (3), and take another vector space J with $\dim(J) = \dim(I)$. Consider the set of pairs (A, B) in $(I \otimes W \otimes U) \times (J^* \otimes W \otimes U)$ as pairs of morphisms:

$$A : W^* \otimes \mathscr{E}_2 \to I \otimes \mathscr{E}_3, \qquad B : J \otimes \mathscr{E}_1 \to W^* \otimes \mathscr{E}_2.$$

We define the variety $\hat{\mathscr{Q}}_{X,k}$ by:

$$\hat{\mathscr{Q}}_{X,k} = \{(A, B) \in (I \otimes W \otimes U) \times (J^* \otimes W \otimes U) \mid A\,B = 0\}.$$

The relevant open piece $\hat{\mathscr{Q}}^{\circ}_{X,k}$ of $\hat{\mathscr{Q}}_{X,k}$ is:

$$\hat{\mathscr{Q}}^{\circ}_{X,k} = \{(A, B) \in \hat{\mathscr{Q}}_{X,k} \mid A \text{ is surjective and } B \text{ is injective}\}.$$

The group $\hat{G}_k$ is defined as $\hat{G}_k = \mathrm{GL}(I) \times \mathrm{GL}(W) \times \mathrm{GL}(J)$. An element (a, b, c) of $\hat{G}_k$ operates on $(A, B) \in \hat{\mathscr{Q}}^{\circ}_{X,k}$ on the left by $(a, b, c).(A, B) = (aAb^{\mathrm{t}}, b^{-\mathrm{t}}Bc^{-1})$, which still lies in $\hat{\mathscr{Q}}^{\circ}_{X,k}$. Clearly, $\hat{G}_k$ also acts on $\hat{\mathscr{Q}}^{\circ}_{X,k}$.

Now we fix an isomorphism $u : J \to I^*$. Note that, given $A \in \mathscr{Q}^{\circ}_{X,k}$, we can associate with A the pair $(A, D\,A^{\mathrm{t}}u)$ which of course lies in $\hat{\mathscr{Q}}^{\circ}_{X,k}$. We get a closed embedding $\Phi : \mathscr{Q}^{\circ}_{X,k} \hookrightarrow \hat{\mathscr{Q}}^{\circ}_{X,k}$. Next, given a pair $(\xi, \eta) \in G_k$, we associate with (ξ, η) the element $(\xi, \eta, u^{-1}\xi^{-\mathrm{t}}u)$ of $\hat{G}_k$, so that G_k is a closed subgroup of $\hat{G}_k$. We observe that the action of $\hat{G}_k$ on $\hat{\mathscr{Q}}^{\circ}_{X,k}$ is compatible with that G_k on $\mathscr{Q}^{\circ}_{X,k}$ under these inclusions. Indeed, for all $A \in \mathscr{Q}^{\circ}_{X,k}$ and all $(\xi, \eta) \in G_k$ we have:

$$(\xi, \eta).\Phi(A) = (\xi A\eta^{\mathrm{t}}, \eta^{-\mathrm{t}}D A^{\mathrm{t}}\xi^{\mathrm{t}}u) = (\xi A\eta^{\mathrm{t}}, D\eta A^{\mathrm{t}}\xi^{\mathrm{t}}u) = \Phi((\xi, \eta).A),$$

which in turn follows from $\eta^{\mathrm{t}}D\eta = D$.

From the proof of Lemma 2.2, Proposition 3.2 and Proposition 4.3, we see that any element in the moduli space $\mathrm{MI}_X(k)$ comes from a pair (A, B) (with in fact $B = D\,A^{\mathrm{t}}$), identified up to $\hat{G}_k$-action. By the standard argument involving Beilinson-type spectral sequences, we obtain that $\mathrm{MI}_X(k)$ is isomorphic to the quotient $\hat{\mathscr{Q}}_{X,k}$ by the action of $\hat{G}_k$, where the quotient map associates with (A, B) the cohomology sheaf $\ker(A)/\mathrm{Im}(B)$ of the monad obtained by (A, B). The same argument as in [LP94] shows that the points of $\hat{\mathscr{Q}}_{X,k}$ are stable for the action of $\hat{G}_k$, and one can prove, just like in [LP79, Proposition

24, part 2] that the action of $\hat{G}_k$ on $\hat{\mathscr{Q}}_{X,k}$ is proper. This says that the quotient of $\hat{\mathscr{Q}}_{X,k}$ by the $\hat{G}_k$-action is geometric. Moreover, rephrasing [LP79, Pages 233-236], we see that $\hat{\mathscr{Q}}_{X,k}/\hat{G}_k$ does represent the moduli functor of families of instanton bundles, for the monad construction works well in families.

Finally, we see that this quotient is identified with the quotient of $\mathscr{Q}_{X,k}$ by the group G_k. This holds if, for any point (A, B) of $\hat{\mathscr{Q}}_{X,k}$, we can find a point A_0 of $\mathscr{Q}_{X,k}$ such that (A, B) lies in the $\hat{G}_k$-orbit of $\Phi(A_0)$. To check this explicitly, given $(A, B) \in \hat{\mathscr{Q}}_{X,k}$, note that, from the proof of Lemma 2.2, Proposition 3.2 and Proposition 4.3, we see that there are isomorphisms $w : W^* \to W$ and $v : J \to I^*$ such that $B = w^{-1}A^{\mathrm{t}}v$, where $w^{\mathrm{t}} = (-1)^{r_X+1}w$. Of course w^{-1} is congruent to D, so that there is an isomorphism $\zeta \in \mathrm{GL}(W)$ such that $w^{-1} = \zeta^{\mathrm{t}}D\zeta$. Then, setting, $A_0 = A\zeta^{\mathrm{t}}$, we have:

$$(1, \zeta, u^{-1}v).(A, B) = (A\zeta^{\mathrm{t}}, \zeta^{-\mathrm{t}}Bv^{-1}u) = \Phi(A_0),$$

indeed $\zeta^{-\mathrm{t}}Bv^{-1}u = D\zeta A^{\mathrm{t}}u$ follows from $B = w^{-1}A^{\mathrm{t}}v$ and $w^{-1} = \zeta^{\mathrm{t}}D\zeta$.

$\square$

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