THE CM REPRESENTATION TYPE OF PROJECTIVE VARIETIES

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Abstract. We show that all projective integral varieties which are not cones and have a Cohen-Macaulay coordinate ring are of wild CM type, except for a few cases which we completely classify.

Introduction

A classical result in representation theory of quivers is Gabriel's theorem, stating that a connected quiver supports only finitely many irreducible representations (i.e., of indecomposable modules over the associated path algebra) if and only if it is of type A, D, E. The classification of tame quivers as *Euclidean graphs*, or *extended Dynkin diagrams*, of type $\tilde{A}$, $\tilde{D}$, $\tilde{E}$ came shortly afterwards. Remarkably, any other quiver supports arbitrarily large families of indecomposable representations, i.e., to be of *wild representation type*.

In algebraic geometry, the relevant problem in representation theory of algebras concerns the complexity of the category of maximal Cohen-Macaulay modules over the coordinate ring $\boldsymbol{k}[X]$ of a closed subvariety $X \subset \mathbb{P}^n$ defined over an algebraically closed field $\boldsymbol{k}$. For $\dim(X) > 0$, assuming $\boldsymbol{k}[X]$ to be Cohen-Macaulay (so X is said to be arithmetically Cohen-Macaulay, briefly aCM), these in turn correspond to aCM sheaves, namely coherent sheaves $\mathcal{E}$ on X without intermediate cohomology i.e. satisfying $\mathrm{H}^i(X, \mathcal{E}(t)) = 0$ for all t and $0 < i < \dim(X)$. For hypersurfaces, in view of [Eis80] these modules also correspond to matrix factorizations, which in turn are related to mirror symmetry, cf. [Orl04].

In this sense, reduced projective aCM varieties of finite CM type have been classified in [EH88], cf. also [BGS87, AR87, Knö87, Her78]. Their list (for positive dimension) consists of rational normal curves, projective spaces, smooth quadrics, the Veronese surface in $\mathbb{P}^5$ and the cubic scroll in $\mathbb{P}^4$. This ties in with Grothendieck's classical result on the splitting of vector bundles over $\mathbb{P}^1$, [Gro57], in turn rooted in ideas of Segre, [Seg84].

Our goal here is to tell what happens to other varieties. Smooth cubic surfaces are of wild CM type according to the main point of [CH11]. Some more varieties have been proved to fall in this class, for instance all Segre varieties besides the CM finite ones (cf. [CMRPL12]), del Pezzo surfaces (cf. [MRPL12, CKM13]), positive-dimensional hypersurfaces of degree at least 4 and some complete intersections (cf. [DT14]) some Fano varieties (cf. [MRPL14]), the triple Veronese embedding of any variety (cf. [MR15]).

As for CM tame projective varieties, besides smooth elliptic curves (by seminal work of Atiyah, [Ati57], also related to classical work of Segre, cf [Seg86]) and rational chains of type $\tilde{A}$ (cf. [DG01, DG93], cf. also [BD04]), the only known examples were given in [FM14] and consist of smooth quartic surface scrolls in $\mathbb{P}^5$. Here we prove the following result.

Theorem. *Let $X \subset \mathbb{P}^n$ be a closed integral aCM subvariety of positive dimension which is not a cone. Then X is of wild CM type unless X is one of the following:*

i) *a linear subspace;*
ii) *a smooth quadric;*
iii) *a rational normal curve;*
iv) *a smooth or nodal curve of arithmetic genus 1;*

2010 *Mathematics Subject Classification.* 14F05; 13C14; 14J60; 16G60.
Key words and phrases. aCM vector sheaves and bundles, Ulrich sheaves, MCM modules, Graded Cohen-Macaulay rings, Representation type.
Both authors partially supported by GEOLMI ANR-11-BS03-0011.

v) a smooth rational surface scroll of degree $d \le 4$*;*
vi) the Veronese surface in $\mathbb{P}^5$.

Our result implies a strong version of the finite-tame-wild trichotomy for the class of varieties appearing in the previous theorem, namely that any such variety X falls in exactly one of the following classes:

Finite: there are only finitely many indecomposable aCM sheaves over X up to isomorphism. This happens in cases (i), (ii), (iii), (v) for $d = 3$ and (vi).

Tame: for any given rank r, the parameter space of indecomposable non-isomorphic aCM sheaves of rank at most r consists of the union of finitely many isolated points and connected curves of genus 0 (in case (v) and $d = 4$) or 1 (in case (iv)).

Wild: X is of wild CM type in the algebraic sense and hence in the geometric sense, i.e. X admits families of arbitrarily large dimension of indecomposable non-isomorphic aCM sheaves.

It should be noted that, although our assumptions on the variety X are rather mild, a different behaviour might be expected when relaxing them. For instance a quadric cone of corank 1 is CM countable, cf. [BGS87]; several other examples of this kind exist over local rings, cf. for instance [LW12] for a detailed picture, see also [Sto13]. More varieties of tame type appear from germs of elliptic singularities, cf. [Kah89] or from non-isolated affine surface singularities, see also [BD10].

Let us indicate the strategy of our proof. Let Y be a variety of the class under examination: an integral closed aCM subscheme of $\mathbb{P}^n$ which is not a cone. The first step is to isolate some essential datum in order to build large families of indecomposable non-isomorphic MCM modules on $\boldsymbol{k}[Y]$. In Theorem 1 we establish that this should be a pair of aCM sheaves $\mathcal{A}$ and $\mathcal{B}$ on Y, whose only endomorphisms are homotheties (i.e. $\mathcal{A}$ and $\mathcal{B}$ are *simple*), having no non-trivial maps in either direction $\mathcal{A} \to \mathcal{B}$ or $\mathcal{B} \to \mathcal{A}$ and such that the extension space $\mathrm{Ext}^1_Y(\mathcal{B},\mathcal{A})$ is sufficiently large, namely of dimension $w \ge 3$.

Indeed, from this datum we construct a fully faithful exact functor from the category $\mathbf{Rep}_\Upsilon$ of finite-dimensional representations of the Kronecker quiver $\Upsilon = \Upsilon_w$ with w arrows to the category $\mathbf{MCM}_{\boldsymbol{k}[Y]}$ of MCM modules over $\boldsymbol{k}[Y]$. This quiver should be seen as parametrizing aCM sheaves appearing as extension of copies of $\mathcal{A}$ and $\mathcal{B}$. In turn, by a standard argument, the existence of such functor suffices to prove CM wildness of Y.

However, constructing the sheaves $\mathcal{A}$ and $\mathcal{B}$ turns out to be quite complicated in general. For this we need our next result, Theorem 2, which shows how to deduce CM wildness of X from CM wildness of Y when Y is a linear section of X of codimension $c > 0$. For this, we need to further assume that $\mathcal{A}$ and $\mathcal{B}$ are Ulrich sheaves, i.e. their modules of global sections have the maximal number of generators. We see this as a further prove of the importance of these sheaves, cf. [ESW03, ES09]. We also need the extra assumption $\mathrm{H}^0(Y,\omega_Y(m-c)) \neq 0$, which turns out to hold everywhere except for varieties of minimal degree, where in turn the CM representation type can be analyzed directly. Note that all non-wild varieties are of minimal degree, with the exception of elliptic curves.

The idea of Theorem 2 is that taking the c-th syzygy $\Omega^c_{\boldsymbol{k}[X]}$ of the $\boldsymbol{k}[X]$-module of an MCM module L over $\boldsymbol{k}[Y]$ one obtains an MCM module over $\boldsymbol{k}[X]$ and that this entails no essential loss of information if L is Ulrich. In fact, for $\Omega^c_{\boldsymbol{k}[X]}$ to be a functor we need to pass to the stable category $\underline{\mathbf{MCM}}_{\boldsymbol{k}[X]}$ where we quotient out by morphisms factoring through a free module. The point is that the stable syzygy functor $\underline{\mathbf{\Omega}}^c$ is fully faithful on Ulrich modules. The proof of this fact relies on a duality argument and on cohomology vanishing of Ulrich sheaves.

The next result, Theorem 3, shows how to put these two ingredients together. Indeed, by resolving over $\boldsymbol{k}[X]$ the module of global sections of the universal extension of the sheaves $\mathcal{B}$ by $\mathcal{A}$ over Y needed for Theorem 1 and taking its c-th syzygy, we get a functor $\mathbf{Rep}_\Upsilon \to \mathbf{MCM}_X$ whose stabilization is fully faithful by Theorem 2. Then the functor itself is not quite fully faithful, nevertheless is a *representation embedding*, i.e. it sends non-isomorphic (resp. irreducible) representations to non-isomorphic (resp. indecomposable) modules, and this suffices to show CM wildness of X.

Summing up, it only remains to construct the Ulrich sheaves $\mathcal{A}$ and $\mathcal{B}$ as above over a linear section Y of X, which we take to be of dimension 1 when the sectional genus p of X is at least 2, or of dimension 2 for $p \leq 1$. This is the content of §5, which we now review.

We first take care of the case $p = 0$, i.e. of varieties of minimal degree. Here, our assumptions combined with del Pezzo and Bertini's classification say that X is a rational normal scroll or the Veronese surface in $\mathbb{P}^5$, and these varieties are quickly dealt with. Note that here is where all CM-finite cases appear, together with the two sporadic cases given by the two rational quartic surface scrolls.

Next, the case $p \geq 2$ is quite easily seen to provide only CM wild varieties, as $\mathcal{A}$ and $\mathcal{B}$ can be taken to be sufficiently general bundles of rank 2 over Y of slope $\deg(Y) + g - 1$, where g is the geometric genus of Y.

Finally, in the case $p = 1$ our proof of the existence of $\mathcal{A}$ and $\mathcal{B}$ is based on a detailed study of Ulrich sheaves on surfaces of almost minimal degree. When Y smooth (i.e. Y is a del Pezzo surface) or even just normal, this can be carried out via the well-known Hartshorne-Serre construction associated with general sets of $\deg(Y) + 2$ points of Y. On the other hand, when Y is not normal, we take $\mathcal{A}$ and $\mathcal{B}$ to be Ulrich sheaves of rank 1 which are not locally free, in fact the main point is to bound from below the number of independent global sections of the first local $\mathcal{E}xt$ sheaf between $\mathcal{A}$ and $\mathcal{B}$. To accomplish this, according to the classification of these surfaces, it suffices to settle a finite number of cases. In the main one (when Y is normalized by a scroll) we define $\mathcal{A}$ and $\mathcal{B}$ to be direct images of certain line bundles via the normalization map $\bar{Y} \to Y$, and reduce the result to counting the intersection points of the moving part of the associated curves.

1. CM wild varieties

Let $\boldsymbol{k}$ be an algebraically closed field, and let $S = \boldsymbol{k}[x_0, \ldots, x_n]$ be the graded polynomial ring in $n + 1$ indeterminates. Write $\boldsymbol{k}[X]$ for the homogeneous coordinate ring of a closed subscheme X of $\mathbb{P}^n := \mathbf{Proj}(S)$, and $\mathcal{O}_X(1)$ for the restriction to X of $\mathcal{O}_{\mathbb{P}^n}(1)$.

1.1. aCM varieties and modules.

We first recall some basic terminology for various Cohen-Macaulay properties of varieties, sheaves and modules.

Definition 1. Let $X \subset \mathbb{P}^n$ be a closed subscheme of dimension m. Then X is arithmetically Cohen-Macaulay (aCM) if the ring $\boldsymbol{k}[X]$ of X is Cohen-Macaulay, i.e. if $\boldsymbol{k}[X]$ has a graded S-free resolution of length $n - m$.

Given a closed subscheme $X \subset \mathbb{P}^n$, we set $R := \boldsymbol{k}[X]$ and let $\mathbf{MCM}_R$ be the category of finitely generated maximal Cohen-Macaulay (MCM) modules over R. If X is an aCM variety of positive dimension, then the sheafification functor provides an equivalence between $\mathbf{MCM}_R$ and the full subcategory of $\mathbf{Coh}_X$ consisting of arithmetically Cohen-Macaulay (aCM) sheaves, i.e. coherent sheaves whose R-module of global sections is MCM. Given M in $\mathbf{MCM}_R$ of positive rank, the number of independent minimal generators of M is at most $\deg(X)\,\mathrm{rk}(M)$. For the corresponding sheaf $\mathcal{E}$ over X, assuming $0 = \dim_{\boldsymbol{k}} \mathrm{H}^0(X, \mathcal{E}(-1))) < \dim_{\boldsymbol{k}} \mathrm{H}^0(X, \mathcal{E})$, we have:

$$\dim_{\boldsymbol{k}} \mathrm{H}^0(X, \mathcal{E}) \leq \deg(X)\,\mathrm{rk}(\mathcal{E}). \tag{1.1}$$

We say that $\mathcal{E}$ is an *Ulrich sheaf* on X and that M is an Ulrich module over R if equality is attained in (1.1). Put $\mathbf{Ulr}_R$ for the full subcategory of $\mathbf{MCM}_R$ consisting of Ulrich modules.

We say that a coherent sheaf $\mathcal{E}$ over X is *simple* if its only endomorphisms are homotheties. Given two coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ over X, we write $\chi(\mathcal{E}, \mathcal{F})$ for the alternating sum of the dimension of $\mathrm{Ext}^i_X(\mathcal{E}, \mathcal{F})$. We abbreviate $\chi(\mathcal{F}) = \chi(\mathcal{O}_X, \mathcal{F})$.

1.2. CM wildness.

We will consider a couple of related notions of CM wildness of a variety. Algebraically this means that, for any finitely generated associative $\boldsymbol{k}$-algebra Λ, the category of MCM modules over $R = \boldsymbol{k}[X]$ contains, in some sense, the category $\mathbf{FMod}_\Lambda$ of left Λ-modules which have finite dimension over $\boldsymbol{k}$. Let us give the precise definitions.

Definition 2. Let $X \subset \mathbb{P}^n$ be an equidimensional closed subscheme. For any finitely generated associative $\boldsymbol{k}$-algebra Λ, and any (R,Λ)-bimodule $\boldsymbol{M}$, flat over Λ, define the functor:

$$\Phi_{\boldsymbol{M}} : \mathbf{FMod}_\Lambda \to \mathbf{Mod}_R,$$
$$N \mapsto \boldsymbol{M} \otimes_\Lambda N.$$

The variety X is of *wild CM type* if, for any Λ as above, there exists $\boldsymbol{M}$ such that $\Phi_{\boldsymbol{M}}$ takes values in $\mathbf{MCM}_R$ and is a *representation embedding*, i.e.:

a) the module N is decomposable whenever $\Phi_{\boldsymbol{M}}(N)$ is;
b) for any pair (N, N') of modules in $\mathbf{FMod}_\Lambda$, we have:

$$\Phi_{\boldsymbol{M}}(N) \simeq \Phi_{\boldsymbol{M}}(N') \Leftrightarrow N \simeq N'.$$

The variety X is of *wild Ulrich type* if moreover:

c) for any N in $\mathbf{FMod}_\Lambda$, $\Phi_{\boldsymbol{M}}(N)$ is Ulrich.

The variety X is said to be *strictly CM wild* if for any Λ as above there is $\boldsymbol{M}$ such that $\Phi_{\boldsymbol{M}}$ takes values in $\mathbf{MCM}_R$ and is fully faithful, i.e.:

$$\mathrm{Hom}_\Lambda(N, N') \simeq \mathrm{Hom}_R(\Phi(N), \Phi(N'))_0.$$

If moreover $\Phi_{\boldsymbol{M}}(N)$ is Ulrich for all N, then X is *strictly Ulrich wild.*

Remark 1. The following facts are well-known.

i) If the subscheme X is strictly CM (resp. Ulrich) wild, then it is of wild CM (resp. Ulrich) type.
ii) If X is of wild CM type, then for any $r \in \mathbb{N}$ there are families of dimension at least r consisting of indecomposable aCM sheaves on X, all non-isomorphic to one another. In other words, X is of wild CM type in the geometric sense. If X is of wild Ulrich type, these families can be taken to consist of Ulrich sheaves.
iii) Any fully faithful exact functor $\Phi : \mathbf{FMod}_\Lambda \to \mathbf{Mod}_R$ is of the form $\Phi_{\boldsymbol{M}}$ for some Λ-flat (R,Λ)-bimodule $\boldsymbol{M}$.

Let $w \geq 1$ be an integer and consider the Kronecker quiver $\Upsilon = \Upsilon_w$ with two vertices and w arrows from the first vertex to the second. Write $\mathbf{Rep}_\Upsilon$ for the Abelian category of finite-dimensional $\boldsymbol{k}$-representations of Υ.

iv) To check that X is strictly CM wild (resp., of wild CM type), it suffices to construct a fully faithful exact functor (resp., a representation embedding):

$$\Phi : \mathbf{Rep}_\Upsilon \to \mathbf{MCM}_R$$

where $\Upsilon = \Upsilon_w$ is the Kronecker quiver with $w \geq 3$. If moreover $\Phi(\mathcal{R})$ is Ulrich for any $\mathcal{R}$ in $\mathbf{Rep}_\Upsilon$, then X is strictly Ulrich wild (resp., of wild Ulrich type).

2. Strict CM wildness from extensions

One of the main tools that we use to build large families of MCM modules (or aCM sheaves) is to start with two such sheaves and build extensions in a systematic way. This goes as follows. Let $X \subset \mathbb{P}^n$ be a closed subscheme, and let $\mathcal{A}$ and $\mathcal{B}$ be two coherent sheaves on X such that:

$$\mathrm{Ext}^1_X(\mathcal{B}, \mathcal{A}) \neq 0.$$

Set $W = \mathrm{Ext}^1_X(\mathcal{B}, \mathcal{A})$ and consider the projective space $\mathbb{P}W$ of lines through the origin in W. Then, over $X \times \mathbb{P}W$, there is a universal extension:

$$0 \to \mathcal{A} \boxtimes \mathcal{O}_{\mathbb{P}W} \to \mathcal{U} \to \mathcal{B} \boxtimes \mathcal{O}_{\mathbb{P}W}(-1) \to 0,$$

where we write p and q for the projections from $X \times \mathbb{P}W$ to X and $\mathbb{P}W$, and for $\mathcal{E} \in \mathbf{Coh}_X$ and $\mathcal{F} \in \mathbf{Coh}_{\mathbb{P}W}$, we set $\mathcal{E} \boxtimes \mathcal{F} = p^*(\mathcal{E}) \otimes q^*(\mathcal{F})$. Then we consider:

$$\Phi_{\mathcal{U}} = \mathbf{R}p_*(q^*(-) \otimes \mathcal{U}) : \mathbf{D}^b(\mathbf{Coh}_{\mathbb{P}W}) \to \mathbf{D}^b(\mathbf{Coh}_X).$$

It is clear that:

$$\Phi_{\mathcal{U}}(\mathcal{O}_{\mathbb{P}W}) \simeq \mathcal{A}, \qquad \Phi_{\mathcal{U}}(\Omega_{\mathbb{P}W}(1)) \simeq \mathcal{B}[-1].$$

Set $w = \dim_{\boldsymbol{k}} \mathrm{Ext}^1_X(\mathcal{B},\mathcal{A})$ and consider the Kronecker quiver $\Upsilon = \Upsilon_w$. Then, the natural isomorphism $W \simeq \mathrm{Hom}_{\mathbb{P}W}(\Omega_{\mathbb{P}W}(1),\mathcal{O}_{\mathbb{P}W})$ provides an equivalence:

$$\Xi : \mathbf{D}^b(\mathbf{Rep}_\Upsilon) \simeq \langle \Omega_{\mathbb{P}W}(1),\mathcal{O}_{\mathbb{P}W}\rangle \subset \mathbf{D}^b(\mathbf{Coh}_{\mathbb{P}W}). \tag{2.1}$$

Explicitly, this is described as follows. Choose a basis $(e_1,\ldots,e_w)$ of $W = \mathrm{Ext}^1_X(\mathcal{B},\mathcal{A})$. Let $\mathcal{R}$ be a representation of Υ having dimension vector (b,a). Then $\mathcal{R}$ corresponds to the choice of w linear maps $m_1,\ldots,m_w : \boldsymbol{k}^b \to \boldsymbol{k}^a$. Take the element:

$$\xi = \sum m_i \otimes e_i \in \mathrm{Hom}_{\boldsymbol{k}}(\boldsymbol{k}^b,\boldsymbol{k}^a)\otimes W. \tag{2.2}$$

Then, under $W \simeq \mathrm{Hom}_{\mathbb{P}W}(\Omega_{\mathbb{P}W}(1),\mathcal{O}_{\mathbb{P}W})$, we obtain from ξ a morphism:

$$M : \Omega_{\mathbb{P}W}(1)^b \to \mathcal{O}^a_{\mathbb{P}W}. \tag{2.3}$$

The cone of M is the element of $\mathbf{D}^b(\mathbf{Coh}_{\mathbb{P}W})$ associated with $\mathcal{R}$ via Ξ. This is directly extended to morphisms.

Theorem 1. *Let $X \subset \mathbb{P}^n$ be a closed subscheme and let $\mathcal{A}$ and $\mathcal{B}$ be simple coherent sheaves on X satisfying:*

$$\mathrm{Hom}_X(\mathcal{A},\mathcal{B}) = \mathrm{Hom}_X(\mathcal{B},\mathcal{A}) = 0.$$

Set $w = \dim_{\boldsymbol{k}} \mathrm{Ext}^1_X(\mathcal{B},\mathcal{A})$, assume $w \neq 0$, and put $\Upsilon = \Upsilon_w$. Then the restriction Φ of $\Phi_{\mathcal{U}} \circ \Xi$ to $\mathbf{Rep}_\Upsilon$ is a fully faithful exact functor:

$$\Phi : \mathbf{Rep}_\Upsilon \to \mathbf{Coh}_X.$$

Proof. Let us first give a more explicit description of Φ. At the level of objects, given a representation $\mathcal{R}$ of the quiver Υ with dimension vector (b,a) let $(m_1,\ldots,m_w)$ be the w linear maps associated with $\mathcal{R}$ and let ξ be as in (2.2). Then $\Phi(\mathcal{R})$ fits as middle term of a representative of the extension class corresponding to ξ:

$$0 \to \mathcal{A}^a \to \Phi(\mathcal{R}) \to \mathcal{B}^b \to 0.$$

Let us now check that this is well-defined on morphisms. Let $\mathcal{S}$ be another representation of Υ, of dimension vector (d,c), corresponding to the linear maps $(n_1,\ldots,n_w)$. A morphism $\lambda : \mathcal{R} \to \mathcal{S}$ of representations is given by a pair (β,α) of linear maps $\alpha : \boldsymbol{k}^a \to \boldsymbol{k}^c$ and $\beta : \boldsymbol{k}^b \to \boldsymbol{k}^d$ such that:

$$\alpha m_i = n_i \beta, \qquad \text{for all } i = 1,\ldots,w. \tag{2.4}$$

Recall that $\mathrm{End}_X(\mathcal{A}) = \langle \mathrm{id}_\mathcal{A}\rangle_{\boldsymbol{k}}$, and consider the map of coherent sheaves $\alpha_\mathcal{A} = \alpha \otimes \mathrm{id}_\mathcal{A} : \mathcal{A}^a \to \mathcal{A}^c$. Then, it is well-known that this map defines a morphism of extensions:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}^a & \longrightarrow & \Phi(\mathcal{R}) & \longrightarrow & \mathcal{B}^b & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_\mathcal{A}} & & \downarrow{\scriptstyle \phi} & & \| & & \\ 0 & \longrightarrow & \mathcal{A}^c & \xrightarrow{i} & \mathcal{D} & \xrightarrow{p} & \mathcal{B}^b & \longrightarrow & 0 \end{array} \tag{2.5}$$

for a certain sheaf $\mathcal{D}$ representing the element:

$$\sum \alpha m_i \otimes e_i \in \mathrm{Ext}^1_X(\mathcal{B}^b,\mathcal{A}^c).$$

Analogously $\beta_\mathcal{B} = \beta \otimes \mathrm{id}_\mathcal{B} : \mathcal{B}^b \to \mathcal{B}^d$ defines:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}^c & \xrightarrow{i'} & \mathcal{D}' & \xrightarrow{p'} & \mathcal{B}^b & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \beta_\mathcal{B}} & & \\ 0 & \longrightarrow & \mathcal{A}^c & \longrightarrow & \Phi(\mathcal{S}) & \longrightarrow & \mathcal{B}^d & \longrightarrow & 0 \end{array} \tag{2.6}$$

with upper row representing an extension class in:

$$\sum n_i \beta \otimes e_i \in \mathrm{Ext}^1_X(\mathcal{B}^b,\mathcal{A}^c).$$

Now we can use (2.4) to see that the lower extension of (2.5) is the same as the upper one (2.6). Then the morphisms ϕ' and ϕ of extensions compose to a map μ from $\Phi(\mathcal{R})$ to $\Phi(\mathcal{S})$.

It is clear that this construction agrees with the definition of Φ. Indeed, the representation $\mathcal{R}$ goes via Ξ to the cone of the matrix M of (2.3), which in turn is sent by $\Phi_{\mathcal{U}}$ to the cone of:

$$\Phi_{\mathcal{U}}(M) : \mathcal{B}^b[-1] \to \mathcal{A}^a.$$

By construction this cone is represented by the extension class $\Phi(\mathcal{R})$.

Now it only remains to use this setup to show that the functor Φ is fully faithful. To check this, first let λ be a morphism $\mathcal{R} \to \mathcal{S}$ represented by a pair (β, α) of linear maps, and assume that the corresponding morphism $\mu : \Phi(\mathcal{R}) \to \Phi(\mathcal{S})$ is zero. Then the composition of $\alpha_{\mathcal{A}}$ and of the inclusion $\mathcal{A}^c \hookrightarrow \Phi(\mathcal{S})$ vanishes, so we get $\alpha = 0$. Dually we obtain $\beta = 0$ and hence $\lambda = 0$ so Φ is faithful, i.e. we have injectiveness of the natural map:

$$\mathrm{Hom}_{\Upsilon}(\mathcal{R}, \mathcal{S}) \to \mathrm{Hom}_X(\Phi(\mathcal{R}), \Phi(\mathcal{S})). \tag{2.7}$$

Let us now study its surjectiveness. Given a morphism $\mu : \Phi(\mathcal{R}) \to \Phi(\mathcal{S})$, we compose μ on one side with the projection $\Phi(\mathcal{S}) \to \mathcal{B}^d$, and with the injection $\mathcal{A}^a \to \Phi(\mathcal{R})$ on the other side. We obtain thus a map $\mathcal{A}^a \to \mathcal{B}^d$, which must vanish since $\mathrm{Hom}_X(\mathcal{A}, \mathcal{B}) = 0$. We deduce that μ defines maps $\mathcal{A}^a \to \mathcal{A}^c$ and $\mathcal{B}^b \to \mathcal{B}^d$, which must be of the form $\alpha \otimes \mathrm{id}_{\mathcal{A}}$ and $\beta \otimes \mathrm{id}_{\mathcal{B}}$ by the assumption that $\mathcal{A}$ and $\mathcal{B}$ are simple.

It is clear that the pair (β, α) defines a morphism $\lambda : \mathcal{R} \to \mathcal{S}$, and we wish to see that λ maps to μ via (2.7). To check this, consider a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}^a & \xrightarrow{i} & \mathcal{D} & \xrightarrow{p} & \mathcal{B}^b & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \mathcal{A}^c & \xrightarrow{i'} & \mathcal{D}' & \xrightarrow{p'} & \mathcal{B}^d & \longrightarrow & 0 \end{array} \tag{2.8}$$

Since $p' \circ \rho = 0$, we have $\mathrm{Im}\, \rho \subseteq \mathcal{A}^c$. But $\rho i = 0$ implies that ρ factors as:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\quad\rho\quad} & \mathcal{A}^c \\ & \searrow \quad \nearrow{\scriptstyle \bar{\rho}} & \\ & \mathcal{B}^b = \mathcal{D}/\mathcal{A}^a & \end{array}$$

If $\rho \neq 0$, this would give a nonzero map $\bar{\rho} : \mathcal{B}^b \to \mathcal{A}^c$, contradicting $\mathrm{Hom}_X(\mathcal{B}, \mathcal{A}) = 0$. We conclude that $\rho = 0$. Since the image of λ via (2.7) and μ differ by a map ρ fitting into (2.8), we finally get that Φ is fully faithful. □

We deduce a criterion for an aCM variety to be strictly CM wild.

Corollary 1. *In the hypothesis of Theorem 1, we have:*

i) if $w \geq 3$ and $\mathcal{A}$ and $\mathcal{B}$ are aCM, then X is strictly CM wild;
ii) if moreover $\mathcal{A}$ and $\mathcal{B}$ are Ulrich, then X is strictly Ulrich wild.

Proof. By construction of the functor Φ of Theorem 1, the sheaf $\Phi(\mathcal{R})$ associated with a representation $\mathcal{R}$ of Υ is aCM (resp., Ulrich) if $\mathcal{A}$ and $\mathcal{B}$ are aCM (resp., Ulrich). We conclude using Remark 1. □

Remark 2. The hypothesis $\mathrm{Hom}_X(\mathcal{B}, \mathcal{A}) = 0$ in Theorem 1 is necessary. Indeed, if $\mathrm{Hom}_X(\mathcal{B}, \mathcal{A}) \neq 0$ we could consider the map:

$$\phi : \mathcal{D} \xrightarrow{p} \mathcal{B} \to \mathcal{A} \xrightarrow{i} \mathcal{D}$$

This map ϕ is not zero, and makes the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A} & \xrightarrow{i} & \mathcal{D} & \xrightarrow{p} & \mathcal{B} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \mathcal{A} & \xrightarrow{i'} & \mathcal{D} & \xrightarrow{p'} & \mathcal{B} & \longrightarrow & 0 \end{array}$$

Notice however that any ϕ fitting in such a diagram will be nilpotent.

As an explicit example, let $X \subset \mathbb{P}^{m+1}$ be an hypersurface of degree d and $Z \subset X$ an AG subscheme of codimension two and index i_Z, where:

$$i_Z := \max\{s \mid \mathrm{H}^{m-1}(X, \mathcal{I}_{Z/X}(s)) \neq 0\}.$$

Define $e := i_Z + m + 2 - d$ and assume that $e < 0$ so that $\mathrm{Hom}_X(\mathcal{I}_{Z/X}(e), \mathcal{O}_X) \neq 0$. Let $\mathcal{D}$ be the non-trivial extension of $\mathcal{I}_{Z/X}(e)$ by $\mathcal{O}_X$ (one can show that this extension exists and is unique up to a nonzero scalar, by the definition of e and by Serre duality).

When Z is not a complete intersection inside X, $\mathcal{D}$ is indecomposable. Anyway $\mathcal{D}$ is an aCM sheaf of rank 2 over X which is never simple, as it always admits a nilpotent endomorphism. The conclusion of Theorem 1 clearly fails in this case.

3. Stable syzygies of Ulrich modules

Let $X \subset \mathbb{P}^n$ be a closed aCM subscheme of dimension $m \geq 1$, let Y be a linear section of X of codimension $c < m$. Set $T := \boldsymbol{k}[Y]$, $R := \boldsymbol{k}[X]$ and write ω_Y for the dualizing sheaf of Y. Looking at a finitely generated graded module L over T as a graded module over R, we take its minimal graded R-free resolution:

$$0 \leftarrow L \leftarrow F_0 \xleftarrow{d_1} F_1 \leftarrow \cdots \leftarrow F_{\ell-1} \xleftarrow{d_\ell} F_\ell \leftarrow \cdots \tag{3.1}$$

Write $\Omega^\ell_R(L)$ for the ℓ-th syzygy of L over R, i.e. $\Omega^\ell_R(L) = \mathrm{Im}(d_\ell)$. It is well-known that, if L is MCM over T, then $\Omega^\ell_R(L)$ is MCM over R for $\ell \geq c$.

Let $\underline{\mathbf{MCM}}_R$ the stable category of maximal Cohen-Macaulay modules over R. Given M, M' in $\mathbf{MCM}_R$, we write $\underline{\mathrm{Hom}}_R(M, M')$ for the morphisms in the this category, namely the morphisms of degree 0 from M to M', modulo the ideal of morphisms (of degree zero) that factor through a free R-module. We write Π for the stabilization functor:

$$\Pi : \mathbf{MCM}_R \to \underline{\mathbf{MCM}}_R.$$

For $\ell \geq c$, we have also the ℓ-th syzygy stable functor:

$$\underline{\Omega}^\ell : \mathbf{MCM}_T \to \underline{\mathbf{MCM}}_R, \qquad L \mapsto \Omega^\ell_R(L).$$

Theorem 2. *Assume* $\mathrm{H}^0(Y, \omega_Y(m - c - 1)) \neq 0$. *Then the restriction to* $\mathbf{Ulr}_T$ *of the* c*-th stable syzygy functor provides a fully faithful embedding:*

$$\underline{\Omega}^c : \mathbf{Ulr}_T \to \underline{\mathbf{MCM}}_R.$$

The following lemma will be one of the keys of our analysis. Given a finitely generated graded R-module M, $\langle M_{\leq d}\rangle$ denotes the graded submodule of M generated by the elements of degree at most d of M. We also write $M^* := \mathrm{Hom}_R(M, R)$.

Lemma 1. *Fix the hypothesis as in Theorem 2, let* L *be an Ulrich module over* T*, and set* $M := \Omega^c_R(L)$. *Then we have a functorial exact sequence:*

$$0 \to \langle M^*_{\leq 1-c}\rangle \to M^* \to \mathrm{Hom}_T(L, T(c)) \to 0.$$

Proof. Recall that L is generated in a single degree, and that the number of minimal generators of L equals $\alpha_0 = \deg(X)\,\mathrm{rk}(L)$. In other words, we can assume $F_0 \simeq R^{\alpha_0}$. By the minimality of the resolution, we have, for $i \geq 1$:

$$F_\ell \simeq \bigoplus_{j \geq \ell} R(-j)^{\alpha_{\ell,j}}, \qquad \text{for some integers } \alpha_{\ell,j}.$$

Also, X and Y are not linear subspaces, so that L has no free summands.

The coherent sheaf over the variety X associated with L is of the form $i_*(\mathcal{L})$, where $\mathcal{L}$ is an Ulrich sheaf over Y and $i : Y \hookrightarrow X$ is the obvious inclusion. Then sheafifying (3.1) we get an exact complex:

$$0 \leftarrow i_*\mathcal{L} \leftarrow \mathcal{F}_0 \xleftarrow{d_1} \mathcal{F}_1 \leftarrow \cdots \leftarrow \mathcal{F}_{\ell-1} \xleftarrow{d_\ell} \mathcal{F}_\ell \leftarrow \cdots \tag{3.2}$$

where $\mathcal{F}_\ell \simeq \bigoplus_{j\geq \ell} \mathcal{O}_X(-j)^{\alpha_{\ell,j}}$. Set $\mathcal{M}$ for the sheafification of M.

Now apply $\mathcal{H}om_X(-, \omega_X)$ to (3.2) and use $\mathcal{E}xt^k_X(i_*\mathcal{L}, \omega_X) = 0$ for $k < c$. Note also that Grothendieck duality gives:

$$\mathcal{E}xt^c_X(i_*\mathcal{L}, \omega_X) \simeq i_*\mathcal{H}om_Y(\mathcal{L}, \omega_Y).$$

We obtain a long exact sequence:

$$0 \to \mathcal{F}_0^\vee \otimes \omega_X \to \cdots \to \mathcal{F}_{c-2}^\vee \otimes \omega_X \xrightarrow{d_{c-1}^*} \mathcal{F}_{c-1}^\vee \otimes \omega_X \xrightarrow{\pi} \mathcal{M}^\vee \otimes \omega_X \to i_* \mathcal{H}om_Y(\mathcal{L}, \omega_Y) \to 0.$$

Next, we apply $\mathcal{H}om_X(\omega_X, -)$ to this sequence and use that X is Cohen-Macaulay so that $\mathcal{H}om_X(\omega_X, \omega_X) \simeq \mathcal{O}_X$. Also, we easily see that:

$$\mathcal{H}om_X(\omega_X, i_*\mathcal{H}om_Y(\mathcal{L}, \omega_Y)) \simeq \mathcal{H}om_Y(i^*\omega_X, \mathcal{H}om_Y(\mathcal{L}, \omega_Y)) \simeq \mathcal{H}om_Y(\mathcal{L}, \mathcal{H}om_Y(i^*\omega_X, \omega_Y)),$$

as the first isomorphism is obvious, while the second one follows from a standard exchange argument of homomorphism groups. Moreover, since Y is cut in X by c hyperplanes forming a regular sequence, adjunction theory says that $\omega_Y \simeq i^*\omega_Y(c)$. Therefore, using $\mathcal{H}om_Y(\omega_Y, \omega_Y) \simeq \mathcal{O}_Y$ we get:

$$\mathcal{H}om_Y(\mathcal{L}, \mathcal{H}om_Y(i^*\omega_X, \omega_Y)) \simeq \mathcal{H}om_Y(\mathcal{L}, \mathcal{O}_Y(c)).$$

Putting together, we get the long exact sequence:

$$0 \to \mathcal{F}_0^\vee \to \cdots \to \mathcal{F}_{c-2}^\vee \xrightarrow{d_{c-1}^*} \mathcal{F}_{c-1}^\vee \xrightarrow{\pi} \mathcal{M}^\vee \to i_*\mathcal{H}om_Y(\mathcal{L}, \mathcal{O}_Y(c)) \to 0.$$

Taking global sections of this sequence, we just get the result of applying $\mathrm{Hom}_R(-, R)$ to (3.1), i.e,. the long exact sequence:

$$0 \to F_0^* \to \cdots \to F_{c-2}^* \xrightarrow{d_{c-1}^*} F_{c-1}^* \xrightarrow{\pi} M^* \to \mathrm{Hom}_T(L, T(c)) \to 0. \tag{3.3}$$

Now comes the main point, namely that $\mathrm{Hom}_T(L,T)_1 = 0$. To see this, recall that $\mathrm{Hom}_T(L,T)_1 \simeq \mathrm{Hom}_Y(\mathcal{L}, \mathcal{O}_Y(1))$ and that $\mathcal{L}$ is Ulrich on Y as well as $\mathcal{H}om_Y(\mathcal{L}, \omega_Y)$. Then, by [ESW03, Proposition 2.1] we get:

$$\mathrm{Hom}_Y(\mathcal{L}, \omega_Y(m-c)) \simeq \mathrm{H}^{m-c}(Y, \mathcal{L}(c-m))^* = 0.$$

Note that, by the assumption $\mathrm{H}^0(Y, \omega_Y(m-c-1)) \neq 0$ we get an embedding:

$$\mathcal{O}_Y(1) \hookrightarrow \omega_Y(m-c).$$

Therefore, $\mathrm{Hom}_Y(\mathcal{L}, \omega_Y(m-c)) = 0$ implies $\mathrm{Hom}_Y(\mathcal{L}, \mathcal{O}_Y(1)) = 0$.

We have thus established that $\mathrm{Hom}_T(L, T(c))$ contains no element of degree $\leq 1-c$. Also, we may write:

$$F_{c-1}^* = R(c-1)^{\alpha_{c-1,c-1}} \oplus R(c)^{\alpha_{c-1,c}} \oplus \cdots$$

Then, (3.3) says that F_{c-1}^* generates all the elements of M^* of degree at most $1-c$, i.e. the image of π is the submodule $\langle M^*_{\leq 1-c} \rangle$ of M^*. This is clearly functorial, and the lemma is now proved. □

Proof of Theorem 2. Let L and N be two Ulrich modules over T. Our goal will be to describe two mutually inverse maps:

$$\mathrm{Hom}_T(L, N)_0 \leftrightarrows \underline{\mathrm{Hom}}_R(\Omega_R^c(L), \Omega_R^c(N))_0.$$

Set $M = \Omega_R^c(L)$ and $P = \Omega_R^c(N)$. First, let $\varphi : L \to N$ belong to $\mathrm{Hom}_T(L, N)_0$. Consider the minimal graded free resolutions of L and N over R and choose a lifting of φ to these resolutions:

$$\begin{array}{ccccccccccc} 0 & \leftarrow & L & \leftarrow & F_0 & \leftarrow & \cdots & \leftarrow & F_{c-1} & \leftarrow & M \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_0} & & & & \downarrow{\scriptstyle \varphi_{c-1}} & & \vdots{\scriptstyle \tilde{\varphi}} \\ 0 & \leftarrow & N & \leftarrow & G_0 & \leftarrow & \cdots & \leftarrow & G_{c-1} & \leftarrow & P \end{array} \tag{3.4}$$

The morphism $\tilde{\varphi}$ induced on the c-th syzygy modules gives the class $\bar{\varphi}$ in $\underline{\mathrm{Hom}}_R(M, P)_0$. This does not depend on the choice of the lifting φ_i, as any other choice would provide a map $\tilde{\varphi}'$ such that $\tilde{\varphi} - \tilde{\varphi}'$ factors through a free module.

Conversely, given $\bar{\psi} \in \underline{\mathrm{Hom}}_R(M, P)_0$, we choose a representative $\psi : M \to P$ and take its dual $\psi^* : P^* \to M^*$. Since ψ^* is homogeneous of degree 0, it maps elements of degree at most $1-c$

in P^* to elements of degree at most $1-c$ in M^*. By Lemma 1 we obtain a diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \langle M^*_{\leq 1-c}\rangle & \longrightarrow & M^* & \longrightarrow & \mathrm{Hom}_T(L,T(c)) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \psi^*} & & \uparrow{\scriptstyle \psi^*} & & \vdots\uparrow{\scriptstyle \hat{\psi}} & & \\
0 & \longrightarrow & \langle P^*_{\leq 1-c}\rangle & \longrightarrow & P^* & \longrightarrow & \mathrm{Hom}_T(N,T(c)) & \longrightarrow & 0
\end{array}
\tag{3.5}
$$

We wish to associate with $\bar{\psi}$ the morphism $\hat{\psi}^* : L \to N$. To do this, we have to check that $\hat{\psi}$ does not depend on the choice of the representative ψ of $\bar{\psi}$. By definition any other representative differs from ψ by a map $\zeta : M \to P$ that factors through a free module, which we call F, i.e. $\zeta = \zeta_2\zeta_1$ with $\zeta_1 : M \to F$ and $\zeta_2 : F \to P$. Therefore ζ^* factors through F^* and again by Lemma 1 we get the commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \langle M^*_{\leq 1-c}\rangle & \longrightarrow & M^* & \longrightarrow & \mathrm{Hom}_T(L,T(c)) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \zeta_1^*} & & \uparrow{\scriptstyle \zeta_1^*} & & \vdots\uparrow & & \\
0 & \longrightarrow & \langle F^*_{\leq 1-c}\rangle & \longrightarrow & F^* & & {\scriptstyle \hat{\zeta}}\,\vdots & & \\
 & & \uparrow{\scriptstyle \zeta_2^*} & & \uparrow{\scriptstyle \zeta_2^*} & & \vdots & & \\
0 & \longrightarrow & \langle P^*_{\leq 1-c}\rangle & \longrightarrow & P^* & \longrightarrow & \mathrm{Hom}_T(N,T(c)) & \longrightarrow & 0
\end{array}
$$

Call G the quotient $F^*/\langle F^*_{\leq 1-c}\rangle$. The diagram says that $\hat{\zeta}$ factors through G.

Now observe that G is a free module. Indeed, any direct summand of F takes the form $R(a)$ for some $a \in \mathbb{Z}$, and:

$$
\langle R(a)_{\leq 1-c}\rangle = \begin{cases} R(a), & \text{if } a \geq c-1, \\ 0, & \text{if } a < c-1. \end{cases}
\tag{3.6}
$$

Therefore G is the direct sum of all summands $R(a)$ of F^* with $a < c-1$, hence G is free. But $\mathrm{Hom}_T(N,T(c))$ is a torsion module, So it admits no non-trivial morphism with target in G, and therefore $\hat{\zeta} = 0$.

Let us now check that these maps are mutually inverse. Given $\varphi \in \mathrm{Hom}_T(L,N)_0$, we consider a representative $\psi := \tilde{\varphi}$ of the class $\bar{\varphi}$. Dualizing (3.4) we obtain a commutative diagram:

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & F^*_{c-1} & \longrightarrow & M^* & \longrightarrow & \mathrm{Hom}_T(L,T(c)) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \varphi^*_{c-1}} & & \uparrow{\scriptstyle \psi^*} & & \vdots\uparrow{\scriptstyle \varphi^*} & & \\
\cdots & \longrightarrow & G^*_{c-1} & \longrightarrow & P^* & \longrightarrow & \mathrm{Hom}_T(N,T(c)) & \longrightarrow & 0
\end{array}
$$

This diagram is the extension of (3.5) to a minimal resolution of $\langle P^*_{\leq 1-c}\rangle$ and $\langle M^*_{\leq 1-c}\rangle$. This says that $\hat{\psi} = \varphi^*$, so $\hat{\psi}^* = \varphi$.

Conversely, let ψ be a representative of $\bar{\psi} \in \underline{\mathrm{Hom}}_R(M,P)_0$ and set $\varphi = \hat{\psi}^*$. Lift the map $\langle P^*_{\leq 1-c}\rangle \to \langle M^*_{\leq 1-c}\rangle$ induced by ψ to the minimal graded free resolutions of these modules and dualize to obtain:

$$
\begin{array}{ccccccccccccc}
0 & \longleftarrow & L & \longleftarrow & F_0 & \longleftarrow & \cdots & \longleftarrow & F_{c-1} & \longleftarrow & M & \longleftarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi_0} & & & & \downarrow{\scriptstyle \psi_{c-1}} & & \downarrow{\scriptstyle \psi} & & \\
0 & \longleftarrow & N & \longleftarrow & G_0 & \longleftarrow & \cdots & \longleftarrow & G_{c-1} & \longleftarrow & P & \longleftarrow & 0
\end{array}
$$

Then ψ is induced by a lifting of $\varphi : L \to N$ to the minimal resolutions of L and N. Our proof is thus complete. □

We isolate the following easy consequence of Lemma 1.

Lemma 2. *If L is an Ulrich module L over T, then $\Omega^c_R(L)$ has no free summands.*

Proof. Suppose $\Omega^c_R(L) = M \oplus F$, with F a non-zero free direct summand. Then by Lemma 1 we have:

$$
0 \to \langle M^*_{\leq 1-c}\rangle \oplus \langle F^*_{\leq 1-c}\rangle \xrightarrow{\pi} M^* \oplus F^* \to \mathrm{Hom}_T(L,T(c)) \to 0,
$$

with π diagonal. Since obviously L has no free summand, (3.6) says that the restriction of π is an isomorphism between $\langle F^*_{\leq 1-c}\rangle$ and $F^*,$. Therefore, looking at (3.3) we see that d_c is surjective onto F, which contradicts minimality of the resolution (3.1). □

Example 1. Let $X \subset \mathbb{P}^{m+1}$ be a hypersurface of degree d and Y be a linear section of codimension c of X. Then, by matrix factorization the minimal graded free resolution of an Ulrich module L of rank r on $T = \boldsymbol{k}[Y]$ reads:

$$0 \leftarrow L \leftarrow T^{rd} \leftarrow T(-1)^{rd} \leftarrow T(-d)^{rd} \leftarrow \cdots \tag{3.7}$$

Since $L(-d) \simeq \ker(T(-1)^{rd} \to T^{rd})$, this yields a resolution:

$$0 \leftarrow \mathrm{Hom}_T(L, T(d-1)) \leftarrow T^{rd} \leftarrow T(-1)^{rd} \leftarrow T(-d)^{rd} \leftarrow \cdots \tag{3.8}$$

Combining (3.7) with the Koszul resolution of Y in X we get a resolution over $R = \boldsymbol{k}[X]$:

$$0 \leftarrow L \leftarrow R^{rd} \leftarrow R(-1)^{rd(c+1)} \leftarrow R(-2)^{rd(c+\binom{c}{2})} \oplus R(-d)^{rd} \leftarrow \cdots \tag{3.9}$$

The k-th term F_k of this resolution looks as follows (here $\varepsilon \in \{0,1\}$):

$$F_k = \bigoplus_{2h+\varepsilon+j=k} R(-(j+hd+\varepsilon))^{\binom{c}{j}rd}.$$

Let $M = \Omega^c_R(L)$. The resolution of the dualized syzygy M^* starts with:

$$\cdots \to R(c-d)^{rd(c+1)} \oplus F^*_{c-2} \to R(c-d+1)^{rd} \oplus F^*_{c-1} \to M^* \to 0.$$

We may now remove from this the dual of the truncation at $M = \Omega^c_R(L)$ of (3.9), which is to say, by Lemma 1, the resolution of $\langle M^*_{1-c}\rangle$. The residual strand recovers precisely (3.8), twisted by $R(c-d+1)$. The two strands of the resolution do not mix if $d > 2$.

Remark 3. The assumption $\mathrm{H}^0(Y, \omega_Y(m-c-1)) \neq 0$ is crucial, as shown by the following example. Take $X = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ and let Y be a cubic scroll obtained as hyperplane section of X. Note that in this case $\mathrm{H}^0(Y, \omega_Y(1)) = 0$. If F is a line in Y and H is the class of a hyperplane in Y, then $\mathcal{L} = \mathcal{O}_Y(H-F)$ is an Ulrich line bundle on Y, whose sheafified minimal $\mathcal{O}_X$-resolution reads:

$$0 \leftarrow i_*\mathcal{L} \leftarrow \mathcal{O}_X^4 \leftarrow \mathcal{O}_X^9(-1) \leftarrow \mathcal{O}_X^{18}(-2) \leftarrow \cdots$$

The sheafified first syzygy $\mathcal{M}$ of $\mathcal{L}$ has a resolution:

$$0 \leftarrow \mathcal{M}^* \leftarrow \mathcal{O}_X^5 \leftarrow \mathcal{O}_X^5(-1) \leftarrow \mathcal{O}_X^9(-2) \leftarrow \mathcal{O}_X^{18}(-3) \leftarrow \cdots$$

In this case, $\mathcal{H}om_Y(\mathcal{L}, \mathcal{O}_Y(1)) \simeq \mathcal{O}_Y(F)$ appears as cokernel of a map $\mathcal{O}_X^4 \to \mathcal{M}$, but this cannot be singled out by degree reasons only from the first map of the above resolution.

4. CM wildness from syzygies of Ulrich extensions

Let us fix the setup for this section. Let $X \subset \mathbb{P}^n$ be an aCM subscheme, put $R = \boldsymbol{k}[X]$ and let Y be a linear section of X of codimension c and dimension $m \geq 1$. Let $\mathcal{A}$ and $\mathcal{B}$ be two simple Ulrich sheaves on Y. Set $W = \mathrm{Ext}^1_Y(\mathcal{B}, \mathcal{A})$, $w = \dim_{\boldsymbol{k}} W$, and assume $w \neq 0$. Put $\Upsilon = \Upsilon_w$. Over $Y \times \mathbb{P}W$, there is a universal extension:

$$0 \to \mathcal{A} \boxtimes \mathcal{O}_{\mathbb{P}W} \to \mathcal{U} \to \mathcal{B} \boxtimes \mathcal{O}_{\mathbb{P}W}(-1) \to 0.$$

Take the sheafified minimal graded free resolutions of $\mathcal{A}$ and $\mathcal{B}$ as $\mathcal{O}_X$-modules, pull-back via p to $X \times \mathbb{P}W$, and use the mapping cone construction to build a minimal graded free resolution of $\mathcal{U}$ over $X \times \mathbb{P}W$:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{A} \boxtimes \mathcal{O}_{\mathbb{P}W} & \longrightarrow & \mathcal{U} & \longrightarrow & \mathcal{B} \boxtimes \mathcal{O}_{\mathbb{P}W}(-1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle d_0} & & \uparrow & & \\
0 & \longrightarrow & \mathcal{F}_0 \boxtimes \mathcal{O}_{\mathbb{P}W} & \longrightarrow & \mathcal{H}_0 & \longrightarrow & \mathcal{G}_0 \boxtimes \mathcal{O}_{\mathbb{P}W}(-1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{F}_{c-1} \boxtimes \mathcal{O}_{\mathbb{P}W} & \longrightarrow & \mathcal{H}_{c-1} & \longrightarrow & \mathcal{G}_{c-1} \boxtimes \mathcal{O}_{\mathbb{P}W}(-1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle d_c} & & \uparrow & & \\
0 & \longrightarrow & \mathcal{F}_c \boxtimes \mathcal{O}_{\mathbb{P}W} & \longrightarrow & \mathcal{H}_c & \longrightarrow & \mathcal{G}_c \boxtimes \mathcal{O}_{\mathbb{P}W}(-1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Here, $\mathcal{H}_i = \mathcal{F}_i \boxtimes \mathcal{O}_{\mathbb{P}W} \oplus \mathcal{G}_i \boxtimes \mathcal{O}_{\mathbb{P}W}(-1)$. Set:

$$\mathcal{V} = \operatorname{Im}(d_c).$$

Then we consider:

$$\Phi_{\mathcal{V}} = \mathbf{R}p_*(q^*(-) \otimes \mathcal{V}) : \mathbf{D}^b(\mathbf{Coh}_{\mathbb{P}W}) \to \mathbf{D}^b(\mathbf{Coh}_X).$$

The following lemma is clear.

Lemma 3. *Let $T = \boldsymbol{k}[Y]$, and set L and N for the modules of global sections of $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{M}$ and $\mathcal{N}$ be the sheafifications of $\Omega^c_R(L)$ and $\Omega^c_R(N)$. Then:*

$$\Phi_{\mathcal{V}}(\mathcal{O}_{\mathbb{P}W}) \simeq \mathcal{M}, \qquad \Phi_{\mathcal{V}}(\Omega_{\mathbb{P}W}(1)) \simeq \mathcal{N}[-1].$$

Now consider the equivalence Ξ of (2.1). Then the restriction of $\Phi_{\mathcal{V}} \circ \Xi$ to $\mathbf{Rep}_\Upsilon$, composed with the global sections functor, gives an exact functor:

$$\Psi : \mathbf{Rep}_\Upsilon \to \mathbf{MCM}_R.$$

Theorem 3. *Assume* $\operatorname{Hom}_Y(\mathcal{A}, \mathcal{B}) = \operatorname{Hom}_Y(\mathcal{B}, \mathcal{A}) = 0$ *and* $\mathrm{H}^0(Y, \omega_Y(m - c - 1)) \neq 0$. *Then Ψ is a representation embedding. In particular if $w \geq 3$ then X is of wild CM type.*

Proof. By construction we have the commutative diagram of functors:

$$
\begin{array}{ccc}
\mathbf{Rep}_\Upsilon & \xrightarrow{\ \Psi\ } & \mathbf{MCM}_R \\
\downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Pi} \\
\mathbf{Ulr}_T & \xrightarrow{\ \underline{\Omega}^c_R\ } & \underline{\mathbf{MCM}}_R
\end{array}
$$

We proved in Theorem 2 that $\underline{\Omega}^c_R$ is fully faithful, and in Theorem 1 that Φ is also fully faithful. So the same happens to $\underline{\Omega}^c_R \circ \Phi$ and therefore to $\Pi \circ \Psi$.

Therefore, if $\mathcal{R}$ and $\mathcal{S}$ are two representations of Υ such that $\Psi(\mathcal{R}) \simeq \Psi(\mathcal{S})$, we see that $\Pi(\Psi(\mathcal{R})) \simeq \Pi(\Psi(\mathcal{S}))$ and hence $\mathcal{R} \simeq \mathcal{S}$ by full faithfulness.

Moreover, if $\Psi(\mathcal{R})$ is decomposable, then $\operatorname{Hom}_R(\Psi(\mathcal{R}), \Psi(\mathcal{R}))_0$ contains a non-trivial idempotent ψ. The class $\bar{\psi}$ is also an idempotent, which is trivial if and only if the direct summand of $\Psi(\mathcal{R})$ associated with ψ is free. But this cannot happen by Lemma 2, again by full faithfulness of $\Pi \circ \Psi$, $\bar{\psi}$ corresponds to a non-trivial idempotent of $\mathcal{R}$, so $\mathcal{R}$ is also decomposable. This finishes the proof that Ψ is a representation embedding. The consequence that X is of wild CM type is clear. □

Example 2. The first class of varieties where it is a priori unknown how to construct large families of aCM bundles is given by general cubic hypersurfaces of dimension $m \geq 4$. We can do this with Theorem 3. Indeed, start with a cubic hypersurface X in $\mathbb{P}^{m+1}$, say smooth in codimension 3, and take a smooth surface section Y. Then we may take $\mathcal{A} = \mathcal{O}_Y(A)$ and $\mathcal{B} = \mathcal{O}_Y(B)$, where A and B are twisted cubics curves in Y meeting at 5 points, cf. [Fae08]; these sheaves will satisfy the assumptions of Theorem 3. In the next section we will see how to deal in a similar fashion with any variety besides the non-wild varieties listed in the main result.

5. Proof of the main result

We prove here the main theorem of the introduction. Let $X \subset \mathbb{P}^n$ be an integral closed aCM subvariety of dimension $m \geq 1$ and degree d, and let Y be an integral one-dimensional linear section of X. We argue on the arithmetic genus p of Y, i.e. the *sectional genus* of X. We also introduce the Δ-*genus* of X, defined as $\Delta(X) = d - m + n - 1$; it is well-known that $\Delta(X) \geq 0$.

5.1. **Minimal degree.** In case $p = 0$ we have $\Delta(X) = 0$, which means that X has minimal degree. According to the classification of del Pezzo and Bertini of these varieties, cf. [EH87], we have that X is smooth, for otherwise X would be a cone. In this case X is of finite CM type (cf. [AR87, EH88]) if it is a linear space (cf. [Hor64]), or a smooth quadric (see [Knö87]), or the Veronese surface in $\mathbb{P}^5$, or a smooth cubic scroll in $\mathbb{P}^4$ (see [AR89]), or a rational normal curve. Also, X is of tame CM type if it is a rational normal surface scroll of degree 4, cf. [FM14]. Besides these cases, X is strictly Ulrich wild, as we see by applying Theorem 1 to the Ulrich line bundles constructed in [MR13, FM14].

5.2. **Sectional genus at least 2.** Let us look at the case of (arithmetic) genus $p \geq 2$. Consider the normalization $\sigma : \bar{Y} \to Y$ of Y, where $\bar{Y}$ is a smooth irreducible curve of genus g.

We first consider an Ulrich sheaf $\mathcal{L}_0$ of rank one on Y. According to [ESW03], to construct $\mathcal{L}_0$ on Y we need to pick a sufficiently general line bundle $\bar{\mathcal{L}}_0$ of degree $d+g-1$ on $\bar{Y}$ and define $\mathcal{L}_0 = \sigma_*(\bar{\mathcal{L}}_0)$. Since Y is integral, the sheaf $\mathcal{L}_0$ is simple, being reflexive of rank 1.

Now we consider two general flat deformations $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{L}_0$ in the compactified Jacobian of Y. These are locally free sheaves of rank 1 on Y, both having the same degree as $\mathcal{L}_0$, namely again $d+g-1$. The condition that $\mathcal{L}_k$ is Ulrich amounts to $\mathrm{H}^i(Y, \mathcal{L}_k(-1)) = 0$ for $i = 0, 1$ (cf. [ESW03, Th. 4.3]). By assumption these vanishings hold for $k = 0$, hence by semicontinuity also for $k = 1, 2$, so that $\mathcal{L}_1$ and $\mathcal{L}_2$ are Ulrich line bundles on Y. Since the dimension of the compactified Jacobian of Y is $p \geq 2$, we may assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are not isomorphic. Therefore, since the $\mathcal{L}_i$ are invertible and have the same Hilbert polynomial, by irreducibility of Y we deduce $\mathrm{Hom}_Y(\mathcal{L}_i, \mathcal{L}_j) = 0$ for $i \neq j$.

Next, since $\mathcal{L}_1$ and $\mathcal{L}_2$ are locally free of rank 1, we have $\chi(\mathcal{L}_i, \mathcal{L}_i) = 1 - p$ for $i = 1, 2$. Therefore, we may choose $\mathcal{A}$ and $\mathcal{B}$ as two sheaves corresponding to non-trivial elements of $\mathrm{Ext}^1_Y(\mathcal{L}_i, \mathcal{L}_i)$, and write:

$$0 \to \mathcal{L}_1 \to \mathcal{A} \to \mathcal{L}_1 \to 0, \qquad 0 \to \mathcal{L}_2 \to \mathcal{B} \to \mathcal{L}_2 \to 0.$$

It is clear that $\mathcal{A}$ and $\mathcal{B}$ are locally free Ulrich sheaves of rank 2. Also it is easy to check that $\mathcal{A}$ and $\mathcal{B}$ are simple and satisfy:

$$\mathrm{Hom}_Y(\mathcal{A}, \mathcal{B}) = \mathrm{Hom}_Y(\mathcal{B}, \mathcal{A}) = 0.$$
$$\chi(\mathcal{A}, \mathcal{B}) = \chi(\mathcal{B}, \mathcal{A}) = 4(1-p).$$

We obtain the following:

$$\dim_{\boldsymbol{k}} \mathrm{Ext}^1_Y(\mathcal{A}, \mathcal{B}) = \dim_{\boldsymbol{k}} \mathrm{Ext}^1_Y(\mathcal{B}, \mathcal{A}) = 4(p-1).$$

Note that the non vanishing condition of Theorem 2 reduces to $\mathrm{H}^0(Y, \omega_Y) \neq 0$, which is obviously true. Therefore X is of wild representation type by Theorem 3. From Theorem 1 we get:

Corollary 2. *An integral curve of arithmetic genus at least* 2 *is of strictly wild Ulrich type.*

5.3. **Sectional genus 1, normal del Pezzo surfaces.** Finally, let us consider the case of sectional genus $p = 1$. Since X is aCM, it turns out that the Δ-genus of X is also one, cf. [Fuj90, page 45], namely X is of almost minimal degree. According to Fujita's terminology, X is a *del Pezzo variety.*

5.3.1. *General discussion and the case of elliptic curves.* This time, X is of tame CM type if it is a smooth elliptic curve (see [Ati57]) or a rational curve of arithmetic genus 1 with only an ordinary double point (see [DG01]), which settles the case $m = 1$.

For higher dimension, we have to work with a surface section Y of X instead of a curve section. So we let Y be a general linear section of X of dimension 2, in particular Y is an integral projective aCM surface which is not a cone, having degree d in $\mathbb{P}^d$. These surfaces constitute a generalization of classical del Pezzo surfaces to allow certain singularities. They are arithmetically Gorenstein, and in fact $\omega_Y \simeq \mathcal{O}_Y(-1)$. In particular Y satisfies the non vanishing condition of Theorem 2, namely $\mathrm{H}^0(Y, \omega_Y(1)) \neq 0$.

Surfaces of almost minimal degree, and actually even higher dimensional varieties of almost minimal degree, are completely classified (cf. [Fuj82] for the case of normal varieties, and [Rei94, BS07] for the non-normal case). We will actually rely on the complete classification of (singular) del Pezzo surfaces of degree 3 and 4. We refer to [LPS11, LPS12] for this theory, rooted in work of Schläfli and Cayley, see for instance [Abh60, BW79].

For our purpose, we have to study in two different ways the case of normal surfaces and that of non-normal ones. Indeed, in the normal case we can treat Ulrich bundles of rank 2 in view of the good knowledge of sets of points in Y and Hartshorne-Serre correspondence, while we may not rely on rank-1 Ulrich bundles (one reason is that they do not even exist for cubic surfaces with an E_6 singularity, cf. [Dol12, Theorem 9.3.6]). On the other hand for the non-normal case we do use rank-1 Ulrich sheaves coming from the normalization $\bar{Y}$, which are under control since $\bar{Y}$ is a surface of minimal degree.

5.3.2. *Hartshorne-Serre correspondence for normal del Pezzo surfaces.* We assume from now on in this subsection that Y *is (locally) normal.* This case is treated via the Hartshorne-Serre construction in the spirit for instance of [CH11], where we only need to modify the argument slightly so that it applies to singular surfaces too. So let us consider a set $Z \subset Y \subset \mathbb{P}^d$ of $d+2$ points in general linear position. We have:

$$\mathrm{Ext}^1_Y(\mathcal{I}_{Z/Y}(2), \mathcal{O}_Y)^* \simeq \mathrm{Ext}^1_Y(\mathcal{O}_Y, \mathcal{I}_{Z/Y}(1)) \simeq \mathrm{H}^1(Y, \mathcal{I}_{Z/Y}(1)) \simeq \boldsymbol{k},$$

where we used that $\omega_Y \simeq \mathcal{O}_Y(-1)$ and that Z spans the whole $\mathbb{P}^d$. Through Serre correspondence, a non-zero element $\eta \in \boldsymbol{k} \simeq \mathrm{H}^1(Y, \mathcal{I}_{Z/Y}(1))$ provides a coherent sheaf $\mathcal{F}$ of rank 2 that fits into the short exact sequence:

$$0 \to \mathcal{O}_Y \to \mathcal{F}_Z \to \mathcal{I}_{Z/Y}(2) \to 0. \tag{5.1}$$

Given a set of points Z of $d+2$ points of Y, write $[Z]$ for the corresponding element of the Hilbert scheme $\mathrm{Hilb}^{d+2}(Y)$ of subschemes of length $d+2$ of Y. In the next lines, stability is always with respect to $\mathcal{O}_Y(1)$.

Lemma 4. *Let $Y \subset \mathbb{P}^d$ be an integral normal aCM surface of degree d.*

i) If Z is a reduced set of $d+2$ points in general linear position then $\mathcal{F}_Z$ is Ulrich.
ii) If moreover Z is disjoint from the singular locus of Y then $\mathcal{F}_Z$ is locally free.
iii) If $[Z]$ is general enough in $\mathrm{Hilb}^{d+2}(Y)$, then the sheaf $\mathcal{F}_Z$ is stable.

Proof. The fact that $\mathcal{F}_Z$ is Ulrich follows from the form of the minimal graded free resolution of the ideal of Z, which can be extracted from [MRPL12], so (i) is clear. The fact that $\mathcal{F}_Z$ is locally free is immediate when Y is smooth, by the Auslander-Buchsbaum formula. If Y is singular and Z does not meet the singular locus of Y, then Z is locally a complete intersection. On the other hand, we have:

$$\eta \in \mathrm{Ext}^1_Y(\mathcal{I}_{Z/Y}(2), \mathcal{O}_Y) \simeq \mathrm{H}^0(Y, \mathcal{E}xt^1_Y(\mathcal{I}_{Z/Y}(2), \mathcal{O}_Y)),$$

so for any $x \in Y$, the element $\eta_x \in \mathcal{E}xt^1_Y(\mathcal{I}_{Z/Y}(2), \mathcal{O}_Y))_x$ represents a non-trivial extension of $(\mathcal{F}_Z)_x$. Moreover, taking the dual of the short exact sequence:

$$0 \to \mathcal{I}_{Z/Y}(2) \to \mathcal{O}_Y(2) \to \mathcal{O}_Z(2) \to 0$$

we deduce:

$$\mathcal{E}xt^1_Y(\mathcal{I}_{Z/Y}(2), \mathcal{O}_Y) \simeq \mathcal{E}xt^2_Y(\mathcal{O}_Z, \mathcal{O}_Y) \simeq \omega_Z.$$

and therefore $\mathcal{E}xt^1_Y(\mathcal{I}_{Z/Y}(2), \mathcal{O}_Y)_x$ is generated by η_x. Then we can apply [OSS80, Lemma 5.1.2] to conclude that $(\mathcal{F}_Z)_x$ is free. We have proved (ii).

The proof of (iii) will be settled by a dimension count. If the Ulrich bundle $\mathcal{F}_Z$ is not stable, a straightforward modification of [CHGS12], relying on the fact that $\omega_Y \simeq \mathcal{O}_Y(-1)$, shows that $\mathcal{F}_Z$ is an extension

$$0 \to \mathcal{A} \to \mathcal{F}_Z \to \mathcal{B} \to 0,$$

where $\mathcal{A}$ and $\mathcal{B}$ are Ulrich sheaves of rank 1. By construction, the global section $s \in \mathrm{H}^0(Y, \mathcal{F}_Z)$ associated with Z vanishes in codimension two. Therefore s cannot lie in $\mathrm{H}^0(Y, \mathcal{A}) \subset \mathrm{H}^0(Y, \mathcal{F}_Z)$, since s would then vanish in codimension 1. We can construct the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & \mathcal{O}_Y & = & \mathcal{O}_Y & & \\
 & & & & \downarrow{\scriptstyle s} & & \downarrow & & \\
0 & \to & \mathcal{A} & \to & \mathcal{F}_Z & \to & \mathcal{B} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{A} & \to & \mathcal{I}_{Z/Y}(2) & \to & \mathcal{T} & \to & 0
\end{array}$$

where $\mathcal{T}$ is a torsion sheaf defined by the diagram. We can then deduce that:

$$\mathrm{H}^0(Y, \mathcal{I}_{Z/Y} \otimes \mathcal{A}^\vee(2)) \neq 0,$$

namely Z lies on a divisor D from the linear system $|\mathcal{A}^\vee(2)|$. Since $\mathcal{A}^\vee(2)$ is also an Ulrich sheaf (again because $\omega_Y \simeq \mathcal{O}_Y(-1)$), the dimension of the linear system $|\mathcal{A}^\vee(2)|$ is $d-1$. Therefore, the family of subschemes of Y consisting of $(d+2)$-tuples of points lying on a divisor of class $\mathcal{A}^\vee(2)$ is at most:

$$\dim \mathrm{Hilb}^{d+2}(D) + d - 1 = 2d + 1 < 2d + 4 = \dim \mathrm{Hilb}^{d+2}(Y).$$

Then $\mathcal{F}_Z$ is stable as soon as $[Z]$ lies away from any of the proper closed algebraic subsets of $\mathrm{Hilb}^{d+2}(Y)$ defined by each Ulrich sheaf of rank 1 over Y. Now the statement is clear since all of these sheaves have discrete parameter space. Indeed, any such sheaf corresponds to the linear equivalence class of a Weil divisor of Y, and since Y is a normal regular surface, the divisor class group of Y is discrete. □

This construction can be performed in quite a general setup. For instance for cubic surfaces one even knows, if $\boldsymbol{k}$ is not algebraically closed, the degree of the field extension needed to construct $\mathcal{E}_Z$, cf. [Tan14]. However what we need is the following result.

Proposition 1. *Let $\mathcal{E}$, $\mathcal{F}$ be non-isomorphic stable Ulrich bundles coming from 5.1. Then:*

$$\mathrm{Hom}_Y(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{F}, \mathcal{E}) = 0,$$
$$\dim_{\boldsymbol{k}} \mathrm{Ext}^1_Y(\mathcal{E}, \mathcal{F}) = 4.$$

Proof. The first two statements concerning morphisms are clear since we can suppose that $\mathcal{E}$ and $\mathcal{F}$ are stable with the same slope. For the last statement, since $\mathcal{E}$ is locally free, we have $\mathrm{Ext}^1_Y(\mathcal{E}, \mathcal{F}) \simeq \mathrm{H}^1(Y, \mathcal{E}^\vee \otimes \mathcal{F})$. Tensoring the short exact sequence 5.1 by $\mathcal{E}^\vee$ and considering the associated long exact sequence of global sections, since $\mathrm{H}^1(Y, \mathcal{E}^\vee) = \mathrm{H}^2(Y, \mathcal{E}^\vee) = 0$, we get an isomorphism

$$\mathrm{Ext}^1_Y(\mathcal{E}, \mathcal{F}) \simeq \mathrm{H}^1(Y, \mathcal{E}^\vee \otimes \mathcal{I}_{Z/Y}(2)). \tag{5.2}$$

On the other hand, the exact sequence defining $Z \subset Y$ twisted by $\mathcal{O}_Y(2)$ reads:

$$0 \to \mathcal{I}_{Z/Y}(2) \to \mathcal{O}_Y(2) \to \mathcal{O}_Z(2) \to 0. \tag{5.3}$$

Taking into account that $\mathcal{E}^\vee \simeq \mathcal{E}(-2)$ we obtain $\mathrm{H}^0(Y, \mathcal{E}^\vee \otimes \mathcal{I}_{Z/Y}(2)) = 0$. Then, tensoring (5.3) by $\mathcal{E}^\vee$ taking global sections and combining with (5.2) we get:

$$0 \to \mathrm{H}^0(Y, \mathcal{E}^\vee(2)) \to \mathrm{H}^0(Y, \mathcal{E}^\vee \otimes \mathcal{O}_Z(2)) \to \mathrm{Ext}^1_Y(\mathcal{E}, \mathcal{F}) \to 0.$$

Now we know that $\mathcal{E}^\vee(2) \simeq \mathcal{E}$ has $2d$ independent global sections, as $\mathcal{E}$ is Ulrich. On the other hand $\mathcal{E}^\vee \otimes \mathcal{O}_Z(2)$ is just a vector space of rank 2 concentrated at $d+2$ points, hence $\dim_{\boldsymbol{k}} \mathrm{H}^0(Y, \mathcal{E}^\vee \otimes \mathcal{O}_Z(2)) = 2d + 4$. We conclude that $\dim_{\boldsymbol{k}} \mathrm{Ext}^1_Y(\mathcal{E}, \mathcal{F}) = 4$. □

We may argue that this proposition says that any surface Y of almost minimal degree supports infinitely many Ulrich bundles of rank 2 with determinant $\mathcal{O}_Y(2)$, in particular any cubic surface admits (over an algebraically closed field) infinitely many pfaffian representations. The normal case is now complete by virtue of Theorem 3.

5.4. **Del Pezzo surfaces normalized by a rational normal scroll.** This time Y is a non-normal aCM surface of degree d in $\mathbb{P}^d$. We let $\bar{Y}$ be the normalization of Y and $\sigma : \bar{Y} \to Y$ the normalization map. It turns out that these surfaces are completely classified (cf. [Rei94]), and that $\bar{Y}$ is a rational normal scroll or a Veronese surface. Also, the locus where σ is not an isomorphism is a line $L \subset Y$ and $\bar{L} = \sigma^* L$ is a conic in $\bar{Y}$. In this subsection we are going to treat the case when $\bar{Y}$ is a rational normal scroll leaving the case of the Veronese surface for the next subsection.

So let us consider the two non-isomorphic rank-1 Ulrich bundles over $\bar{Y}$, normalized so that they have d independent global section, where $d = d_Y$ is the degree of $\bar{Y}$ as a rational normal scroll, which is the same as the degree of Y. These line bundles are $\mathcal{O}_{\bar{Y}}((d-1)F)$, where F is a *fibre* of the scroll map $\bar{Y} \to \mathbb{P}^1$, i.e. F is just a line of the scroll, and $\mathcal{O}_{\bar{Y}}(H - F)$, where H is the hyperplane section of $\bar{Y}$. Let us write $A = (d-1)F$ and $B = H - F$. We have:

$$\dim_{\boldsymbol{k}} \mathrm{Ext}^1_{\bar{Y}}(\mathcal{O}_{\bar{Y}}(A), \mathcal{O}_{\bar{Y}}(B)) = d - 2.$$

Set $\mathcal{A} = \sigma_* \mathcal{O}_{\bar{Y}}(A)$ and $\mathcal{B} = \sigma_* \mathcal{O}_{\bar{Y}}(B)$. These are Ulrich sheaves of rank 1 on Y. We also have $\mathrm{Ext}^2_{\bar{Y}}(\mathcal{O}_{\bar{Y}}(A), \mathcal{O}_{\bar{Y}}(B)) = 0$, so the local-to-global spectral sequence degenerates to:

$$\mathrm{Ext}^1_Y(\mathcal{A}, \mathcal{B}) \simeq \mathrm{Ext}^1_{\bar{Y}}(\mathcal{O}_{\bar{Y}}(A), \mathcal{O}_{\bar{Y}}(B)) \oplus \mathrm{H}^0(Y, \mathcal{E}xt^1_Y(\mathcal{A}, \mathcal{B})),$$

and similarly:

$$\mathrm{Ext}^1_Y(\mathcal{B}, \mathcal{A}) \simeq \mathrm{H}^0(Y, \mathcal{E}xt^1_Y(\mathcal{B}, \mathcal{A})).$$

Therefore, in order to conclude by Theorem 3, we only need to prove that in the cases $d = 3, 4$ either $\mathrm{H}^0(Y, \mathcal{E}xt^1_Y(\mathcal{B}, \mathcal{A}))$ has dimension at least 3 or that $\mathrm{H}^0(Y, \mathcal{E}xt^1_Y(\mathcal{A}, \mathcal{B}))$ has dimension at least $5 - d$. Luckily, these two cases are completely classified up to projective equivalence, cf. [LPS11, LPS12].

5.4.1. *The non-normal locus.* We said that Y is not normal along a line L. We have:

$$0 \to \mathcal{O}_Y \to \sigma_* \mathcal{O}_{\bar{Y}} \to \mathcal{O}_L(-1) \to 0. \tag{5.4}$$

In order to see this notice that the restriction of σ to $\bar{L} \to L$ is $2:1$. Therefore $\sigma_* \mathcal{O}_{\bar{Y}}$ is locally free of rank 2 over L, and $(\sigma_* \mathcal{O}_{\bar{Y}})|_L \simeq \sigma_* \mathcal{O}_{\bar{L}}$, so we get an exact sequence:

$$0 \to \mathcal{O}_L \to \sigma_* \mathcal{O}_{\bar{L}} \to Q \to 0,$$

where Q is a line bundle on L. From the previous exact sequence we have $\chi(Q) = 0$, namely $Q \simeq \mathcal{O}_L(-1)$.

This gives with no difficulty the isomorphism:

$$\mathcal{E}xt^1_Y(\sigma_* \mathcal{O}_{\bar{Y}}, \sigma_* \mathcal{O}_{\bar{Y}}) \simeq \mathcal{E}xt^1_Y(\mathcal{O}_L, \mathcal{O}_L). \tag{5.5}$$

Next we may compute a locally free resolution of $\mathcal{O}_L$ (in fact the sheafified minimal graded free resolution of the module of global sections of $\mathcal{O}_L$) over Y, relying on the classification appearing in [LPS11, LPS12]. This reads, when $d = 4$:

$$0 \leftarrow \mathcal{O}_L \leftarrow \mathcal{O}_Y \leftarrow \mathcal{O}_Y(-1)^3 \leftarrow \mathcal{O}_Y(-2)^5 \leftarrow \mathcal{O}_Y(-3)^7 \leftarrow \cdots,$$

and for $d = 3$:

$$0 \leftarrow \mathcal{O}_L \leftarrow \mathcal{O}_Y \xleftarrow{d_1} \mathcal{O}_Y(-1)^2 \xleftarrow{d_2} \mathcal{O}_Y(-2) \oplus \mathcal{O}_Y(-3) \leftarrow \mathcal{O}_Y(-4)^2 \leftarrow \cdots \tag{5.6}$$

This gives:

$$\mathcal{E}xt^1_Y(\mathcal{O}_L,\mathcal{O}_L) \simeq \mathcal{O}_L \oplus \mathcal{O}_L(1), \qquad \text{for } d = 4, \tag{5.7}$$

$$\mathcal{E}xt^1_Y(\mathcal{O}_L,\mathcal{O}_L) \simeq \mathcal{O}_L(1)^2, \qquad \text{for } d = 3. \tag{5.8}$$

We may also note that $\mathcal{E}xt^i_Y(\mathcal{O}_L,\mathcal{O}_L)$ is locally free of rank 2 on L for all $i > 0$.

5.4.2. *Curves associated with Ulrich bundles.* A general global section of $\mathcal{O}_{\bar{Y}}(B)$ vanishes along a rational normal curve B of degree $d-1$ in $\bar{Y}$, while any non-zero global section of $\mathcal{O}_{\bar{Y}}(A)$ vanishes along the union A of $d-1$ fibres, which are pairwise disjoint lines contained in $\bar{Y}$. All this is clear from the description of divisors on $\bar{Y}$, independently on the characteristic of $\boldsymbol{k}$. It turns out that the zero locus of a general global section of $\mathcal{A}$ is a curve $\hat{A}$ which is the union of σA and L, and the same happens to $\mathcal{B}$ with obvious notation. We write $A' = \sigma A$ and $B' = \sigma B$, so $\hat{A} = L \cup A'$ and $\hat{B} = L \cup B'$. Actually B' cuts L at a single point p_0, while A' cuts L at $d-1$ points $p_1, \ldots, p_{d-1}$, one for each component of A'. Both $\hat{A}$ and $\hat{B}$ are aCM curves in Y.

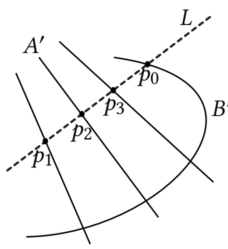

Figure 1. Curves associated with Ulrich bundles for a quartic surface normalized by a rational scroll.

When B is smooth (which happens if the defining section of $\mathcal{O}_{\bar{Y}}(B)$ is general enough), setting p (resp. p') for a point of B (resp. of B'), we get:

$$\mathcal{O}_{\bar{Y}}(B)|_B \simeq \mathcal{O}_B((d-2)p), \qquad \mathcal{B}|_{B'} \simeq \mathcal{O}_{B'}((d-2)p')$$

We also have:

$$\mathcal{O}_{\bar{Y}}(A)|_A \simeq \mathcal{O}_A, \qquad \mathcal{A}|_{A'} \simeq \mathcal{O}_{A'}.$$

There is an exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_Y & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{A}|_{\hat{A}} & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \sigma_*\mathcal{O}_{\bar{Y}} & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{A}|_{A'} & \longrightarrow & 0 \end{array} \tag{5.9}$$

This induces:

$$0 \to \mathcal{O}_L(-1) \to \mathcal{A}|_{\hat{A}} \to \mathcal{O}_{A'} \to 0. \tag{5.10}$$

Similarly we get:

$$0 \to \mathcal{O}_L(-1) \to \mathcal{B}|_{\hat{B}} \to \mathcal{O}_{B'}((d-2)p') \to 0. \tag{5.11}$$

We also have that the curves A' and B' intersect transversely at $d-1$ points, none of which lies in L, when A and B are general enough.

5.4.3. *Computation of Ext sheaves.* Our goal now is to give a lower bound for the number of independent global sections of the coherent sheaf $\mathcal{E}xt^1_Y(\mathcal{A},\mathcal{B})$. We know already that this sheaf is supported on L.

First, we replace each instance of A in the diagram (5.9) by B and we apply $\mathcal{H}om_Y(\mathcal{A},-)$ to the first row of the resulting exact commutative diagram. Since $\mathcal{A}$ is locally CM we get an isomorphism:

$$\mathcal{E}xt^1_Y(\mathcal{A},\mathcal{B}) \simeq \mathcal{E}xt^1_Y(\mathcal{A},\mathcal{B}|_{\hat{B}}), \tag{5.12}$$

Next, applying $\mathcal{H}om_Y(-,\mathcal{B}|_{\hat{B}})$ to the second row of (5.9) we get a long exact sequence:

$$\cdots \to \mathcal{E}xt^i_Y(\mathcal{O}_{A'},\mathcal{B}|_{\hat{B}}) \xrightarrow{\varsigma_i} \mathcal{E}xt^i_Y(\mathcal{A},\mathcal{B}|_{\hat{B}}) \to \mathcal{E}xt^i_Y(\sigma_*\mathcal{O}_{\tilde{Y}},\mathcal{B}|_{\hat{B}}) \to \cdots$$

Note that, for all i, the sheaf $\mathcal{E}xt^i_Y(\mathcal{O}_{A'},\mathcal{B}|_{\hat{B}})$ is supported at $A' \cap \hat{B}$ and so has finite length. For $i = 2$ we need something more precise, namely the next lemma.

Lemma 5. *The sheaf $\mathcal{E}xt^2_Y(\mathcal{O}_{A'},\mathcal{B}|_{\hat{B}})$ has length $d-1$.*

Proof. We first note that (5.11) easily implies $\mathcal{E}xt^2_Y(\mathcal{O}_{A'},\mathcal{B}|_{\hat{B}}) \simeq \mathcal{E}xt^2_Y(\mathcal{O}_{A'},\mathcal{O}_L(-1))$ since A' and B meet transversely along B', i.e. away from the non-normal locus L. In turn, this sheaf is obviously isomorphic to $\mathcal{E}xt^2_Y(\mathcal{O}_{A'},\mathcal{O}_L)$. So we would like to check that $\mathcal{E}xt^2_Y(\mathcal{O}_{A'},\mathcal{O}_L)$ has length $d-1$ by showing that it has rank 1 over each of the points $p_1,\ldots,p_{d-1}$ of $L \cap A'$.

Looking at each point p_i we have to show that the line $A_i \subset Y$ defined as the irreducible component of A' passing through p_i satisfies $\mathcal{E}xt^2_Y(\mathcal{O}_{A_i},\mathcal{O}_L) \simeq \mathcal{O}_{p_i}$. The situation is local, and the cases $d = 3$ and $d = 4$ do not differ in this sense, so we just look at $d = 3$. We consider the sheafified minimal graded free resolution of the module of global sections of $\mathcal{O}_{A_i}$, which in turn looks precisely like (5.6) with L replaced with A_i. Of course, $\Omega^1(\mathcal{O}_{A_i}) \simeq \mathcal{I}_{A_i/Y}$, and we see that $\Omega^2(\mathcal{O}_{A_i}) \simeq \mathcal{I}_{C_i/Y}(-1)$, where C_i is the residual conic of A_i with respect to a plane section of Y containing A_i. Write ι_j for the restriction of d_j to $\Omega^j(\mathcal{O}_{A_i})$.

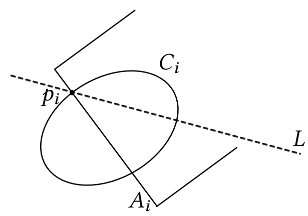

FIGURE 2. Lines L and A_i with residual conic C_i.

Next we observe that, since A_i meets L transversely at p_i, the sheaf $\mathcal{H}om_Y(\mathcal{I}_{A_i/Y},\mathcal{O}_L)$ is just the dual over L of $\mathcal{I}_{p_i/L}$, that is $\mathcal{O}_L(1)$. The curve C_i also meets L transversely at p_i, so:

$$\mathcal{H}om_Y(\mathcal{I}_{A_i/Y},\mathcal{O}_L) \simeq \mathcal{O}_L(1) \simeq \mathcal{H}om_Y(\mathcal{I}_{C_i/Y},\mathcal{O}_L), \tag{5.13}$$

Now we apply then $\mathcal{H}om_Y(-,\mathcal{O}_L)$ to our resolution of $\mathcal{O}_{A_i}$. We first note that:

$$\mathcal{E}xt^2_Y(\mathcal{O}_{A_i},\mathcal{O}_L) \simeq \mathcal{E}xt^1_Y(\mathcal{I}_{A_i/Y},\mathcal{O}_L).$$

Then, using (5.13) we get an exact sequence:

$$0 \to \mathcal{O}_L(1) \to \mathcal{O}_L(1)^2 \xrightarrow{\mathcal{H}om_Y(\iota_2,\mathcal{O}_L)} \mathcal{O}_L(2) \to \mathcal{E}xt^2_Y(\mathcal{O}_{A_i},\mathcal{O}_L) \to 0$$

Now it clearly turns out that $\mathcal{E}xt^2_Y(\mathcal{O}_{A_i},\mathcal{O}_L) \simeq \mathcal{O}_{p_i}$, which finishes the proof. □

To continue our computation, we observe that the sheaf $\mathcal{E}xt^1_Y(\mathcal{A},\mathcal{B})$ is locally free of rank 2 over L. This follows from (5.5), (5.7) and (5.8) since $\mathcal{O}_{\tilde{Y}}(B)$ and $\mathcal{O}_{\tilde{Y}}(A)$ locally behave like $\mathcal{O}_{\tilde{Y}}$, so $\mathcal{B}$ and $\mathcal{A}$ locally behave like $\sigma_*\mathcal{O}_{\tilde{Y}}$. Then $\varsigma_i = 0$ for $i = 1,2$, because we have just seen that the middle sheaf is locally free over L in view of (5.12) and the leftmost sheaf has finite length.

Set $\epsilon = \chi(\mathcal{E}xt^1_Y(\sigma_*O_{\bar{Y}},\mathcal{B}|_{\hat{B}}))$. We have shown via Lemma 5:

$$\dim_{\boldsymbol{k}} \operatorname{Ext}^1_Y(\mathcal{A},\mathcal{B}) \geq \chi(\mathcal{E}xt^1_Y(\mathcal{A},\mathcal{B})) \geq \epsilon - d + 1.$$

It remains to compute ϵ. Observe that in order to finish we have to show $\epsilon \geq 4$. To get this, we apply $\mathcal{H}om_Y(\sigma_*O_{\bar{Y}},-)$ to (5.10) after replacing A with B. Using (5.4), it is easy to observe that:

$$\mathcal{E}xt^1_Y(\sigma_*O_{\bar{Y}},O_L(-1)) \simeq \mathcal{E}xt^1_Y(O_L,O_L).$$

Note that $\mathcal{E}xt^i_Y(\sigma_*O_{\bar{Y}},\mathcal{B}|_{B'})$ is supported at B' for all i and $\mathcal{E}xt^i_Y(\sigma_*O_{\bar{Y}},O_L)$ is locally free over L, so we see get an exact sequence:

$$0 \to \mathcal{E}xt^1_Y(O_L,O_L) \to \mathcal{E}xt^1_Y(\sigma_*O_{\bar{Y}},\mathcal{B}|_{\hat{B}}) \to \mathcal{E}xt^1_Y(\sigma_*O_{\bar{Y}},O_{B'}) \to 0.$$

Now, in view of (5.8) and (5.7), it suffices to compute $\chi(\mathcal{E}xt^1_Y(\sigma_*O_{\bar{Y}},O_{B'})) \geq 1$. But using (5.4) we see that:

$$\mathcal{E}xt^1_Y(\sigma_*O_{\bar{Y}},O_{B'}) \simeq \mathcal{E}xt^1_Y(O_L(-1),O_{B'}),$$

and this is just a skyscraper sheaf over the point $p_0 = L \cap B'$. The proof is now finished.

Remark 4. According to [LPS11, LPS12] there are 2 inequivalent integral non-normal cubic surfaces which are not cones in characteristic different from 2, and one more in characteristic 2; the normalization of all of them is a smooth cubic scroll in $\mathbb{P}^4$. Also, there are three inequivalent integral non-normal surfaces of degree 4 in $\mathbb{P}^4$ whose normalization is a quartic scroll and which are not cones, one of them appearing in characteristic 2. All these cases can be treated with the method we outlined. The computations sketched here can also be carried out explicitly with Macaulay2, [GS], cf. the webpage `http://dfaenzi.perso.math.cnrs.fr/code/cm-wild`

5.5. **Quartic del Pezzo surfaces normalized by the Veronese surface.** We still have to study the case when $\bar{Y}$ is the Veronese surface. Again there is a line $L \subset Y$ which is the non-normal locus of Y. In this case, the two quadrics threefolds Q_1, Q_2 whose equations generate I_Y can be chosen so that Q_1 is singular along a line $M \subset \mathbb{P}^4$. The intersection $M \cap Y$ consists a double point $\vec{q}$ whose reduced structure q lies in L. Also, Q_2 is a cone over a point $p \in L$. Then, Q_1 admits a unique linear symmetric determinantal representation by a 2×2 matrix whose cokernel $\mathcal{L}_1$ is a rank-1 Ulrich sheaf on Q_1 with 2 independent global sections, corresponding to a pencil of planes Λ_1 in Q_1. On the other hand Q_2 supports two non-isomorphic rank-1 Ulrich sheaves $\mathcal{L}_2$ and $\mathcal{L}_2'$, one for each of the two distinct pencils of planes Λ_2 and Λ_2' in Q_2. We define:

$$\mathcal{A} = \mathcal{L}_1 \otimes \mathcal{L}_2, \qquad \mathcal{B} = \mathcal{L}_1 \otimes \mathcal{L}_2'.$$

One can check now that $\mathcal{A}$ and $\mathcal{B}$ are rank-1 Ulrich sheaves on Y, having each 4 independent global sections, as an easy application of [ESW03, Proposition 2.1].

Moreover, $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic, as one can see by comparing the curves associated with them. Indeed, the curves associated with $\mathcal{A}$ corresponding to decomposable tensors in $\mathrm{H}^0(Y,\mathcal{A}) \simeq \mathrm{H}^0(Q_1,\mathcal{L}_1) \otimes \mathrm{H}^0(Q_2,\mathcal{L}_2)$ are reducible of the form $\Lambda_1 \cap Q_2 \cup \Lambda_2 \cap Q_1$, and these curves cannot appear as zero locus of global sections in $\mathrm{H}^0(Y,\mathcal{B})$, as Λ_2' would lie in the same pencil as Λ_2. It follows that:

$$\operatorname{Hom}_Y(\mathcal{A},\mathcal{B}) = 0 = \operatorname{Hom}_Y(\mathcal{B},\mathcal{A}),$$

as any morphism $\mathcal{A} \to \mathcal{B}$ would be an isomorphism, since the source and target are torsionfree of rank 1 with the same Hilbert polynomial, and the variety Y is integral.

It remains to compute $\operatorname{Ext}^1_Y(\mathcal{A},\mathcal{B})$. We would like to show that this vector space has dimension 3, which suffices to conclude by Theorem 3. Note that $\mathcal{L}_1$ is locally free away from M, while $\mathcal{L}_2$ and $\mathcal{L}_2'$ are locally free except at p, so that $\mathcal{A}$ is locally free away from p and q. The same happens to $\mathcal{B}$, in particular $\mathcal{E}xt^1_Y(\mathcal{A},\mathcal{B})$ is supported (set-theoretically) at p and q so it has finite length.

By our analysis of reducible curves appearing as zero loci of sections of decomposable tensors in $\mathcal{A}$, we see that the only singularity of such a curve is, when Λ_1 and Λ_2 are general enough, the point $\Lambda_1 \cap \Lambda_2$. Therefore a general global section of $\mathcal{A}$ will vanish on a smooth irreducible curve, actually a rational normal quartic A in Y, and one has:

$$0 \to \mathcal{O}_Y \to \mathcal{A} \to \omega_A \to 0, \qquad 0 \to \mathcal{O}_Y \to \mathcal{B} \to \omega_B \to 0, \tag{5.14}$$

where ω_A and ω_B are invertible on A and B.

Any such curve A or B contains $\vec{q}$ and p. Also, two curves A and B corresponding to sufficiently general global sections of $\mathcal{A}$ and $\mathcal{B}$ intersect at the length-3 subscheme consisting of the fixed points $\vec{q}$ and p, and moreover at 3 more simple points p_1,p_2,p_3 which move along with the choice of A and B. This says that the sheaf $\mathcal{E}xt^1_Y(\omega_A,\omega_B)$ has length 6 and is supported scheme-theoreticaly at $\vec{q}$, p, p_1, p_2, p_3.

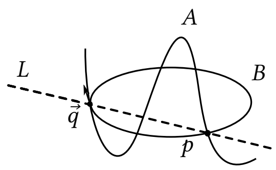

Figure 3. Curves associated with Ulrich bundles for a quartic surface normalized by the Veronese surface.

Actually this can already be seen on the reducible curves by direct computation. So:

$$\dim_{\boldsymbol{k}} \operatorname{Ext}^1_Y(\omega_A,\omega_B) = 6. \tag{5.15}$$

Now applying $\operatorname{Hom}_Y(-,\mathcal{B})$ to the first of (5.14) we get:

$$0 \to \mathrm{H}^0(Y,\mathcal{B}) \to \operatorname{Ext}^1_Y(\omega_A,B) \to \operatorname{Ext}^1_Y(\mathcal{A},\mathcal{B}) \to 0.$$

Recall that $\mathcal{B}$ is Ulrich and has thus 4 independent global sections. On the other hand, one immediately computes $\operatorname{Ext}^1_Y(\omega_Y,\mathcal{O}_Y) \simeq \boldsymbol{k}$ and $\operatorname{Ext}^2_Y(\omega_Y,\mathcal{O}_Y) = 0$, while of course $\operatorname{Hom}_Y(\omega_A,\omega_B) = 0$. So applying $\operatorname{Hom}_Y(\omega_A,-)$ to the second sequence of (5.14) one gets:

$$0 \to \boldsymbol{k} \to \operatorname{Ext}^1_Y(\omega_A,\mathcal{B}) \to \operatorname{Ext}^1_Y(\omega_A,\omega_B) \to 0.$$

Using these sequences and (5.15) we finally obtain $\dim_{\boldsymbol{k}} \operatorname{Ext}^1_Y(\mathcal{A},\mathcal{B}) = 3$.

In characteristic 2, there is another surface Y of degree 4, not projectively equivalent to Y, whose normalization is the Veronese surface, cf. case 8 of [LPS12]. However this case can be treated in the same way as the present one, so we omit the details.

Corollary 3. *An integral aCM surface of almost minimal degree which is not a cone is of strictly Ulrich wild type.*

5.6. **Further remarks.** The hypothesis that X is integral and not a cone can be removed for sectional genus at least 3 thanks to the following remark. In order to state it, we will use the notion of h-vector, cf. [Mig98] for a thorough exposition.

Given an aCM closed subscheme $X \subset \mathbb{P}^n$ of dimension $m \geq 1$, the Hilbert function of a linear Artinian reduction of X is eventually zero and therefore it is encoded on a finite sequence of positive integers $(1,h_1,\ldots,h_s)$. This sequence is known as the *h-vector* of X. Its length is $s+1$. The sectional genus p of X can be recovered by the following formula:

$$p = \sum_{i=2}^{s}(i-1)h_i.$$

Lemma 6. *Let $X \subset \mathbb{P}^n$ be an aCM projective scheme of positive dimension and of sectional genus at least 3. Then X is of wild representation type.*

Proof. Set $R := \boldsymbol{k}[X]$, so that R is a graded CM ring. We have $p \geq 3$ so either $s \geq 3$ or $h_2 \geq 3$; assume $s \geq 3$. Let $0 \neq f \in R$ be a homogeneous element of degree a and $Y = X \cap \mathbb{V}(f)$. Then the Hilbert series $H_{\boldsymbol{k}[X]}$ and $H_{\boldsymbol{k}[Y]}$ of $\boldsymbol{k}[X]$ and $\boldsymbol{k}[Y]$ are related as follows:

$$H_{\boldsymbol{k}[Y]}(t) = (1-t^a)H_{\boldsymbol{k}[X]}(t).$$

Suppose now $m = 1$. Let $f_1, f_2 \in R$ (respectively, $g_1, g_2 \in R$) be a R-regular sequences with $\deg f_i = 1$ (respectively $\deg g_i = 2$). Consider then $V = \boldsymbol{k}[X]/(f_1, f_2)$ and $W = \boldsymbol{k}[X]/(g_1, g_2)$. The Hilbert series of V and W correspond, respectively, to the standard h-vector of X and to the weighed one with weight $(2,2)$. Now, the Hilbert functions H_V and H_W of V and of W are related by:

$$H_W(t) = (1+t)^2 H_V(t).$$

In particular, the weighted h-vector $(1, k_1, \ldots, k_t)$ of weight $(2,2)$ of X and the h-vector $(1, h_1, \ldots, h_s)$ of X are related by:

$$k_i = h_i + 2h_{i-1} + h_{i-2}.$$

Therefore our claim follows from [DT14, Theorem 2.1], as well as the cases $m > 1$ (since the h-vector does not change by linear section) and $h_2 \geq 3$. □

In fact, we have a more detailed information: if $g_i := f_i^2$, then the resolution of $V = \boldsymbol{k}[X]/(f_1, f_2)$ over $W = \boldsymbol{k}[Y]/(g_1, g_2)$ reads:

$$0 \leftarrow V \leftarrow W \leftarrow W(-1)^2 \leftarrow W(-2)^3 \leftarrow \cdots$$

Then one easily gets the already mentioned relation between the Hilbert functions of V and W. We thank Aldo Conca for pointing out to us this fact.

Acknowledgments. The second named author would like to thank to the University of Bourgogne for their hospitality during the stay when part of this work was done.

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