Triple planes with $p_g = q = 0$

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Abstract

We show that general triple planes with $p_g = q = 0$ belong to at most 12 families, that we call surfaces of type I, ..., XII, and we prove that the corresponding Tschirnhausen bundle is direct sum of two line bundles in cases I, II, III, whereas is a rank 2 Steiner bundle in the remaining cases.

We also provide existence results and explicit descriptions for surfaces of type I, ..., VII, recovering all classical examples and discovering several new ones. In particular, triple planes of type VII provide counterexamples to a wrong claim made in 1942 by Bronowski.

Finally, in the last part of the paper we discuss some moduli problems related to our constructions.

Introduction

A *triple plane* is a finite triple cover $f : X \longrightarrow \mathbb{P}^2$. Let $B \subset \mathbb{P}^2$ be the branch locus of f; then we say that f is a *general* triple plane if the following conditions are satisfied:

(*i*) f is unramified over $\mathbb{P}^2 \setminus B$;

(*ii*) $f^*B = 2R + R_0$, where R is irreducible and non-singular and R_0 is reduced;

(*iii*) $f_{|R} : R \longrightarrow B$ coincides with the normalization map of B.

The aim of this paper is to classify those smooth, projective surfaces X with $p_g(X) = q(X) = 0$ that arise as general triple planes. Some result in this direction were obtained by Du Val in [**DuVal33**] and [**DuVal35**], where he described the triple planes with branch curve of degree at most 14. Du Val's paper are written in the old (and sometimes difficult to read) "classical" language and make use of ad-hoc constructions based on synthetic projective geometry. The methods that we propose here are completely different, in fact they are a mixture of adjunction theory and vector bundles techniques. This allows us to treat this problem in a unified way.

The first cornerstone in our work is the general structure theorem for triple covers given in [**Mi85**] and [**CasEk96**]. More precisely, we relate the existence of a triple cover $f : X \to \mathbb{P}^2$ to the existence of a "sufficiently general" element $\eta \in H^0(Y, S^3\mathcal{E}^* \otimes \wedge^2\mathcal{E})$, where $\mathcal{E}$ is a rank 2 vector bundle on $\mathbb{P}^2$ such that $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}$. It is called the *Tschirnhausen bundle* of the cover, and the pair $(\mathcal{E}, \eta)$ completely encodes the geometry of f. For instance, setting $b := -c_1(\mathcal{E})$ and $h := c_2(\mathcal{E})$, the branch curve B has degree $2b$ and contains $3h$ ordinary cusps as only singularities, see Proposition 2.4.

So we can try to study triple planes with $p_g = q = 0$ by analyzing their Tschirnhausen bundles. In fact, we show that these triple planes can be classified in (at most) 12 families, that we call surfaces of type I, II,..., XII. We are also able to show that surfaces of type I, II, ..., VII actually exist. In the cases I, II, ..., VI we rediscover (in the modern language) the examples described by Du Val. On the other hand, not only the triple planes of type VII (which have sectional genus equal to 6 and branch locus of degree 16) are completely new, but they also provide explicit counterexamples to a wrong claim made by Bronowski in [**Br42**], see Remark 2.8.

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A key point in our analysis is the fact that in cases I, II, III the bundle $\mathcal{E}$ splits as the sum of two line bundles, whereas in the remaining cases IV, ...,XII it is indecomposable and it has minimal free resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1-b)^{b-4} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(2-b)^{b-2} \longrightarrow \mathcal{E} \longrightarrow 0.$$

This shows that $\mathcal{E}$ is a so-called *Steiner bundle* (see Subsection 1.4 for more details on this topic), so we can use all the known results about Steiner bundles in order to get information on X. For instance, in cases VI and VII the geometry of the triple plane is strictly related to the existence of *unstable lines* for $\mathcal{E}$, see Subsections 3.6, 3.7.

The second main ingredient in our classification procedure is adjunction theory, see [**SoV87**], [**Fu90**]. For example, if we write $H = f^*L$, where $L \subset \mathbb{P}^2$ is a general line, we prove that the divisor $D = K_X + 2H$ is very ample (Proposition 2.9), so we consider the corresponding adjunction mapping

$$\varphi_{|K_X+D|} \colon X \longrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(K_X + D))).$$

Iterating the adjunction process if necessary, we can achieve further information about the geometry of X. Furthermore, when $b \geq 7$ a more refined analysis of the adjunction map allows us to start the process with $D = H$, see Remark 2.18.

As a by-product of our classification, it turns out that triple planes $f : X \to \mathbb{P}^2$ with sectional genus $0 \leq g(H) \leq 5$ (i.e., those of type I, ..., VI) can be realized via an embedding of X into $\mathrm{Gr}(1, \mathbb{P}^3)$ as a surface of bidegree $(3, n)$, such that the triple covering f is induced by the projection from a general element of the family of planes of $\mathrm{Gr}(1, \mathbb{P}^3)$ that are n-secant to X. In this way, we relate our work to the work of Gross [**G93**], see Remarks 3.2, 3.4, 3.6, 3.10, 3.12, 3.15. On the other hand, this is not true for surfaces of type VII: here the only case where the triple cover is induced by an embedding in the Grassmannian is (VII.7), where X is a *Reye congruence*, namely an Enriques surface having bidegree $(3, 7)$ in $\mathrm{Gr}(1, \mathbb{P}^3)$, see Remark 3.19.

We have not been able so far to use our method beyond case VII; thus the existence of surfaces of type VIII, ..., XII is still an open problem. Furthermore, there are some interesting unsettled questions also in case VII, especially regarding the number of unstable conic for the Tschirnhausen bundle, see Subsection 3.7.2 for more details. We plan to investigate them in the future.

Let us explain now how this work is organized. In Section 1 we set up notation and terminology and we collect the background material which is needed in the sequel of the paper. In particular, we recall the theory of triple covers based on the study of the Tschirnhausen bundle (Theorems 1.2 and 1.3) and we state the main results on adjunction theory for surfaces (Theorem 1.4).

In Section 2 we start the analysis of general triple planes $f : X \longrightarrow \mathbb{P}^2$ with $p_g(X) = q(X) = 0$. We compute the numerical invariants (degree of the branch locus, number of its cusps, K_X^2, sectional genus) for the surfaces in the 12 families I, ..., XII (Proposition 2.11) and we describe their Tschirnhausen bundle (Theorem 2.12).

Section 3 is devoted to the detailed description of surfaces of type I, ..., VII. This description leads to a complete classification in cases I, ..., VI (Propositions 3.1, 3.3, 3.5, 3.7, 3.11, 3.14) whereas in case VII we provide many examples, leaving only few cases unsolved (Proposition 3.17).

Finally, in Section 4 we study some moduli problems related to our constructions.

Part of the computations in this paper was carried out by using the Computer Algebra System `Macaulay2`, see [**Mac2**]. All the scripts can be downloaded from the web-page of the second author:

`https://sites.google.com/site/francescopolizzi/publications`.

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1 Preliminaries

1.1 Notation and conventions

We work over the field $\mathbb{C}$ of complex numbers. By "*surface*" we mean a projective, non-singular surface S, and for such a surface $\omega_S = \mathcal{O}_S(K_S)$ denotes the canonical class, $p_g(S) = h^0(S, K_S)$ is the geometric genus, $q(S) = h^1(S, K_S)$ is the irregularity and $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ is the holomorphic Euler-Poincaré characteristic. We write $P_m(S) = h^0(S, mK_S)$ for the m-th plurigenus of S.

Following [**Har92**, Lecture 8], if $k \leq n$ are positive integers we denote by $S_{k,n}$ the rational normal scroll of type (k, n) in $\mathbb{P}^{k+n+1}$. In particular, a cone over a rational normal curve $C \subset \mathbb{P}^n$ may be taught of as the scroll $S_{0,n} \subset \mathbb{P}^{n+1}$.

For $n \geq 1$, we write $\mathbb{F}_n$ for the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$; every divisor in $\text{Pic}(\mathbb{F}_n)$ can be written as $a\mathfrak{c}_0 + b\mathfrak{f}$, where $\mathfrak{f}$ is the fibre of the $\mathbb{P}^1$-bundle map $\mathbb{F}_n \to \mathbb{P}^1$ and $\mathfrak{c}_0$ is the unique section with negative self-intersection, namely $\mathfrak{c}_0^2 = -n$. Note that the morphism $\mathbb{F}_n \to \mathbb{P}^{n+1}$ associated with the tautological linear system $|\mathfrak{c}_0 + n\mathfrak{f}|$ contracts $\mathfrak{c}_0$ to a point and is an isomorphism outside $\mathfrak{c}_0$, so its image is the cone $S_{0,n}$.

For $n = 0$, the surface $\mathbb{F}_0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$; every divisor in $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ is written as $a_1L_1 + a_2L_2$, where the L_i are the two rulings.

The blow-up of $\mathbb{P}^2$ at the points $p_1, \ldots, p_k$ is denoted by $\mathbb{P}^2(p_1, \ldots, p_k)$. If $\sigma \colon \widetilde{X} \to X$ is the blow-up of a surface X at k points, with exceptional divisors $E_1, \ldots, E_k$, and L is a line bundle on X, we will write $L + \sum a_iE_i$ instead of $\sigma^*L + \sum a_iE_i$.

We will usually write the Chern classes of vector bundles on $\mathbb{P}^2$ as integers. We use everywhere the symbol "$*$" for the dual and "S" for the symmetric powers. So, for instance, the dual of a vector bundle $\mathcal{E}$ is indicated by $\mathcal{E}^*$ and its second symmetric power by $S^2\mathcal{E}$.

1.2 Triple covers and sections of vector bundles

A *triple cover* is a finite flat morphism $f \colon X \to Y$ of projective varieties, whose fibres have degree 3; we assume that X and Y are smooth and irreducible. With such cover is associated an exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow 0, \tag{1}$$

where $\mathcal{E}$ is a vector bundle of rank 2 on Y, called the *Tschirnhausen bundle* of f.

Proposition 1.1. *The following holds:*

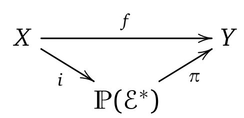
- **(*i*)** $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}$;
- **(*ii*)** $f_*\omega_X = \omega_Y \oplus (\mathcal{E}^* \otimes \omega_Y)$;
- **(*iii*)** $f_*\omega_X^2 = S^2\mathcal{E}^* \otimes \omega_Y^2$.

Proof. The trace map yields a splitting of sequence (1), hence (i) follows. Duality for finite flat morphisms implies $f_*\omega_X = (f_*\mathcal{O}_X)^* \otimes \omega_Y$, hence we obtain (ii). For (iii) see [**Pa89**, Lemma 8.2]. □

In order to reconstruct f from $\mathcal{E}$ we need some extra data, namely a section of $S^3\mathcal{E}^* \otimes \wedge^2\mathcal{E}$. Moreover, we can naturally see X as sitting into the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E}^*)$ over Y. This is the content of the next two results, see [**Mi85**, **CasEk96**, **FS01**].

Theorem 1.2. *Any triple cover $f \colon X \longrightarrow Y$ is determined by rank 2 vector bundle $\mathcal{E}$ on Y and a global section $\eta \in H^0(Y, S^3\mathcal{E}^* \otimes \wedge^2\mathcal{E})$, and conversely. Moreover, if Y is smooth and $S^3\mathcal{E}^* \otimes \wedge^2\mathcal{E}$ is globally generated, its general section η defines a triple cover $f \colon X \longrightarrow Y$ with X smooth.*

Theorem 1.3. *Let $f \colon X \longrightarrow Y$ be a triple cover, with X and Y smooth varieties. Then there exists a unique embedding $i \colon X \longrightarrow \mathbb{P}(\mathcal{E}^*)$ such that the following diagram commutes:*



According to Theorem 1.2*, this embedding induces an isomorphism of X with the zero-scheme $D_0(\eta) \subset \mathbb{P}(\mathcal{E}^*)$ of a global section η of the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E}^*)}(3) \otimes \pi^*(\wedge^2 \mathcal{E})$.*

1.3 Adjunction theory

We refer to [**So79**, **SoV87**, **LaPa84**, **DES93**, **BeSo95**] for basic material on adjunction theory.

Theorem 1.4. *Let $X \subset \mathbb{P}^n$ be a smooth surface and D its hyperplane class. Then $|K_X + D|$ is non-special and has dimension $N = g(D) + p_g(X) - q(X) - 1$. Moreover*

(A) *$|K_X + D| = \emptyset$ if and only if*

- **(1)** *$X \subset \mathbb{P}^n$ is a scroll over a curve of genus $g(D) = q(X)$ or*
- **(2)** *$X = \mathbb{P}^2$, $D = \mathcal{O}_{\mathbb{P}^2}(1)$ or $D = \mathcal{O}_{\mathbb{P}^2}(2)$.*

(B) *If $|K_X + D| \neq \emptyset$ then $|K_X + D|$ is base-point free. In this case $(K_X + D)^2 = 0$ if and only if*

- **(3)** *X is a Del Pezzo surface and $D = -K_X$ (in particular X is rational) or*
- **(4)** *$X \subset \mathbb{P}^n$ is a conic bundle.*

If $(K_X + D)^2 > 0$ then the adjunction map

$$\varphi_{|K_X+D|} : X \longrightarrow X_1 \subset \mathbb{P}^N$$

defined by the complete linear system $|K_X + D|$ is birational onto a smooth surface X_1 of degree $(K_X + D)^2$ and blows down precisely the (-1)-curves E on X with $DE = 1$, unless

- **(5)** *$X = \mathbb{P}^2(p_1, \dots, p_7)$, $\quad D = 6L - \sum_{i=1}^7 2E_i$,*
- **(6)** *$X = \mathbb{P}^2(p_1, \dots, p_8)$, $\quad D = 6L - \sum_{i=1}^7 2E_i - E_8$,*
- **(7)** *$X = \mathbb{P}^2(p_1, \dots, p_8)$, $\quad D = 9L - \sum_{i=1}^8 3E_i$,*
- **(8)** *$X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is an indecomposable rank 2 vector bundle over an elliptic curve and $D = 3B$, where B is an effective divisor on X with $B^2 = 1$.*

We can apply Theorem 1.4 repeatedly, obtaining a sequence of surfaces and adjunction maps

$$X =: X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \xrightarrow{\varphi_n} X_n \longrightarrow \cdots$$

At each step we must control the numerical data arising from the adjunction process. We have

$$(D_{i+1})^2 = (K_{X_i} + D_i)^2, \quad K_{X_{i+1}} D_{i+1} = (K_{X_i} + D_i) K_{X_i}.$$

For the computation of

$$(K_{X_{i+1}})^2 = (K_{X_i})^2 + \alpha_i$$

we also need to know the number α_i of *exceptional lines* on X_i, i.e. the number of smooth curves $E \subset X_i$ such that $K_{X_i} E = E^2 = -1$, $ED = 1$. Notice that by the Hodge Index Theorem ([**H77**, Exercise 1.9 p. 368]) we have

$$\det \begin{pmatrix} (D_i)^2 & K_{X_i} D_i \\ K_{X_i} D_i & (K_{X_i})^2 \end{pmatrix} \leq 0$$

and the equality holds if and only if K_{X_i} and D_i are numerically dependent.

Proposition 1.5. *Let $E \subset X_{n-1}$ be a curve contracted by the n-th adjunction map $\varphi_n : X_{n-1} \longrightarrow X_n$, and set $\psi := \varphi_{n-1} \circ \varphi_{n-2} \circ \cdots \circ \varphi_1$ and $E^* := \psi^* E$. Then we have*

$$(E^*)^2 = -1, \quad K_X E^* = -1, \quad DE^* = n.$$

Proof. Since E is contracted by φ_n, we have $E^2 = -1$, $K_{X_{n-1}}E = -1$, $D_{n-1}E = 1$. The map ψ is birational, so $(E^*)^2 = E^2 = -1$. Moreover

$$\psi_* K_X = K_{X_{n-1}}, \quad \psi_* D = D_{n-1} - (n-1)K_{X_{n-1}}.$$

Applying the projection formula we obtain

$$K_X E^* = (\psi_* K_X)E = K_{X_{n-1}}E = -1;$$
$$DE^* = (\psi_* D)E = (D_{n-1} - (n-1)K_{X_{n-1}})E = n.$$

□

1.4 Steiner bundles

1.4.1 Steiner sheaves and their projectivization

Let U, V and W be $\mathbb{C}$-vector spaces. Consider the projective spaces $\mathbb{P}(V) = \mathrm{Proj}(V^*)$ and $\mathbb{P}(U) = \mathrm{Proj}(U^*)$, and identify V and U with $H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))$ and $H^0(\mathbb{P}(U), \mathcal{O}_{\mathbb{P}(U)}(1))$, respectively. An element ϕ of $U \otimes V \otimes W$ gives rise to two maps

$$W^* \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \xrightarrow{M_\phi} U \otimes \mathcal{O}_{\mathbb{P}(V)}, \qquad W^* \otimes \mathcal{O}_{\mathbb{P}(U)}(-1) \xrightarrow{N_\phi} V \otimes \mathcal{O}_{\mathbb{P}(U)}. \tag{2}$$

Set $\mathcal{F} := \operatorname{coker} M_\phi$. We say that $\mathcal{F}$ is a *Steiner sheaf*, and we denote its projectivization by $\mathbb{P}(\mathcal{F})$; this is a projective bundle precisely when $\mathcal{F}$ is locally free. Let $p \colon \mathbb{P}(\mathcal{F}) \to \mathbb{P}(V)$ be the bundle map and $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(\xi)$ be the tautological relatively ample line bundle on $\mathbb{P}(\mathcal{F})$, so that

$$H^0(\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(\xi)) \simeq H^0(\mathbb{P}(V), \mathcal{F}) \simeq U.$$

Since $\mathcal{F}$ is a quotient of $U \otimes \mathcal{O}_{\mathbb{P}(V)}$, we get a natural embedding

$$\mathbb{P}(\mathcal{F}) \subset \mathbb{P}(U \otimes \mathcal{O}_{\mathbb{P}(V)}) \simeq \mathbb{P}(V) \times \mathbb{P}(U).$$

The map q associated with the linear system $|\mathcal{O}_{\mathbb{P}(\mathcal{F})}(\xi)|$ is just the restriction to $\mathbb{P}(\mathcal{F})$ of the second projection from $\mathbb{P}(V) \times \mathbb{P}(U)$. On the other hand, setting $\ell := p^*(\mathcal{O}_{\mathbb{P}(V)}(1))$, the linear system $|\mathcal{O}_{\mathbb{P}(\mathcal{F})}(\ell)|$ is naturally associated with the map p. In this procedure the roles of U and V can be reversed. In other words, setting $\mathcal{G} = \operatorname{coker} N_\phi$, we get a second Steiner sheaf, this time on $\mathbb{P}(U)$, and a second projective bundle $\mathbb{P}(\mathcal{G})$ with maps p' and q' to $\mathbb{P}(U)$ and $\mathbb{P}(V)$, respectively. So we have two incidence diagrams

$$\begin{array}{ccccccc} & \mathbb{P}(\mathcal{F}) & & & & \mathbb{P}(\mathcal{G}) & \\ {}^{p}\swarrow & & \searrow^{q} & & {}^{p'}\swarrow & & \searrow^{q'} \\ \mathbb{P}(V) & & \mathbb{P}(U), & & \mathbb{P}(U) & & \mathbb{P}(V). \end{array}$$

The link between $\mathbb{P}(\mathcal{F})$ and $\mathbb{P}(\mathcal{G})$ is provided by the following

Proposition 1.6. *Let $\phi \in U \otimes V \otimes W$ and set $m = \dim W$. Then:*

- **(i)** *the schemes $\mathbb{P}(\mathcal{F})$ and $\mathbb{P}(\mathcal{G})$ are both identified with the same m-fold linear section of $\mathbb{P}(V) \times \mathbb{P}(U)$. Moreover, under this identification, $q = p'$ and $p = q'$;*
- **(ii)** *for any non-negative integer k, there are natural isomorphisms :*

$$p_* q^*(\mathcal{O}_{\mathbb{P}(U)}(k)) \simeq S^k \mathcal{F}, \qquad q_* p^*(\mathcal{O}_{\mathbb{P}(V)}(k)) \simeq S^k \mathcal{G}.$$

Proof. Set $M := M_\phi$. By construction, the scheme $\mathbb{P}(\mathcal{F})$ is defined as the set

$$\mathbb{P}(\mathcal{F}) = \{([v], [u]) \in \mathbb{P}(V) \times \mathbb{P}(U) \mid u \in \operatorname{coker} M_v\},$$

where $v: V \to \mathbb{C}$ (resp. $u: U \to \mathbb{C}$) is a 1-dimensional quotient of V (resp. of U) and $M_v: W^* \to U$ is the evaluation of M at the point $[v]$. Now, we get that u is defined on $\operatorname{coker} M_v$ if and only if $u \circ M_v = 0$. This clearly amounts to require $(u \circ M_v)(w) = 0$ for all $w \in W^*$, that is $u \otimes v \otimes w(\phi) = 0$ for all $w \in W^*$. Summing up, we have

$$\mathbb{P}(\mathcal{F}) = \{([v], [u]) \mid u \otimes v \otimes w(\phi) = 0 \text{ for all } w \in W^*\}. \tag{3}$$

The same argument works for $\mathbb{P}(\mathcal{G})$ by interchanging the roles of v and v, hence $\mathbb{P}(\mathcal{F})$ and $\mathbb{P}(\mathcal{G})$ are both identified with the same subset of $\mathbb{P}(V) \times \mathbb{P}(U)$. Since each element w_i of a basis of W^* gives a linear equation of the form $u \otimes v \otimes w_i(\phi) = 0$, we have that $\mathbb{P}(\mathcal{F})$ is an m-fold linear section (of codimension m or smaller) of $\mathbb{P}(V) \times \mathbb{P}(U)$.

Note that, in view of the identification above, the map p is just the projection from $\mathbb{P}(V) \times \mathbb{P}(U)$ onto $\mathbb{P}(V)$, restricted to the set given by (3). The same holds for q', hence we are allowed to identify p and q'. Analogously, both q and p' are given as projections onto the factor $\mathbb{P}(V)$. We have thus proved (i). Now let us check (ii). For any non-negative integer k we have

$$q^*(\mathcal{O}_{\mathbb{P}(U)}(k)) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{F})}(k\xi),$$
$$p^*(\mathcal{O}_{\mathbb{P}(V)}(k)) \simeq (q')^*(\mathcal{O}_{\mathbb{P}(V)}(k)) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{G})}(k\xi'),$$

where ξ' is the tautological relatively ample line bundle on $\mathbb{P}(\mathcal{G})$. Therefore the claim follows from the canonical isomorphisms

$$p_*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(k\xi)) \simeq S^k\mathcal{F}, \qquad p'_*(\mathcal{O}_{\mathbb{P}(\mathcal{G})}(k\xi')) \simeq S^k\mathcal{G}.$$

□

Remark 1.7. We can rephrase the content of Proposition 1.6 by using coordinates as follows. Let us consider bases

$$\{z_i\}, \{x_j\}, \{y_k\}$$

for U, V, W, respectively. With respect to these bases, the tensor $\phi \in U \otimes V \otimes W$ will correspond to a trilinear form

$$\phi = \sum a_{ijk} z_i x_j y_k.$$

Then ϕ induces two linear maps of graded vector spaces

$$W^* \otimes S(V)[-1] \to U \otimes S(V), \quad W^* \otimes S(U)[-1] \to V \otimes S(U),$$

both defined as

$$w \otimes \Phi \to \Big(\sum a_{ijk} z_i x_j y_k(w)\Big)\Phi,$$

where $\Phi \in S(V)$, resp. $\Phi \in S(U)$. The sheafification of these maps gives precisely the two maps of vector bundles M_ϕ and N_ϕ written in (2), whose defining matrices of linear forms are

$$\Big(\sum\nolimits_j a_{ijk} x_j\Big)_{ik} \quad \text{and} \quad \Big(\sum\nolimits_i a_{ijk} z_i\Big)_{jk},$$

respectively. Thus the Steiner sheaves $\mathbb{P}(\mathcal{F})$ and $\mathbb{P}(\mathcal{G})$ are both identified with the zero locus of the following m bilinear equations in $\mathbb{P}(V) \times \mathbb{P}(U)$:

$$\sum\nolimits_{i,j} a_{ij1} z_i x_j = 0, \ldots, \sum\nolimits_{i,j} a_{ijm} z_i x_j = 0.$$

1.4.2 Unstable lines

Let us assume now $\dim V = 3$ and consider a Steiner bundle $\mathcal{F}$ of rank 2 on $\mathbb{P}^2 = \mathbb{P}(V)$. To be consistent with the notation that will appear later, we set $\dim U = b-2$ and $\dim W = b-4$, for $b \geq 4$, and we write $\mathcal{F}_b$ instead of $\mathcal{F}$. The minimal resolution of $\mathcal{F}_b$ is then

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{b-4} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}^{b-2} \longrightarrow \mathcal{F}_b \longrightarrow 0, \tag{4}$$

where M is a $(b-2)\times(b-4)$ matrix of linear forms.

Definition 1.8. *A line $L \subset \mathbb{P}^2$ is said to be* unstable *for $\mathcal{F}_b$ if*

$$\mathcal{F}_b|_L = \mathcal{O}_L \oplus \mathcal{O}_L(b-4).$$

Here are other useful characterizations of unstable lines.

Lemma 1.9. *The following are equivalent:*

(*i*) *$L \subset \mathbb{P}^2$ is an unstable line for $\mathcal{F}_b$;*

(*ii*) *$H^0(L, \mathcal{F}_b^*|_L) \neq 0$;*

(*iii*) *there is a nonzero global section of $\mathcal{F}_b$ whose vanishing locus contains $b-4$ points of L (counted with multiplicity).*

Proof. We first observe that, given a line $L \subset \mathbb{P}^2$, the restriction of (4) to L yields

$$0 \longrightarrow \mathcal{O}_L(-1)^{b-4} \xrightarrow{M} \mathcal{O}_L^{b-2} \longrightarrow \mathcal{F}_b|_L \longrightarrow 0. \tag{5}$$

Hence $h^0(\mathbb{P}^2, \mathcal{F}_b) = h^0(L, \mathcal{F}_b|_L) = b-2$, in particular all the global sections of $\mathcal{F}_b|_L$ are obtained by restricting the global sections of $\mathcal{F}_b$ to L. We can now prove our chain of implications.

(*i*) $\Leftrightarrow$ **(*ii*)**. There is an integer a such that $\mathcal{F}_b|_L = \mathcal{O}_L(a) \oplus \mathcal{O}_L(b-4-a)$, and since $\mathcal{F}_b$ is globally generated we have $0 \leq a \leq b-4$. Condition **(*i*)** corresponds to $a=0$ or $a=b-4$, and this clearly implies **(*ii*)**. Conversely, if **(*ii*)** holds, then $a \leq 0$ or $a \geq b-4$; this implies either $a=0$ or $a=b-4$, hence **(*i*)** holds.

(*ii*) $\Leftrightarrow$ **(*iii*)**. If **(*ii*)** holds, there is a short exact sequence

$$0 \to \mathcal{O}_L \to \mathcal{F}_b^*|_L \to \mathcal{O}_L(4-b) \to 0,$$

so by dualizing we get

$$0 \to \mathcal{O}_L(b-4) \xrightarrow{\iota} \mathcal{F}_b|_L \to \mathcal{O}_L \to 0. \tag{6}$$

Composing ι with a non-zero map $\mathcal{O}_L \to \mathcal{O}_L(b-4)$, we obtain **(*iii*)**.

Conversely, assume that **(*iii*)** holds. Then there is a section s of $\mathcal{F}_b$ whose vanishing locus Z contains a subscheme of L of length $b-4$. Applying the functor $\otimes_{\mathcal{O}_{\mathbb{P}^2}} \mathcal{O}_L$ to the exact sequence $0 \to \mathcal{I}_Z(b-4) \to \mathcal{O}_{\mathbb{P}^2}(b-4) \to \mathcal{O}_Z \to 0$, we get

$$0 \to \mathrm{Tor}_1^{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_Z, \mathcal{O}_L) \to \mathcal{I}_Z(b-4)|_L \to \mathcal{O}_L(b-4) \to \mathcal{O}_{Z\cap L} \to 0$$

and from this, using the isomorphism $\mathrm{Tor}_1^{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_Z, \mathcal{O}_L) = \mathcal{O}_{Z\cap L}$, we obtain

$$0 \to \mathcal{O}_{Z\cap L} \to \mathcal{I}_Z(b-4)|_L \to \mathcal{I}_{Z\cap L}(b-4) \to 0. \tag{7}$$

Since $\mathcal{O}_{Z\cap L}$ is a torsion sheaf on L, whereas $\mathcal{I}_{Z\cap L} = \mathcal{O}_L(-Z\cap L)$ is locally free, standard facts about coherent sheaves on smooth curves imply that the exact sequence (7) is split. On the other hand, our condition says that the scheme $Z \cap L$ has length $b-4$, so we can write

$$\mathcal{I}_Z(b-4)|_L = \mathcal{I}_{Z\cap L}(b-4) \oplus \mathcal{O}_{Z\cap L} = \mathcal{O}_L \oplus \mathcal{O}_{Z\cap L}. \tag{8}$$

Therefore, restricting to L the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{F}_b \to \mathcal{I}_Z(b-4) \to 0$$

and using (8), we obtain a surjection $\mathcal{F}_b|_L \to \mathcal{O}_L$. Dualizing we get the required nonzero global section of $\mathcal{F}_b^*|_L$. □

The set of unstable lines of $\mathcal{F}_b$ has a natural structure of subscheme of $\mathbb{P}^{2*}$; we denote it by $\mathcal{W}(\mathcal{F}_b)$. Let us give a summary of the behaviour of the unstable lines of $\mathcal{F}_b$ for small values of b.

$b=4$. We have $\mathcal{F}_4 \simeq \mathcal{O}_{\mathbb{P}^2}^2$, so $\mathcal{W}(\mathcal{F}_4)$ is empty.

$b=5$. There is an isomorphism $\mathcal{F}_5 \simeq T_{\mathbb{P}^2}(-1)$. Also, $\mathcal{W}(\mathcal{F}_5) = \mathbb{P}^{2*}$.

$b=6$. The scheme $\mathcal{W}(\mathcal{F}_6)$ is a smooth conic in $\mathbb{P}^{2*}$, and the unstable lines of $\mathcal{F}_6$ are the tangent lines to the dual conic, see [**DK93**].

$b=7$. The scheme $\mathcal{W}(\mathcal{F}_7)$ is either finite of length 6 or consists of a smooth conic in $\mathbb{P}^{2*}$. The former case is the general one, and when it occurs $\mathcal{F}_7$ is called a *logarithmic bundle*. Instead, the latter case occurs if and only if $\mathcal{F}_7$ is a so-called *Schwarzenberger bundle*, whose matrix M, up to a linear change of coordinates, has the form

$$M = {}^{t}\begin{pmatrix} x_0 & x_1 & x_2 & 0 & 0 \\ 0 & x_0 & x_1 & x_2 & 0 \\ 0 & 0 & x_0 & x_1 & x_2 \end{pmatrix}, \tag{9}$$

see [**DK93**, **Val00a**, **Val00b**].

$b \geq 8$. Unstable lines do not always exist in this range. The scheme $\mathcal{W}(\mathcal{F}_b)$ is either finite of length $\leq b-1$ or consists of a smooth conic in $\mathbb{P}^{2*}$. In the latter case, $\mathcal{F}_b$ is a Schwarzenberger bundle, whose matrix M, up to a linear change of coordinates, is a $(b-2)\times(b-4)$ matrix having the same form as (9). We can actually state a more precise result, see again [**DK93**, **Val00a**, **Val00b**].

Proposition 1.10. *If $\mathcal{F}_b$ contains a finite number α of unstable lines, then $0 \leq \alpha \leq b-1$. More precisely, the following holds.*

(*i*) *If $0 \leq \alpha \leq b-2$ then, up to a linear change of coordinates, the matrix M is of type*

$$M = {}^{t}\left(\begin{array}{ccc|c} a_{1,1}H_1 & \cdots & a_{1,\alpha}H_\alpha & \\ \vdots & \vdots & \vdots & M' \\ a_{b-4,1}H_1 & \cdots & a_{b-4,\alpha}H_\alpha & \end{array}\right),$$

for some $(b-2-\alpha)\times(b-4)$ matrix M' of linear forms. In this case the unstable lines are given by

$$H_1 = 0, \quad H_2 = 0, \quad \cdots, \quad H_\alpha = 0.$$

(*ii*) *If $\alpha = b-1$ then $\mathcal{F}_b$ is a logarithmic bundle. In this case, the matrix M is of type*

$$M = {}^{t}\begin{pmatrix} a_{1,1}H_1 & a_{1,2}H_2 & \cdots & a_{1,b-2}H_{b-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{b-4,1}H_1 & a_{b-4,2}H_2 & \cdots & a_{b-4,b-2}H_{b-2} \end{pmatrix},$$

where $H_1, \ldots, H_{b-2}$ are such that the linear form

$$H_{b-1} := \sum_{j=1}^{b-2} a_{i,j}H_j$$

does not depend on $i \in \{1, \ldots, b-4\}$. The unstable lines are given by

$$H_j = 0, \quad j \in \{1, \ldots, b-2\}.$$

Remark 1.11. Proposition 1.10 allows us to give another proof of the implication **(*i*)** $\Rightarrow$ **(*iii*)** in Lemma 1.9. Indeed, we can take a basis $s_1, \ldots, s_{b-2}$ of $H^0(\mathbb{P}^2, \mathcal{F}_b)$ such that the homogeneous ideal I_k of the vanishing locus of s_k is defined by the maximal minors of the matrix obtained by deleting the k-th row of M, namely by $b-3$ forms of degree $b-4$. Assume now that the unstable line L is defined by the equation $H_i = 0$. Then, if $k \neq i$, all the minors defining I_k are divisible by H_i, except the one obtained by deleting the k-th and i-th rows of M; so s_k vanishes at $b-4$ points on L.

The following result will be used in Section 4, see in particular the proof of Theorem 4.2. The reason to denote by α_1 the number of unstable lines will be apparent later, see Remark 1.13.

Proposition 1.12. *Assume $b = 8$ and let $\mathcal{F} := \mathcal{F}_8$ be a Steiner bundle with a finite number α_1 of unstable lines. Then*

$$h^0(\mathbb{P}^2, S^3\mathcal{F}(-2)) \geq \alpha_1. \tag{10}$$

Proof. If $b = 8$, then (4) becomes

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^4 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}^6 \longrightarrow \mathcal{F} \longrightarrow 0. \tag{11}$$

As a preliminary remark we note that, since $H^0(\mathbb{P}^2, \mathcal{F}(-1)) = 0$, any non-zero global section of $\mathcal{F}$ vanishes in codimension 2. Thus $\mathcal{F}$ fits into a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{F} \to \mathcal{I}_Z(4) \to 0, \tag{12}$$

where $Z \subset \mathbb{P}^2$ is a 0-dimensional subscheme of length 10. By using Serre construction ([**OkSchSp88**, Theorem 5.1.1]) we see that this process is reversible, in the sense that starting from a subscheme $Z \subset \mathbb{P}^2$ of length 10 whose homogeneous ideals contains 6 quartics and no cubics (i.e, a *syzygy-general* subscheme), we can find a unique Steiner bundle $\mathcal{F}$ having a global section vanishing on Z, and hence appearing as an extension of $\mathcal{I}_Z(4)$ by $\mathcal{O}_{\mathbb{P}^2}$ as in (12).

The symmetric cube of (12), tensored by $\mathcal{O}_{\mathbb{P}^2}(-2)$, gives

$$0 \to S^2\mathcal{F}(-2) \to S^3\mathcal{F}(-2) \to \mathcal{I}_Z^3(10) \to 0. \tag{13}$$

We claim that $H^2(\mathbb{P}^2, S^2\mathcal{F}(-2)) = 0$. In fact, by Serre duality the last group is isomorphic to $H^0(\mathbb{P}^2, S^2\mathcal{F}^*(-1))$, which naturally injects into $H^0(\mathbb{P}^2, S^2\mathcal{F}^*)$. Since we are working on a field of characteristic 0, the stability of $\mathcal{F}^*$ implies the semi-stability of $S^2\mathcal{F}^*$, so $c_1(S^2\mathcal{F}^*) < 0$ implies $H^0(\mathbb{P}^2, S^2\mathcal{F}^*) = 0$ proving our claim.

By taking the symmetric square of (11) we can compute $\chi(\mathbb{P}^2, S^2\mathcal{F}(-2)) = -6$; so, whenever $h^0(\mathbb{P}^2, S^2\mathcal{F}(-2)) = 0$, we deduce $h^1(\mathbb{P}^2, S^2\mathcal{F}(-2)) = 6$.

If $\alpha_1 = 0$ there is nothing to prove. We analyze separately the cases $1 \leq \alpha_1 \leq 5$, $\alpha_1 = 6$, $\alpha_1 = 7$.

Case $1 \leq \alpha_1 \leq 5$. In this case, by using the form for the matrix M given in Proposition 1.10, with the help of `Macaulay2` we can easily find an example of Steiner bundle $\mathcal{F}$ with $H^0(\mathbb{P}^2, S^2\mathcal{F}(-2)) = 0$, so this vanishing holds for a sufficiently general $\mathcal{F}$. On the other hand, by semi-continuity, it suffices to show (10) for $\mathcal{F}$ general. Therefore it is not restrictive to assume $h^1(\mathbb{P}^2, S^2\mathcal{F}(-2)) = 6$ from now on and so, passing to cohomology in (13), we see that it is enough to prove the inequality

$$h^0(\mathbb{P}^2, \mathcal{I}_Z^3(10)) \geq 6 + \alpha_1. \tag{14}$$

Let $L_1, \ldots, L_{\alpha_1}$ be α_1 general lines in $\mathbb{P}^2$, and let us define the following subschemes of the plane:

- Z_1 is the subscheme of length $\binom{\alpha_1}{2}$ given by the pairwise intersection points of the lines;
- Z_2 is a subscheme of length $\alpha_1(5 - \alpha_1)$ obtained by choosing $4 - (\alpha_1 - 1) = 5 - \alpha_1$ general points on each line;
- Z_3 is a general 0-dimensional subscheme of length $\frac{1}{2}(\alpha_1 - 4)(\alpha_1 - 5)$.

Setting $Z := Z_1 \cup Z_2 \cup Z_3$, we see that Z is a 0-dimensional, syzygy-general subscheme of length 10. By Lemma 1.9 the lines $L_1, \ldots, L_{\alpha_1}$ are unstable for Z and, conversely, the general Steiner bundle $\mathcal{F}$ with α_1 unstable lines can be obtained from a Z of this form by means of the Serre construction. Now, the linear system of plane curves of degree 10 having multiplicity ≥ 3 at Z clearly contains the reducible curves of type

$$L_1 + \cdots + L_{\alpha_1} + C_{10-\alpha_1},$$

where $C_{10-\alpha_1}$ has degree $10-\alpha_1$ and multiplicity $\geq i$ at Z_i. Then we obtain

$$\begin{aligned} h^0(\mathbb{P}^2, \mathcal{I}_Z^3(10)) &\geq h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(10-\alpha_1)) - \ell(Z_1) - 3\,\ell(Z_2) - 6\,\ell(Z_3) \\ &= \binom{10-\alpha_1+2}{2} - \binom{\alpha_1}{2} - 3\alpha_1(5-\alpha_1) - 3(\alpha_1-4)(\alpha_1-5) \\ &= 6+\alpha_1, \end{aligned}$$

which is (14).

Case $\boldsymbol{\alpha_1 = 6}$. In this case, we use reduction of $\mathcal{F}$ along its six unstable lines $L_1, \dots, L_6$, see [**Val00b**]. This yields the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^2 \to \mathcal{F} \to \bigoplus_{i=1}^{6} \mathcal{O}_{L_i} \to 0. \tag{15}$$

From Pieri's formulas (cf. [**W03**, Corollary 2.3.5 p. 62]) we obtain

$$\mathcal{F} \otimes S^2\mathcal{F}(-2) = S^3\mathcal{F}(-2) \oplus \wedge^2\mathcal{F} \otimes \mathcal{F}(-2) = S^3\mathcal{F}(-2) \oplus \mathcal{F}(2), \tag{16}$$

and the fact that L_i is unstable implies

$$S^2\mathcal{F}(-2)|_{L_i} = \mathcal{O}_{L_i}(-2) \oplus \mathcal{O}_{L_i}(2) \oplus \mathcal{O}_{L_i}(6). \tag{17}$$

So, tensoring (15) with $S^2\mathcal{F}(-2)$ we get

$$0 \to (S^2\mathcal{F}(-3))^2 \to S^3\mathcal{F}(-2) \oplus \mathcal{F}(2) \to \bigoplus_{i=1}^{6} \big(\mathcal{O}_{L_i}(-2) \oplus \mathcal{O}_{L_i}(2) \oplus \mathcal{O}_{L_i}(6)\big) \to 0. \tag{18}$$

Using as in the previous case the form for the matrix M given in Proposition 1.10 together with `Macaulay2`, it is not difficult to construct an example of $\mathcal{F}$ with six unstable lines and such that $H^2(\mathbb{P}^2, S^2\mathcal{F}(-3)) = 0$, so this vanishing takes place for the general such a $\mathcal{F}$. Moreover, (11) forces $H^1(\mathbb{P}^2, \mathcal{F}(2)) = 0$. Therefore, passing to cohomology in (18) we get a surjection

$$H^1(\mathbb{P}^2, S^3\mathcal{F}(-2)) \to \bigoplus_{i=1}^{6} H^1(L_i, \mathcal{O}_{L_i}(-2)) \to 0,$$

which in turn implies $h^1(\mathbb{P}^2, S^3\mathcal{F}(-2)) \geq 6$. Since we have $\chi(\mathbb{P}^2, S^3\mathcal{F}(-2)) = 0$, we obtain $h^0(\mathbb{P}^2, S^3\mathcal{F}(-2)) \geq 6$, that is (10).

Case $\boldsymbol{\alpha_1 = 7}$. We use the same approach as in the previous case. This time, the reduction of $\mathcal{F}$ along its seven unstable lines $L_1, \dots, L_7$ amounts to the residue exact sequence, cf. for instance [**Val00b**], so we obtain the short exact sequence

$$0 \to \Omega^1_{\mathbb{P}^2} \to \mathcal{F} \to \bigoplus_{i=1}^{7} \mathcal{O}_{L_i} \to 0. \tag{19}$$

Tensoring (19) with $S^2\mathcal{F}(-2)$ and using (16) and (17), we get

$$0 \to \Omega^1_{\mathbb{P}^2} \otimes S^2\mathcal{F}(-2) \to S^3\mathcal{F}(-2) \oplus \mathcal{F}(2) \to \bigoplus_{i=1}^{7} \big(\mathcal{O}_{L_i}(-2) \oplus \mathcal{O}_{L_i}(2) \oplus \mathcal{O}_{L_i}(6)\big) \to 0. \tag{20}$$

Again with the help of `Macaulay2`, we can construct an example of $\mathcal{F}$ with seven unstable lines and satisfying $H^2(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2} \otimes S^2\mathcal{F}(-2)) = 0$, hence this vanishing holds for a general such $\mathcal{F}$. Therefore, since $H^1(\mathbb{P}^2, \mathcal{F}(2)) = 0$, we obtain via (20) a surjection

$$H^1(\mathbb{P}^2, S^3\mathcal{F}(-2)) \to \bigoplus_{i=1}^{7} H^1(L_i, \mathcal{O}_{L_i}(-2)) \to 0.$$

This, together with $\chi(\mathbb{P}^2, S^3\mathcal{F}(-2)) = 0$, implies $h^0(\mathbb{P}^2, S^3\mathcal{F}(-2)) \geq 7$. □

Remark 1.13. In Propositions 1.10 and 1.12 we denoted the number of unstable lines of the twisted Tschirnhausen bundle by α_1, just like the number of exceptional lines contracted by the first adjunction map (see §1.3). These two numbers are in fact the same for $b \geq 7$, see §2.3.2.

2 General triple planes with $p_g = q = 0$

2.1 General triple planes

A *triple plane* a triple cover $f : X \longrightarrow \mathbb{P}^2$. We denote by H the pullback $H := f^*L$, where $L \subset \mathbb{P}^2$ is a line. The divisor H is ample, because L is ample and f is a finite map.

The Tschirnhausen bundle $\mathcal{E}$ of f is a rank 2 vector bundle on $\mathbb{P}^2$ such that $f_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{E}$. Proposition 1.1 now implies

Proposition 2.1. *Let $f : X \longrightarrow \mathbb{P}^2$ be a triple plane with Tschirnhausen bundle $\mathcal{E}$. Then we have:*

$$\begin{aligned} p_g(X) &= h^0(\mathbb{P}^2, \mathcal{E}^*(-3)), \\ q(X) &= h^1(\mathbb{P}^2, \mathcal{E}^*(-3)), \\ P_2(X) &= h^0(X, 2K_X) = h^0(\mathbb{P}^2, S^2\mathcal{E}^*(-6)). \end{aligned}$$

Definition 2.2. *Let $f : X \longrightarrow \mathbb{P}^2$ be a triple plane and $B \subset \mathbb{P}^2$ its branch locus. We say that f is a* general triple plane *if the following conditions are satisfied:*

(*i*) *f is unramified over $\mathbb{P}^2 \setminus B$;*

(*ii*) *$f^*B = 2R + R_0$, where R is irreducible and non-singular and R_0 is reduced;*

(*iii*) *$f_{|R} : R \longrightarrow B$ coincides with the normalization map of B.*

A useful criterion to check that a triple plane is a general one is provided by the following

Proposition 2.3. *Let $f : X \to \mathbb{P}^2$ be a triple plane with X smooth. Then either f is general or f is a Galois cover. In the last case, f is totally ramified over a smooth branch locus.*

Proof. See [**Tan02**, Theorems 2.1 and 3.2]. □

Hence Theorem 1.2 shows that, if $S^3\mathcal{E}^* \otimes \wedge^2\mathcal{E}$ is globally generated, the cover associated with a general section $\eta \in H^0(\mathbb{P}^2, S^3\mathcal{E}^* \otimes \wedge^2\mathcal{E})$ is a general triple plane as soon as it is not totally ramified.

Since the curve R is the ramification divisor of f and the ramification is simple, we have

$$K_X = f^*K_{\mathbb{P}^2} + R = -3H + R. \tag{21}$$

Moreover, by [**Mi85**, Proposition 4.7 and Lemma 4.1], we obtain

Proposition 2.4. *Let $f : X \longrightarrow \mathbb{P}^2$ be a general triple plane with Tschirnhausen bundle $\mathcal{E}$ and define*

$$b := -c_1(\mathcal{E}), \quad h := c_2(\mathcal{E}).$$

Then the branch curve B has degree $2b$ and contains $3h$ ordinary cusps and no further singularities. Moreover the cusps are exactly the points where f is totally ramified.

Moreover, in view of [**Mi85**, Lemma 5.9] and [**CasEk96**, Corollary 2.2], we have the following information on the curves R and R_0.

Proposition 2.5. *The two curves R and R_0 are both smooth and isomorphic to the normalization of B. Furthermore, they are tangent at the preimages of the cusps of B and they do not meet elsewhere. Finally, the ramification divisor R is very ample on X.*

This allows us to compute the intersection numbers of R and R_0 as follows.

Proposition 2.6. *We have*

$$R^2 = 2b^2 - 3h, \quad RR_0 = 6h, \quad R_0^2 = 4b^2 - 12h.$$

Proof. From $f^*B = 2R + R_0$ we deduce $(2R+R_0)^2 = 3B^2 = 12b^2$. Moreover projection formula yields

$$R(2R+R_0) = R(f^*B) = (f_*R)B = B^2 = 4b^2.$$

Finally, by Proposition 2.5 it follows $RR_0 = 6h$, so a short calculation concludes the proof. □

Corollary 2.7. *We have* $3h \geq \frac{2}{3}b^2$.

Proof. Since the divisor R is very ample, the Hodge Index theorem implies $R^2R_0^2 \leq (RR_0)^2$ and the claim follows. □

Remark 2.8. Proposition 2.6 and Corollary 2.7 were already established by Bronowski in [**Br42**]. Note that the (very) ampleness of R implies $R^2 > 0$, that is $3h < 2b^2$. In [**Br42**], it is also stated the stronger inequality $3h \leq b^2$, or equivalently $R_0^2 \geq 0$, holds. This is actually false, and counterexamples will be provided by our surfaces of type VII, see Subsection 3.7. Bronowski's mistake is at page 28 of his paper, where he assumes that there exists a curve algebraically equivalent to R_0 and distinct from it; of course, when $R_0^2 < 0$ such a curve cannot exist.

Proposition 2.9. *Let* $f : X \longrightarrow \mathbb{P}^2$ *be a general triple plane. If* $K_X^2 \neq 8$ *then* $D := K_X + 2H$ *is very ample.*

Proof. Since $(2H)^2 = 12$, by [**Fu90**, Theorem 18.5] D is very ample, unless there exists an effective divisor Z such that HZ=1 and $Z^2 = 0$. By projection formula we have

$$1 = HZ = (f^*L)Z = L(f_*Z),$$

hence $f_*Z \subset \mathbb{P}^2$ is a line. On the other hand, $HZ = 1$ implies that the restriction of f to Z is an isomorphism, so Z is a smooth and irreducible rational curve. Since $Z^2 = 0$, the surface X is birationally ruled and Z belongs to the ruling; moreover the argument above shows that all the curves in the ruling are irreducible. So X is isomorphic to $\mathbb{F}_n$ for some n, in particular $K_X^2 = 8$. □

When $D = K_X + 2H$ is very ample on X we can study the adjunction maps associated with D. Using Proposition 1.5, we obtain

Proposition 2.10. *Assume* $K_X^2 \neq 8$ *and let* $\varphi_n : X_{n-1} \longrightarrow X_n$ *be the n-th adjunction map with respect to the very ample divisor* $D = K_X + 2H$. *Then* φ_n *is an isomorphism when n is even, whereas when n is odd* φ_n *contracts exactly the* (-1)*-curves* $E \subset X$ *such that* $HE = (n+1)/2$.

2.2 The numerical invariants and the Tschirnhausen bundle when $p_g = q = 0$

Let $f : X \longrightarrow \mathbb{P}^2$ be a general triple plane and let B be the branch locus of f. Recall that, by Proposition 2.4, the curve B is has degree $2b$ and contains $3h$ ordinary cusps as only singularities.

Proposition 2.11. *Set notation as above, and assume moreover* $\chi(\mathcal{O}_X) = 1$, *that is* $p_g(X) = q(X)$. *Therefore we have at most the following possibilities for the numerical invariants* b, h, K_X^2, $g(H)$:

Case	b	h	K_X^2	$g(H)$
I	2	1	8	0
II	3	2	3	1
III	4	4	−1	2
IV	5	7	−4	3
V	6	11	−6	4
VI	7	16	−7	5
VII	8	22	−7	6
VIII	9	29	−6	7
IX	10	37	−4	8
X	11	46	−1	9
XI	12	56	3	10
XII	13	67	8	11

Table 1: Possible numerical invariants for a general triple plane with $\chi(\mathcal{O}_X) = 1$

Proof. Using the projection formula we obtain

$$HR = (f^*L)R = L(f_*R) = LB = 2b.$$

Since $K_X = -3H + R$ and $H^2 = 3$ it follows $K_X H = 2b - 9$, hence $g(H) = b - 2$. Using the *formule di corrispondenza* ([**Iv70**, Section V]) we infer

$$\begin{cases} 9h + 3 = 4b^2 - 6b + K_X^2 \\ 2h - 4 = b^2 - 3b. \end{cases}$$

Therefore $h = \frac{1}{2}(b^2 - 3b + 4)$ and $b^2 - 15b + 42 - 2K_X^2 = 0$. Imposing that the discriminant of this quadratic equation is non-negative, we get $K_X^2 \geq -7$; on the other hand, the Enriques-Kodaira classification and the Miyaoka-Yau inequality imply that any surface with $p_g = q$ satisfies $K_X^2 \leq 9$, see [**BHPV04**, Chapter VII], so $-7 \leq K_X^2 \leq 9$. Now a case-by-case analysis concludes the proof. □

Note that the previous proof shows that

$$c_1(\mathcal{E}) = -b, \quad c_2(\mathcal{E}) = \frac{1}{2}(b^2 - 3b + 4). \tag{22}$$

From now on, we will restrict ourselves to the case $p_g(X) = q(X) = 0$, that is, in terms of the Tschirnhausen bundle $\mathcal{E}$, we suppose $h^1(\mathbb{P}^2, \mathcal{E}) = 0$ and $h^2(\mathbb{P}^2, \mathcal{E}) = 0$. Furthermore, we will write for brevity "triple plane" instead of "general triple plane", and we will use without further mention the natural isomorphism

$$\mathcal{E}^* \simeq \mathcal{E}(b). \tag{23}$$

Theorem 2.12. *Let $f : X \to \mathbb{P}^2$ be a triple plane with $p_g = q = 0$ and let $\mathcal{E}$ be the corresponding Tschirnhausen bundle. With the notation of Proposition* 2.11*, we have*

- **(*i*)** $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$, *in case* I;
- **(*ii*)** $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$, *in case* II;
- **(*iii*)** $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$, *in case* III;
- **(*iv*)** *in cases* IV,..., XII*, the vector bundle $\mathcal{E}$ is stable and has a minimal free resolution of the form*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1-b)^{b-4} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(2-b)^{b-2} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Hence it is a rank 2 *Steiner bundle on $\mathbb{P}^2$, see Subsection* 1.4.

Proof. The Chern classes of $\mathcal{E}$ are computed in each case by (22). In cases I, II and III, using $h^0(\mathbb{P}^2, \mathcal{E}) = 0$ (for I and II)or $h^0(\mathbb{P}^2, \mathcal{E}(1)) = 0$ (for III), we get that $\mathcal{E}$ decomposes in the desired way (an argument for this is for instance [**FV12**, Lemma 2.4]).

Let us now turn to the cases IV,..., XII, so that $5 \leq b \leq 13$. Set $\mathcal{F} := \mathcal{E}(b-2)$. By (22) we obtain

$$c_1(\mathcal{F}) = b - 4, \quad c_2(\mathcal{F}) = \binom{b-3}{2}. \tag{24}$$

Using Proposition 2.1 we can now calculate the cohomology groups of $\mathcal{F}(-i)$, for $i = 0, 1, 2$. We have

$$h^0(\mathbb{P}^2, \mathcal{F}(-1)) = h^0(\mathbb{P}^2, \mathcal{E}(b-3)) = h^0(\mathbb{P}^2, \mathcal{E}^*(-3)) = p_g(X) = 0,$$
$$h^1(\mathbb{P}^2, \mathcal{F}(-1)) = h^0(\mathbb{P}^2, \mathcal{E}(b-3)) = h^1(\mathbb{P}^2, \mathcal{E}^*(-3)) = q(X) = 0.$$

Using (24) and the Riemann-Roch theorem we deduce $\chi(\mathbb{P}^2, \mathcal{F}(-1)) = 0$, hence $h^2(\mathbb{P}^2, \mathcal{F}(-1)) = 0$. Further, the vanishing of $h^0(\mathbb{P}^2, \mathcal{F}(-1))$ implies $h^0(\mathbb{P}^2, \mathcal{F}(-2)) = 0$ and, since $b \geq 5$, we also have

$$h^2(\mathbb{P}^2, \mathcal{F}(-2)) = h^0(\mathbb{P}^2, \mathcal{F}^*(-1)) = h^0(\mathbb{P}^2, \mathcal{F}(3-b)) = 0. \tag{25}$$

Again by Riemann-Roch, we obtain $h^1(\mathbb{P}^2, \mathcal{F}(-2)) = b-4$.

As in (25) we can check that $h^2(\mathbb{P}^2, \mathcal{F}) = 0$. Given a line L in $\mathbb{P}^2$, let us now tensor with $\mathcal{F}(t)$ the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_L \to 0.$$

By the vanishing we already proved, we infer $h^1(L, \mathcal{F}(-1)|_L) = 0$ and this easily implies $h^1(L, \mathcal{F}(t)|_L) = 0$ for any $t \geq 0$. Therefore we get $h^1(\mathbb{P}^2, \mathcal{F}) = 0$, that in turn yields $h^0(\mathbb{P}^2, \mathcal{F}) = b-2$.

We can now use Beilinson's theorem (see for instance [**Huy06**, Chapter 8.3]) in order to write down a free resolution of $\mathcal{F}$. The Beilinson table of $\mathcal{F}$, displaying the values of $h^j(\mathbb{P}^2, \mathcal{F}(-i))$, is as follows:

	$\mathcal{F}(-2)$	$\mathcal{F}(-1)$	$\mathcal{F}$
h^2	0	0	0
h^1	$b-4$	0	0
h^0	0	0	$b-2$

Table 2: The Beilinson table of $\mathcal{F}$

This gives the resolution of $\mathcal{F}$

$$0 \to H^1(\mathbb{P}^2, \mathcal{F}(-2)) \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to H^0(\mathbb{P}^2, \mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^2} \to \mathcal{F} \to 0,$$

hence the required resolution of $\mathcal{E}$. Setting

$$W := H^1(\mathbb{P}^2, \mathcal{F}(-2))^*, \quad U := H^0(\mathbb{P}^2, \mathcal{F}), \quad \mathbb{P}^2 = \mathbb{P}(V), \tag{26}$$

we see that $\mathcal{F}$ is a Steiner bundle. Stability follows immediately by Hoppe's criterion, see [**Hop84**, Lemma 2.6]. □

Corollary 2.13. *In cases* I, II *and* III *the triple planes* $f: X \longrightarrow \mathbb{P}^2$ *exist and* X *is a rational surface.*

Proof. Let us consider case I. By Theorem 2.12 we have $S^3\mathcal{E}^* \otimes \wedge^2\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1)^4$ which is globally generated, so the triple cover exists by Theorem 1.2. Using Proposition 2.1 we obtain

$$P_2(X) = h^0(\mathbb{P}^2, S^2\mathcal{E}^*(-6)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-4)^3) = 0,$$

hence Castelnuovo's Theorem ([**B83**, Chapter V]) implies that X is a rational surface. The proof in cases II and III is the same. □

2.3 The projective bundle associated with a triple plane

2.3.1 Triple planes and direct images

Let $f: X \to \mathbb{P}^2$ be a triple plane with $p_g = q = 0$ and Tschirnhausen bundle $\mathcal{E}$. We assume $b \geq 5$ and we set $\mathcal{F} := \mathcal{E}(b-2)$ as before; as shown in Theorem 2.12, $\mathcal{F}$ is a *Steiner bundle* of rank 2. The notation in this paragraph is borrowed from Subsection 1.4.

Theorem 1.3 implies that X is a Cartier divisor in $\mathbb{P}(\mathcal{F})$, such that the restriction of $p: \mathbb{P}(\mathcal{F}) \to \mathbb{P}^2$ to X is our covering map f. More precisely, the equality

$$S^3\mathcal{E}^* \otimes \wedge^2\mathcal{E} = S^3\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^2}(6-b) \tag{27}$$

shows that X lies in $|\mathcal{L}|$, with

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(3\xi + (6-b)\ell). \tag{28}$$

Let us consider now the morphism $q: \mathbb{P}(\mathcal{F}) \to \mathbb{P}(U) \simeq \mathbb{P}^{b-3}$. Setting

$$\mathcal{R} := q_*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}((6-b)\ell)),$$

the projection formula yields natural identifications

$$\begin{aligned} H^0(\mathbb{P}^2, S^3\mathcal{E}^* \otimes \wedge^2\mathcal{E}) &\simeq H^0(\mathbb{P}^2, S^3\mathcal{F}(6-b)) \\ &\simeq H^0(\mathbb{P}(\mathcal{F}), \mathcal{L}) \simeq H^0(\mathbb{P}^{b-3}, \mathcal{R}(3)). \end{aligned} \tag{29}$$

In order to get information on the sheaf $\mathcal{R}$, it is useful to consider the Koszul resolution of $\mathbb{P}(\mathcal{F})$ in $\mathbb{P}(V) \times \mathbb{P}(U) \simeq \mathbb{P}^2 \times \mathbb{P}^{b-3}$, which is given by

$$\wedge^\bullet (W^* \otimes \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^{b-3}}(-1,-1)) \to \mathcal{O}_{\mathbb{P}(\mathcal{F})} \to 0 \tag{30}$$

with $W^* = H^1(\mathbb{P}^2, \mathcal{F}(-2))$, see Proposition 1.6 and equation (26). We will write $\mathcal{K}_i$ for the image of the i-th differential

$$d_i : (\wedge^i W^*) \otimes \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^{b-3}}(-i,-i) \to (\wedge^{i-1} W^*) \otimes \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^{b-3}}(-i+1,-i+1)$$

of the complex (30). Moreover, we will often use the relation

$$R^i q_*(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^{b-3}}(n_1, n_2)) = H^i(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n_1)) \otimes \mathcal{O}_{\mathbb{P}^{b-3}}(n_2), \quad i \in \mathbb{N},\ n_1, n_2 \in \mathbb{Z}. \tag{31}$$

We finally define $Y \subset \mathbb{P}^{b-3}$ as the image of q; then the support of $\mathcal{R}$ is contained in Y. In §2.3.2 we shall see that, if $b \geq 6$, the morphism $q\colon \mathbb{P}(\mathcal{F}) \longrightarrow \mathbb{P}^{b-3}$ is generically injective, so $Y \subset \mathbb{P}^{b-3}$ is a (possibly singular) threefold which is generated by the 3-secant lines to the canonical curves of genus $g(H)$ representing in $\mathbb{P}^{b-3}$ the net $|H|$ inducing the triple cover. The threefold Y is defined by the 3×3 minors of the matrix N appearing in the resolution of $q_*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(\ell))$:

$$\mathcal{O}_{\mathbb{P}^{b-3}}(-1)^{b-4} \xrightarrow{N} \mathcal{O}^3_{\mathbb{P}^{b-3}} \to q_*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(\ell)) \to 0.$$

2.3.2 Adjunction maps and projective bundles

We use the notation of §1.4.1. Recall that the canonical line bundle of $\mathbb{P}(\mathcal{F})$ is

$$\omega_{\mathbb{P}(\mathcal{F})} \simeq \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-2\xi + (b-7)\ell), \tag{32}$$

see for instance [**H77**, Ex. 8.4 p. 253]. The following result provides now a link between the adjunction theory and the vector bundles techniques used in this paper.

Lemma 2.14. *Let $f : X \longrightarrow \mathbb{P}^2$ be a triple plane with $p_g(X) = q(X) = 0$. Then $q|_X$ coincides with the first adjoint map $\varphi_{|K_X+H|} : X \longrightarrow \mathbb{P}^{b-3}$ associated with the ample divisor H.*

Proof. Since H is ample, by Kodaira vanishing theorem we have $h^1(X, K_X+H) = h^2(X, K_X+H) = 0$, so Riemann-Roch theorem gives $h^0(X, K_X+H) = g(H) = b-2$. Therefore it suffices to show that

$$\omega_X \otimes \mathcal{O}_X(H) = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(\xi)|_X.$$

The adjunction formula, together with (28)and (32), yields

$$\omega_X = (\omega_{\mathbb{P}(\mathcal{F})} \otimes \mathcal{L})|_X = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(\xi - \ell)|_X.$$

Since $\ell|_X = \mathcal{O}_X(H)$, the claim follows. □

Lemma 2.15. *The morphism $q\colon \mathbb{P}(\mathcal{F}) \to \mathbb{P}^{b-3}$ contracts precisely the negative sections of the Hirzebruch surfaces of the form $\mathbb{P}(\mathcal{F}|_L)$, where L is an unstable line of $\mathcal{F}$. If $b \geq 6$ then q is birational onto its image $Y \subseteq \mathbb{P}^{b-3}$, which is a birationally ruled threefold of degree $\binom{b-4}{2}$.*

Proof. Let L be an unstable line of $\mathcal{F}$. We have $\mathcal{F}|_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L(b-4)$, so $\mathbb{P}(\mathcal{F}|_L)$ is isomorphic to the Hirzebruch surface $\mathbb{F}_{b-4}$. The divisor $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(\xi)$ cuts on $\mathbb{P}(\mathcal{F}|_L)$ the complete linear system $|\mathfrak{c}_0 + (b-4)\mathfrak{f}|$; therefore $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(\xi) \cdot \mathfrak{c}_0 = 0$, that is q contracts $\mathfrak{c}_0$. In particular, this means that the image of $\mathbb{P}(\mathcal{F}|_L)$ via q is a cone $S_{0,b-4} \subset \mathbb{P}^{b-3}$.

Conversely, assume that x_1 and x_2 are points of $\mathbb{P}(\mathcal{F})$ not separated by q. Since $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(\xi)$ is very ample when restricted to the fibres of $p\colon \mathbb{P}(\mathcal{F}) \to \mathbb{P}^2$, the points $p(x_1)$ and $p(x_2)$ must be distinct. Let L be the line joining $p(x_1)$ and $p(x_2)$. Restricting q to $S := \mathbb{P}(\mathcal{F}|_L)$, we see that L must be unstable for $\mathcal{F}$, since otherwise $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(\xi)|_S$ would be very ample on S, hence q would separate x_1 and x_2. Thus $S \simeq \mathbb{F}_{b-4}$ and moreover x_1 and x_2 must both lie on the negative section of S, because $q|_S$ is an embedding outside this section. The same argument goes through if x_2 and x_2 are infinitely near.

Finally, for $b \geq 6$ the subscheme $\mathcal{W}(\mathcal{F}_b)$ of unstable lines has positive codimension in $\mathbb{P}^{2*}$, see §1.4.2. Then q is birational onto its image. This says that the image Y is a birationally ruled threefold in $\mathbb{P}^{b-3}$ (of course for $b = 6$ the image is the whole $\mathbb{P}^3$).

After recalling $c_1(\mathcal{F}) = b-4$ and $c_2(\mathcal{F}) = \binom{b-3}{2}$, the formula for the degree follows from

$$\xi^3 = p^*(c_1(\mathcal{F})^2 - c_2(\mathcal{F}))\xi = (b-4)^2 - \binom{b-3}{2} = \binom{b-4}{2}.$$

□

Lemma 2.16. *Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(3\xi + (6-b)\ell)$ and let $\mathfrak{c}_0$ be the negative section of the Hirzebruch surface $\mathbb{P}(\mathcal{F}|_L)$, where L is an unstable line for $\mathcal{F}$. If $b \geq 7$, then $\mathfrak{c}_0$ is contained in the base locus of $|\mathcal{L}|$.*

Proof. The restriction of $\mathcal{L}$ to $\mathbb{P}(\mathcal{F}|_L)$ is a divisor C linearly equivalent to

$$3(\mathfrak{c}_0 + (b-4)\mathfrak{f}) + (6-b)\mathfrak{f} = 3\mathfrak{c}_0 + (2b-6)\mathfrak{f}.$$

We have $C\mathfrak{c}_0 = 3(4-b) + (2b-6) = 6-b$, so if $b \geq 7$ we have $C\mathfrak{c}_0 < 0$ and this in turn implies that $\mathfrak{c}_0$ is a component of C. Hence $\mathfrak{c}_0$ is contained in every divisor of $|\mathcal{L}|$. □

Let us come back now to our triple planes $f : X \to \mathbb{P}^2$.

Proposition 2.17. *If $b \geq 7$ then the first adjoint map $\varphi_{|K_X+H|}\colon X \longrightarrow \mathbb{P}^{b-3}$ is a birational morphism onto its image $X_1 \subset \mathbb{P}^{b-3}$. Furthermore, X_1 is a smooth surface and $\varphi_{|K_X+H|}$ contracts precisely the (-1)-curves E in X such that $HE = 1$. There is one precisely such curve for each unstable line of $\mathcal{F}$.*

Proof. By Lemma 2.15 the map $q\colon \mathbb{P}(\mathcal{F}) \to \mathbb{P}^{b-3}$ is birational onto its image and contracts precisely the negative sections of $\mathbb{P}(\mathcal{F}|_L)$, where L is an unstable line of $\mathcal{F}$; let E be such a section. In view of Lemma 2.14 we have $\varphi_{|K_X+H|} = q|_X$, and by Lemma 2.16 the curve E is contained in X. Finally, by projection over L we obtain $HE = 1$. By Lemma 2.15 and 2.16, the curves E are precisely the negatives sections $\mathfrak{c}_0$ of the Hirzebruch surfaces $\mathbb{P}(\mathcal{F}|_L)$, where L runs among unstable lines of $\mathcal{F}$. □

Remark 2.18. When $b \geq 7$, Proposition 2.17 will allow us to apply the iterated adjunction process described in Subsection 1.3 starting from $D = H$, even if H is ample but *not* very ample.

Remark 2.19. Proposition 3.11 shows that $\varphi_{|K_X+H|}$ is birational also for $b = 6$ (in this case, X is a cubic surface in $\mathbb{P}^3$ blown up at 9 points, and $\varphi_{|K_X+H|}$ is the blow-down morphism).

3 The classification in cases I, . . . , VII

3.1 Triple planes of type I

In this case the invariants are

$$K_X^2 = 8, \quad b = 2, \quad h = 1, \quad g(H) = 0$$

and the Tschirnhausen bundle splits as $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$. The existence of these triple planes follows from Corollary 2.13, whereas Proposition 3.1 below provides their complete classification.

Proposition 3.1. *Let $f : X \longrightarrow \mathbb{P}^2$ be a triple plane of type* I*. Then X is a cubic scroll $S_{1,2} \subset \mathbb{P}^4$ and f is the projection of X from a general line of $\mathbb{P}^4$.*

Proof. By Proposition 2.5 we know that R is very ample, so we can apply Theorem 1.4. Using Proposition 2.6 and the fact that $g(R) = g(B) = 0$ (note that $B \subset \mathbb{P}^2$ is a tri-cuspidal quartic curve by Proposition 2.11), we obtain

$$\begin{pmatrix} R^2 & K_X R \\ K_X R & K_X^2 \end{pmatrix} = \begin{pmatrix} 5 & -7 \\ -7 & 8 \end{pmatrix},$$

therefore no multiple of K_X can be effective (because $K_X R < 0$) and X is a rational surface (as predicted by Corollary 2.13). Moreover we get

$$\dim |K_X + R| = g(R) - 1 = -1,$$

that is $|K_X + R| = \varnothing$. The condition $K_X^2 = 8$ implies that the X is not isomorphic to $\mathbb{P}^2$, so it must be a rational normal scroll. There are two different kind of rational normal scrolls of dimension 2 and degree 5, namely

- $S_{1,4}$, that is $\mathbb{F}_3$ embedded in $\mathbb{P}^6$ via $|\mathfrak{c}_0 + 4\mathfrak{f}|$;
- $S_{2,3}$, that is $\mathbb{F}_1$ embedded in $\mathbb{P}^6$ via $|\mathfrak{c}_0 + 3\mathfrak{f}|$.

In the former case, using (21) we obtain $H = \mathfrak{c}_0 + 3\mathfrak{f}$, which is not ample on $\mathbb{F}_3$; so this case cannot occur. In the latter case we have $H = \mathfrak{c}_0 + 2\mathfrak{f}$, that is very ample and embeds $\mathbb{F}_1$ in $\mathbb{P}^4$ as a cubic scroll $S_{1,2}$. The triple cover is now obtained by taking the morphism to $\mathbb{P}^2$ associated with a general net of curves inside $|H|$, which corresponds to the projection of $S_{1,2}$ from a general line of $\mathbb{P}^4$. □

Remark 3.2. Another description of triple planes of Type I is the following. Let X' be the Veronese surface, embedded in the Grassmannian $\mathrm{Gr}(1, \mathbb{P}^3)$ as a surface of bidegree (3, 1), see [**G93**, Theorem 4.1 (a)]. There is a family of 1-secant planes to X'; projecting from one of these planes, we obtain a birational model of a triple plane $f : X \to \mathbb{P}^2$ of type I (in fact, X is the blow-up of X' at one point).

3.2 Triple planes of type II

In this case the invariants are

$$K_X^2 = 3, \quad b = 3, \quad h = 2, \quad g(H) = 1$$

and the Tschirnhausen bundle splits as $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$. The existence of these triple planes follows from Corollary 2.13, whereas Proposition 3.3 below provides their complete classification.

Proposition 3.3. *Let $f : X \longrightarrow \mathbb{P}^2$ be a triple plane of type* II. *Then X is a cubic surface in $\mathbb{P}^3$ and f is the projection from a general point $p \notin X$. The branch locus B is a sextic plane curve with six cusps lying on a conic.*

Proof. By Proposition 2.9, the divisor $D := K_X + 2H$ is very ample so we can apply Theorem 1.4. We have

$$\begin{pmatrix} D^2 & K_X D \\ K_X D & K_X^2 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix},$$

hence the map $\varphi_{|D|} : X \longrightarrow \mathbb{P}^3$ is an isomorphism onto a smooth cubic surface, see Proposition 2.9. The statement about the position of the cusps in the branch locus is a well-known classical result, see [**Zar29**]. □

Remark 3.4. Other descriptions of triple planes of type II are the following.

- Let X' be a smooth Del Pezzo surface of degree 5, embedded in $\mathrm{Gr}(1, \mathbb{P}^3)$ as a surface of bidegree (3, 2), see [**G93**, Theorem 4.1 (b)]. There is a family of 2-secant planes to X'; projecting from one of these planes, we obtain a birational model of a triple plane $f : X \to \mathbb{P}^2$ of type II (in fact, X is the blow-up of X' at two points).
- Let X' be a smooth Del Pezzo surface of degree 6, embedded in $\mathrm{Gr}(1, \mathbb{P}^3)$ as a surface of bidegree (3, 3), see [**G93**, Theorem 4.1 (d)]. There is a family of 3-secant planes to X'; projecting from one of these planes, we obtain a birational model of a triple plane $f : X \to \mathbb{P}^2$ of type II (in fact, X is the blow-up of X' at three points).

3.3 Triple planes of type III

In this case the invariants are

$$K_X^2 = -1, \quad b = 4, \quad h = 4, \quad g(H) = 2$$

and the Tschirnhausen bundle splits as $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$. The existence of these triple planes follows from Corollary 2.13, whereas Proposition 3.5 below provides their complete classification.

Proposition 3.5. *Let* $f : X \longrightarrow \mathbb{P}^2$ *be a triple plane of type* III. *Then* X *is a blow-up at 9 points* $\sigma : X \to \mathbb{F}_n$ *of a Hirzebruch surface* $\mathbb{F}_n$*, with* $n \in \{0, 1, 2, 3\}$*, and*

$$H = 2\mathfrak{c}_0 + (n+3)\mathfrak{f} - \sum_{i=1}^{9} E_i. \tag{33}$$

Proof. By Proposition 2.9, the divisor $D := K_X + 2H$ is very ample. We have

$$\begin{pmatrix} D^2 & K_X D \\ K_X D & K_X^2 \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ -3 & -1 \end{pmatrix},$$

in particular $K_X D < 0$ shows that X is a rational surface. The morphism $\varphi_{|D|} : X \longrightarrow X_1 \subset \mathbb{P}^5$ is an isomorphism of X onto its image X_1, which is a surface of degree 7 with $K_{X_1}^2 = -1$. Embedded projective varieties of degree at most 7 are classified in [**Io84**]; in particular, the table at page 148 of that paper shows that X_1 is a blow-up at 9 points $\sigma : X_1 \to \mathbb{F}_n$, with $n \in \{0, 1, 2, 3\}$, and that

$$D = 2\mathfrak{c}_0 + (n+4)\mathfrak{f} - \sum_{i=1}^{9} E_i.$$

Using $2H = D - K_X$, we now obtain (33). □

Remark 3.6. When $n = 0$, the surface X is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at 9 points and a birational model of the triple cover $f : X \longrightarrow \mathbb{P}^2$ is obtained by using the curves of bidegree (2, 3) passing through these points, since (33) becomes $H = 2L_1 + 3L_2 - \sum_{i=1}^{9} E_i$.

When $n = 1$, since $\mathbb{F}_1$ is the blow-up of the plane at one point, we see from (33) that X can be also seen as the blow-up of $\mathbb{P}^2$ at 10 points and that $H = 4L - 2E_{10} - \sum_{i=1}^{9} E_i$.

Another description of triple planes of type III is the following. Let X' be a Castelnuovo surface with $K_{X'}^2 = 2$, embedded in $\mathrm{Gr}(1, \mathbb{P}^3)$ as a surface of bidegree (3, 3), see [**G93**, Theorem 4.1 (*e*)]. So there is a family of 3-secant planes to X'; projecting from one of these planes, we obtain a birational model of a triple cover $f : X \to \mathbb{P}^2$ of type III (in fact, X is the blow-up of X' at three points).

3.4 Triple planes of type IV

In this case the invariants are

$$K_X^2 = -4, \quad b = 5, \quad h = 7, \quad g(H) = 3.$$

By Theorem 2.12, the resolution of $\mathcal{F} = \mathcal{E}(3)$ is

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^3 \longrightarrow \mathcal{F} \longrightarrow 0,$$

hence $\mathcal{F} \simeq T_{\mathbb{P}^2}(-1)$ and (27) implies that $S^3\mathcal{E}^* \otimes \wedge^2\mathcal{E}$ is isomorphic to $S^3(T_{\mathbb{P}^2}(-1)) \otimes \mathcal{O}_{\mathbb{P}^2}(1)$, which is globally generated. By Theorem 1.2 this ensures the existence of triple planes of type IV, whereas Proposition 3.7 below provides their complete classification.

Proposition 3.7. *Let* $f : X \longrightarrow \mathbb{P}^2$ *be a triple plane of type* IV. *Then* X *is a blow-up* $X = \mathbb{P}^2(p_1, \ldots, p_{13})$ *of* $\mathbb{P}^2$ *at* 13 *points which impose only* 12 *conditions on plane quartic curves. Moreover,*

$$H = 4L - \sum_{i=1}^{13} E_i. \tag{34}$$

Proof. We give two proofs. The first one is based on the adjunction theory, the second one on vector bundles techniques.

First proof of Proposition 3.7. Proposition 2.9 shows that the divisor $D := K_X + 2H$ is very ample, so by Proposition 2.9 the first adjunction map

$$\varphi_1 := \varphi_{|K_X+D|} : X \longrightarrow X_1 \subset \mathbb{P}^5$$

is a birational morphism onto a smooth surface X_1 whose intersection matrix is

$$\begin{pmatrix} (D_1)^2 & K_{X_1}D_1 \\ K_{X_1}D_1 & (K_{X_1})^2 \end{pmatrix} = \begin{pmatrix} 4 & -6 \\ -6 & -4+\alpha_1 \end{pmatrix}.$$

In particular $K_{X_1}D_1 < 0$ shows that X_1 (and so X) is a rational surface. We can now check that the adjoint linear system $|K_{X_1} + D_1|$ is empty, so by Theorem 1.4 the surface X_1 is either a rational normal scroll (and in this case $\alpha_1 = 12$) or $\mathbb{P}^2$ (and in this case $\alpha_1 = 13$). Let us exclude the former case. There are two types of smooth quartic rational normal scroll surfaces: $S_{2,2}$, namely $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^5$ by $|L_1 + 2L_2|$, and $S_{1,3}$, namely $\mathbb{F}_2$ embedded in $\mathbb{P}^5$ by $|\mathfrak{c}_0 + 3\mathfrak{f}|$. If $X_1 = \mathbb{P}^1 \times \mathbb{P}^1$ we obtain

$$2H = 5L_1 + 6L_2 - \sum_{i=1}^{12} 2E_i,$$

whereas if $X_1 = \mathbb{F}_2$ we obtain

$$2H = 5\mathfrak{c}_0 + 11\mathfrak{f} - \sum_{i=1}^{12} 2E_i$$

In both cases we have a contradiction, since H must be a divisor with integer coefficients.

It follows that $(X_1, D_1) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, hence $\alpha_1 = 13$ and φ_1 contracts exactly 13 exceptional lines, i.e. X is isomorphic to the blow-up of $\mathbb{P}^2$ at 13 points. Therefore we have

$$X = \mathbb{P}^2(p_1, \ldots, p_{13}), \quad D = 5L - \sum_{i=1}^{13} E_i,$$

which implies $H = 4L - \sum_{i=1}^{13} E_i$.

Finally, since $h^0(X, \mathcal{O}_X(H)) = 3$, it follows that the 13 points $p_1, \ldots, p_{13}$ impose only 12 conditions on plane quartic curves. This completes our first proof.

Second proof of Proposition 3.7. In case IV, the 3-dimensional vector space $U = H^0(\mathbb{P}^2, \mathcal{F})$ is naturally identified with V^* and $\mathbb{P}(\mathcal{F})$ is the point-line incidence correspondence in $\mathbb{P}^2 \times \mathbb{P}^{2*}$, namely a smooth hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^{2*}$, whose Koszul resolution is

$$0 \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^{2*}}(-1,-1) \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^{2*}} \to \mathcal{O}_{\mathbb{P}(\mathcal{F})} \to 0. \tag{35}$$

Twisting (35) by $p^*(\mathcal{O}_{\mathbb{P}^2}(1)) = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^{2*}}(1, 0)$, applying the functor q_* and using (31) we obtain

$$0 \to \mathcal{O}_{\mathbb{P}^{2*}}(-1) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \otimes \mathcal{O}_{\mathbb{P}^{2*}} \to q_*\big(p^*(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{F})}\big) \to 0,$$

so Euler sequence yields

$$\mathcal{R} = q_*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(\ell)) = q_*\big(p^*(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{F})}\big) \simeq T_{\mathbb{P}^{2*}}(-1)$$

and equality (29) implies

$$H^0(\mathbb{P}^2, S^3\mathcal{E}^* \otimes \wedge^2\mathcal{E}) = H^0(\mathbb{P}^{2*}, \mathcal{R}(3)) = H^0(\mathbb{P}^{2*}, T_{\mathbb{P}^{2*}}(2)).$$

Furthermore we have $\mathcal{R}(3) = q_*\mathcal{L}$, where $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(3\xi + \ell)$, and our triple plane X is a smooth divisor in the complete linear system $|\mathcal{L}|$, see (28). Since a global section of $\mathcal{L}$ corresponds to a non-zero morphism $\mathcal{O}_{\mathbb{P}(\mathcal{F})} \to \mathcal{L}$, we obtain a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-3\xi) \to \mathcal{L}(-3\xi) \to \mathcal{O}_X(H) \to 0, \tag{36}$$

and, taking the direct image via q, we get

$$0 \to \mathcal{O}_{\mathbb{P}^{2*}}(-3) \to T_{\mathbb{P}^{2*}}(-1) \to \mathcal{I}_Z(4) \longrightarrow 0, \tag{37}$$

where Z is a 0-dimensional subscheme of $\mathbb{P}^{2*}$ of length $\ell(Z) = c_2(T_{\mathbb{P}^{2*}}(2)) = 13$.

The inclusion $X \simeq \mathbb{P}(\mathcal{O}_X(H)) \hookrightarrow \mathbb{P}(\mathcal{F})$ corresponds to the surjection $\mathcal{L}(-3\xi) \to \mathcal{O}_X(H)$ in (36); then (37) shows that X can be identified with $\mathbb{P}(\mathcal{I}_Z(4))$, embedded in in $\mathbb{P}(T_{\mathbb{P}^{2*}}(-1))$ via the surjection $T_{\mathbb{P}^{2*}}(-1) \to \mathcal{I}_Z(4)$. Hence a birational model for the triple cover map $f : X \to \mathbb{P}^2$ is the rational map $\mathbb{P}^{2*} \dashrightarrow \mathbb{P}^2$ given by the linear system of (dual) quartics through Z.

In fact, exact sequence (37) also implies

$$\begin{aligned} h^0(\mathbb{P}^{2*}, \mathcal{I}_Z(4)) &= h^0(\mathbb{P}^{2*}, T_{\mathbb{P}^{2*}}(-1)) = 3, \\ h^1(\mathbb{P}^{2*}, \mathcal{I}_Z(4)) &= h^2(\mathbb{P}^{2*}, \mathcal{O}_{\mathbb{P}^{2*}}(-3)) = 1, \end{aligned} \tag{38}$$

so the 13 points in the support of Z actually impose only 12 conditions on quartic plane curves. This completes our second proof. □

Remark 3.8. The the Cayley-Bacharach condition (38) on the points on Z implies that the unique non-trivial extension of $\mathcal{I}_Z(4)$ by $\mathcal{O}_{\mathbb{P}^{2*}}(-3)$ is the one given by (37). Indeed, by Serre duality we have

$$\mathrm{Ext}^1(\mathcal{I}_Z(4), \mathcal{O}_{\mathbb{P}^{2*}}(-3)) = \mathrm{Ext}^1(\mathcal{I}_Z(4), \omega_{\mathbb{P}^{2*}}) = H^1(\mathbb{P}^{2*}, \mathcal{I}_Z(4)) = \mathbb{C}.$$

Furthermore, the fact that such an extension is locally free is also a consequence of (38), see [**Fr98**, Theorem 12 p. 39].

Remark 3.9. Looking carefully at the second proof, we see that it works, in principle, only for the *general* surface X of type IV, where general means that it corresponds to a global section of $T_{\mathbb{P}^{2*}}(2)$ vanishing at precisely 13 distinct points (i.e., we are assuming that the scheme Z is reduced and of pure dimension 0). However, we see that sections with special vanishing behaviour cannot give rise to a smooth triple plane, because our first proof shows that we must actually have $\alpha_1 = 13$.

Remark 3.10. A *Bordiga surface* is a smooth surface of degree 6 in $\mathbb{P}^4$, given by the blow up of $\mathbb{P}^2$ at 10 points embedded by the linear system of quartics through them, see [**Ott95**, Capitolo 5]. Then (34) shows that a birational model of a triple plane $f : X \longrightarrow \mathbb{P}^2$ of type IV can be realized as the projection of a Bordiga surface from a 3-secant line.

Furthermore, contracting one of the exceptional divisors in the Bordiga surface, we obtain a rational surface X' with $K^2_{X'} = 0$ that can be embedded in $\mathrm{Gr}(1, \mathbb{P}^3)$ as a surface of bidegree (3, 4), see [**G93**, Theorem 4.1 (f)]. So there is a family of 4-secant planes to X'; projecting from one of these planes, we obtain another birational model of a triple plane $f : X \to \mathbb{P}^2$ of type IV (in fact, X is the blow-up of X' at four points).

3.5 Triple planes of type V

In this case the invariants are

$$K^2_X = -6, \quad b = 6, \quad h = 11, \quad g(H) = 4$$

and by Theorem 2.12 the twisted Tschirnhausen bundle $\mathcal{F}$ has a resolution of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^2 \xrightarrow{M} \mathcal{O}^4_{\mathbb{P}^2} \longrightarrow \mathcal{F} \longrightarrow 0. \tag{39}$$

It is not difficult to write down a resolution of $S^3\mathcal{E}^* \otimes \wedge^2 E = S^3\mathcal{F}$ and to check that it is globally generated. Hence triple planes $f : X \longrightarrow \mathbb{P}^2$ of type V do exist, and Proposition 3.11 below provides their complete classification.

Proposition 3.11. *Let $f : X \longrightarrow \mathbb{P}^2$ be a triple plane of type* V*. Then X is a blow-up $X = \mathbb{P}^2(p_1, \ldots, p_{15})$ of $\mathbb{P}^2$ at 15 points and*

$$H = 6L - \sum_{i=1}^{6} 2E_i - \sum_{j=7}^{15} E_j.$$

Proof. Let us give again two different proofs.

First proof of Proposition 3.11. By Proposition 2.9 the divisor $D := K_X + 2H$ is very ample, and by Proposition 2.9 the first adjunction map

$$\varphi_1 = \varphi_{|K_X+D|} \colon X \longrightarrow X_1 \subset \mathbb{P}^9$$

is birational onto its image X_1, whose intersection matrix is

$$\begin{pmatrix} (D_1)^2 & K_{X_1}D_1 \\ K_{X_1}D_1 & (K_{X_1})^2 \end{pmatrix} = \begin{pmatrix} 12 & -6 \\ -6 & -6+\alpha_1 \end{pmatrix}.$$

In particular $K_{X_1}D_1 < 0$ shows that X_1 (and so X) is a rational surface. Now we apply the second adjunction map $\varphi_2 \colon X_1 \longrightarrow X_2 \subset \mathbb{P}^3$, which is an isomorphism onto its image X_2 (Proposition 2.10), whose intersection matrix is

$$\begin{pmatrix} (D_2)^2 & K_{X_2}D_2 \\ K_{X_2}D_2 & (K_{X_2})^2 \end{pmatrix} = \begin{pmatrix} -6+\alpha_1 & -12+\alpha_1 \\ -12+\alpha_1 & -6+\alpha_1 \end{pmatrix}.$$

This shows that X_2 is a non-degenerate, smooth rational surface in $\mathbb{P}^3$, hence it is either a quadric surface or a cubic surface. If X_2 were a quadric then $(D_2)^2 = 2$, hence $\alpha_1 = 8$ and the intersection matrix would give $(K_{X_2})^2 = 2$, which is a contradiction. Therefore X_2 is a cubic surface S, hence $\alpha_1 = 9$. Moreover X_1 is isomorphic to X_2, so X is the blow-up of S at 9 points. It follows

$$X = \mathbb{P}^2(p_1, \ldots, p_{15}), \quad D = 9L - \sum_{i=1}^{6} 3E_i - \sum_{i=7}^{15} E_j,$$

which implies $H = 6L - \sum_{i=1}^{6} 2E_i - \sum_{i=7}^{15} E_j$. This ends our first proof.

Let us give now our second proof of Proposition 3.11. As a by-product, it will also show that the 9 points that we are blowing up on the cubic surface S are not in general position, but lie on the intersection of S with a twisted cubic curve.

Second proof of Proposition 3.11. Set $\mathbb{P}^3 = \mathbb{P}(U)$. The projective bundle $\mathcal{F}$ is the complete intersection of two divisors of bidegree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^3$, so the Koszul resolution is

$$0 \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(-2,-2) \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(-1,-1)^2 \xrightarrow{d_1} \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}(\mathcal{F})} \to 0. \tag{40}$$

Twisting (40) by $p^*(\mathcal{O}_{\mathbb{P}^2}(1)) = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1, 0)$ and splitting it into short exact sequences, we get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(-1,-2) \longrightarrow \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(0,-1)^2 \longrightarrow \widetilde{\mathcal{K}}_1 \longrightarrow 0,$$
$$0 \to \widetilde{\mathcal{K}}_1 \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1,0) \to \mathcal{O}_{\mathbb{P}(\mathcal{F})}(\ell) \to 0,$$

where $\widetilde{\mathcal{K}}_1 := \mathcal{K}_1 \otimes \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1,0)$ and $\mathcal{K}_1$ is the image of the first differential d_1 of the Koszul complex, see §2.3.1. Applying the functor q_* and using (31), we infer

$$q_*\widetilde{\mathcal{K}}_1 = \mathcal{O}_{\mathbb{P}^3}(-1)^2, \quad R^1q_*\widetilde{\mathcal{K}}_1 = 0,$$

obtaining

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^2 \xrightarrow{N} \mathcal{O}_{\mathbb{P}^3}^3 \longrightarrow q_*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(\ell)) \longrightarrow 0. \tag{41}$$

This shows that we can identify $q_*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(\ell))$ with $\mathcal{I}_C(2)$, the ideal sheaf of quadrics in $\mathbb{P}^3$ containing a twisted cubic C. Note that C is given by the vanishing of the three 2×2 minors of the matrix of linear forms N, and that the resolution (41) coincides with the one obtained by transposing (39) as explained in Subsection 1.4. Therefore we have $\mathcal{G} = q_*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(\ell))$, hence by Proposition 1.6 we infer

$$\mathbb{P}(\mathcal{F}) \simeq \mathbb{P}(\mathcal{G}) \simeq \mathbb{P}(\mathcal{I}_C(2)),$$

that is, $\mathbb{P}(\mathcal{F})$ is isomorphic to the blow-up of $\mathbb{P}^3$ along the twisted cubic C and the morphism $p\colon \mathbb{P}(\mathcal{F}) \to \mathbb{P}^2$ is induced by the net $|\mathcal{I}_C(2)|$.

We also get $\mathcal{R} \simeq q_* \mathcal{O}_{\mathbb{P}(\mathcal{F})} \simeq \mathcal{O}_{\mathbb{P}^3}$, so (29) yields

$$H^0(\mathbb{P}^2, S^3\mathcal{E}^* \otimes \wedge^2\mathcal{E}) = H^0(\mathbb{P}^3, \mathcal{R}(3)) = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)).$$

This means that the choice of the section η in Theorem 1.2 is given by the choice of a cubic surface $S \subset \mathbb{P}^3$. Moreover, from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-3\xi + \ell) \to \mathcal{O}_{\mathbb{P}(\mathcal{F})}(\ell) \to \mathcal{O}_X(H) \to 0$$

it follows that $X \simeq \mathbb{P}(\mathcal{O}_X(H))$ corresponds in the blown-up space $\mathbb{P}(\mathcal{F})$ to the strict transform of S.

Finally, the triple cover map $f : X \to \mathbb{P}^2$, being associated with $|\mathcal{O}_X(H)|$, is defined on S by the linear system of quadrics that containing the intersection $S \cap C$. Such an intersection consists of 9 points, hence we have 9 exceptional divisors $E_7, \ldots, E_{15}$ on X. Identifying S with $\mathbb{P}^2$ blown-up at 6 points, with exceptional divisors $E_1, \ldots, E_6$, its hyperplane section is given by $H_S = 3L - \sum_{i=1}^6 E_i$, so we can write

$$H = 2H_S - \sum_{j=7}^{15} E_j = 6L - \sum_{i=1}^{6} 2E_i - \sum_{j=7}^{15} E_j.$$

This ends our second proof. □

Remark 3.12. A birational model of the cover $f : X \longrightarrow \mathbb{P}^2$ is the projection of a hyperplane section T of a Palatini scroll from a 4-secant line. In fact, T is a surface of degree 7 in $\mathbb{P}^4$ and with $K_T^2 = -2$ (see [**Ott95**, Capitolo 5]), which is isomorphic to $\mathbb{P}^2$ blown up at 11 points and embedded in $\mathbb{P}^4$ by the complete linear system $|6L - \sum_{i=1}^6 2E_i - \sum_{j=7}^{11} E_j|$. Actually, this is the unique non-degenerate, rational surface of degree 7 in $\mathbb{P}^4$, see [**Ok84**, Theorems 4 and 6].

Contracting one of the exceptional divisors E_j in T, we obtain a rational surface X' with $K_{X'}^2 = -1$ that can be embedded in $\mathrm{Gr}(1, \mathbb{P}^3)$ as a surface of bidegree (3, 5), see [**G93**, Theorem 4.1 (g)]. So there is a family of 5-secant planes to X'; projecting from one of these planes, we obtain a birational model of a triple cover $f : X \to \mathbb{P}^2$ of type V (in fact, X is the blow-up of X' at five points).

Remark 3.13. Triple planes of types I, ..., V were previously considered (using completely different, "classical" methods based on synthetic projective geometry) by Du Val in [**DuVal33**].

3.6 Triple planes of type VI

In this case the invariants are

$$K^2 = -7, \quad b = 7, \quad h = 16, \quad g(H) = 5$$

and by Theorem 2.12 the twisted Tschirnhausen bundle $\mathcal{F}$ has a resolution of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^3 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}^5 \longrightarrow \mathcal{F} \longrightarrow 0. \tag{42}$$

The existence and classification of triple planes of type VI are established in Proposition 3.14 below.

Proposition 3.14. *Let $f : X \longrightarrow \mathbb{P}^2$ be a triple plane of type* VI. *Then the morphism $q\colon \mathbb{P}(\mathcal{F}) \longrightarrow \mathbb{P}^4$ is birational onto its image, which is a determinantal cubic threefold $Y \subset \mathbb{P}^4$. The surface X is the blow-up of a Bordiga surface $X_1 \subset Y$ at the six nodes of Y (that belong to X_1). So X is a rational surface; more precisely, it can be seen as the blow-up of $\mathbb{P}^2$ at* 16 *points and the net $|H|$ defining the triple cover f is given by*

$$H = 7L - \sum_{i=1}^{10} 2E_i - \sum_{j=11}^{16} E_j. \tag{43}$$

Proof. Set $\mathbb{P}^4 = \mathbb{P}(U)$. By the results of §2.3.1, the surface X corresponds to a global section

$$\eta \in H^0(\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(3\xi - \ell)) \simeq H^0(\mathbb{P}^4, \mathcal{R}(3)),$$

where $\mathcal{R} = q_*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(-\ell))$. The projective bundle $\mathbb{P}(\mathcal{F})$ is a complete intersection of three divisors of bidegree (1, 1) in $\mathbb{P}^2 \times \mathbb{P}^4$. Tensoring the Koszul resolution (30) of $\mathcal{O}_{\mathbb{P}(\mathcal{F})}$ inside $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^4}$ with $p^*(\mathcal{O}_{\mathbb{P}^2}(-1)) = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^4}(-1, 0)$ and splitting it into short exact sequences, we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^4}(-4, -3) \longrightarrow \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^4}(-3, -2)^3 \longrightarrow \widetilde{\mathcal{K}}_2 \longrightarrow 0, \tag{44}$$

$$0 \longrightarrow \widetilde{\mathcal{K}}_2 \longrightarrow \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^4}(-2, -1)^3 \longrightarrow \widetilde{\mathcal{K}}_1 \longrightarrow 0, \tag{45}$$

$$0 \longrightarrow \widetilde{\mathcal{K}}_1 \longrightarrow \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^4}(-1, 0) \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-\ell) \longrightarrow 0, \tag{46}$$

where $\widetilde{\mathcal{K}}_i := \mathcal{K}_i \otimes \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^4}(-1, 0)$ and $\mathcal{K}_i$ denotes the image of the i-th differential of the Koszul complex, see §2.3. Applying the functor q_* to (44) and using (31), we deduce $q_*\widetilde{\mathcal{K}}_2 = 0$ and we get

$$0 \longrightarrow R^1 q_*\widetilde{\mathcal{K}}_2 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-3)^3 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2)^3 \longrightarrow R^2 q_*\widetilde{\mathcal{K}}_2 \longrightarrow 0. \tag{47}$$

By (46) the sheaf $\widetilde{\mathcal{K}}_1$ injects into $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^4}(-1, 0)$, so we have $q_*\widetilde{\mathcal{K}}_1 = 0$. Therefore, applying q_* to (45), we get

$$R^1 q_*\widetilde{\mathcal{K}}_2 = 0, \quad R^1 q_*\widetilde{\mathcal{K}}_1 = R^2 q_*\widetilde{\mathcal{K}}_2. \tag{48}$$

Finally, applying the functor q_* to (46) we infer

$$\mathcal{R} = q_*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(-\ell)) = R^1 q_*\widetilde{\mathcal{K}}_1. \tag{49}$$

Using (48) and (49), the exact sequence (47) becomes

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-3)^3 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2)^3 \longrightarrow \mathcal{R} \longrightarrow 0,$$

that can be rewritten as

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}^3 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(1)^3 \to \mathcal{R}(3) \longrightarrow 0. \tag{50}$$

We can interpret this by looking at the transposition of the sequence (42) as explained in Subsection 1.4. We obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^3 \xrightarrow{N} \mathcal{O}_{\mathbb{P}^4}^3 \longrightarrow \mathcal{G} \longrightarrow 0, \tag{51}$$

where N is a 3×3 matrix of linear forms, hence the support of $\mathcal{G}$ is the determinantal cubic threefold $Y \subset \mathbb{P}^4$ defined by $\det N = 0$. Note that the support of $\mathcal{R}$ is also Y, because the self-duality of the Koszul complex implies

$$\mathcal{R} = \mathcal{G}^* = \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{G}(3), \mathcal{O}_{\mathbb{P}^4}),$$

where the second equality follows from Grothendieck duality, see [**Gr**]. The threefold Y is singular, expectedly at six points (by Porteous formula, see [**ACGH85**, Chapter II]), and its desingularization is isomorphic to $\mathbb{P}(\mathcal{F})$.

According to §1.4.2, we can now distinguish two cases according to whether $\mathcal{F}$ has either six or infinitely many unstable lines.

(*i*) Assume that $\mathcal{F}$ has six unstable lines. In this case, the singularities of Y are related to the Steiner bundle $\mathcal{F}$ by Lemma 2.15. Indeed, $q\colon \mathbb{P}(\mathcal{F}) \to \mathbb{P}^4$ contracts precisely the sections of negative self-intersection lying in the Hirzebruch surfaces of the form $\mathbb{P}(\mathcal{F}|_L)$, where L is an unstable line for $\mathcal{F}$, and each such a section gives an ordinary double point of Y. These double points are precisely the images of the exceptional lines of X contracted by the first adjunction map $\varphi_{|K_X+H|}\colon X \to \mathbb{P}^4$, see Proposition 2.17, so X is the blow-up of the smooth surface $X_1 := q(X) \subset Y$ along these points.

If $L \subset \mathbb{P}^2$ is a line, then $T_L := q(p^{-1}(L))$ is a ruled surface of degree 3 in $\mathbb{P}^4$, contained in Y. The complete linear system $|T_L|$ in Y has dimension 2; its general element is a smooth scroll of type $S_{1,2}$, and there are six singular elements, corresponding to the six jumping lines of $\mathcal{F}$, consisting of scrolls of type $S_{0,3}$, namely cones over a twisted cubic curve. The six vertices of these cones are precisely the six nodes of Y, and they provide the base locus of $|T_L|$. If we write

$$N = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix},$$

then the net $|T_L|$ is generated by the three determinantal surfaces

$$\begin{aligned} T_1 &= \left\{ v \in \mathbb{P}^4 \mid \operatorname{rank} \begin{pmatrix} n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix} \leq 1 \right\}, \\ T_2 &= \left\{ v \in \mathbb{P}^4 \mid \operatorname{rank} \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{31} & n_{32} & n_{33} \end{pmatrix} \leq 1 \right\}, \\ T_3 &= \left\{ v \in \mathbb{P}^4 \mid \operatorname{rank} \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \end{pmatrix} \leq 1 \right\}. \end{aligned} \tag{52}$$

Let us consider now a non-zero global section $\eta \colon \mathcal{O}_{\mathbb{P}^4} \to \mathcal{R}(3)$ of $\mathcal{R}(3)$, whose cokernel we denote by $\mathcal{H}$. The section η lifts to a map $\mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4}(1)^3$, so by (50) we get an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^4}^4 \xrightarrow{\Sigma} \mathcal{O}_{\mathbb{P}^4}(1)^3 \to \mathcal{H} \longrightarrow 0. \tag{53}$$

The sheaf $\mathcal{H}$ is supported precisely on the surface $X_1 \subset \mathbb{P}^4$. More precisely, this surface is defined by the vanishing of the 3×3 minors of the 3×4 matrix Σ of linear forms appearing in (53), hence it is a Bordiga surface of degree 6, see [**Ott95**, Capitolo 5]. As explained in Remark 3.10, we can see X_1 as the blow-up of $\mathbb{P}^2$ at 10 points, with exceptional divisors $E_1, \ldots, E_{10}$, embedded in $\mathbb{P}^4$ by the linear system $|4L - \sum_{i=1}^{10} E_i|$. On the other hand, by Proposition 2.17 the first adjoint map $\varphi := \varphi_{|K_X + H|} \colon X \to X_1$ is a birational morphism, contracting precisely the six exceptional divisors $E_{11}, \ldots, E_{16}$ on X coming from the the blow-up of X_1 at the six nodes of Y. Hence we obtain

$$\begin{aligned} K_X &= \varphi^* K_{X_1} + \sum_{j=11}^{16} E_j = \varphi^* \Big(-3L + \sum_{i=1}^{10} E_i \Big) + \sum_{j=11}^{16} E_j \quad \text{and} \\ K_X + H &= \varphi^* \mathcal{O}_{X_1}(1) = \varphi^* \Big(4L - \sum_{i=1}^{10} E_i \Big), \end{aligned}$$

so (43) follows.

(*ii*) We assume now that $\mathcal{F}$ has infinitely many unstable lines, i.e. that it is a Schwarzenberger bundle. Up to a change of coordinates, the matrix M defining $\mathcal{F}$ is in this case given by (9), so, using Remark 1.7, one easily finds that the matrix N is

$$N = \begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_4 \end{pmatrix}. \tag{54}$$

The singular locus of Y is the determinantal variety given by the vanishing of the 2×2 minors of the matrix N, and this is now a rational normal curve of degree four $C_4 \subset \mathbb{P}^4$. This curve is also the base locus of the net $|T_L|$, and the morphism $q \colon \mathbb{P}(\mathcal{F}) \to Y$ is the blow-up along C_4.

By [**Val00a**] we have

$$h^0(\mathbb{P}^2, S^2\mathcal{F}(-2)) = 1. \tag{55}$$

This can be explained geometrically as follows. Looking at (52) and (54), we see that the quadric hypersurface $Q \subset \mathbb{P}^4$ defined by $z_1 z_3 - z_2^2 = 0$ contains both the surfaces T_1 and T_3. Therefore $Q' := q^{-1}Q$ is a relative quadric in $\mathbb{P}(\mathcal{F})$ that provides a non-zero global section of $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(2\xi - 2\ell)$, hence a non-zero global section of $S^2\mathcal{F}(-2)$. In addition, one checks that Q' is the exceptional divisor of the blow-up $q \colon \mathbb{P}(\mathcal{F}) \to Y$, so (55) follows.

Now, notice that the multiplication of global sections gives an inclusion

$$H^0(\mathbb{P}^2, S^2\mathcal{F}(-2)) \otimes H^0(\mathbb{P}^2, \mathcal{F}(1)) \subseteq H^0(\mathbb{P}^2, S^3\mathcal{F}(-1)). \tag{56}$$

On the other hand, we can compute

$$h^0(\mathbb{P}^2, \mathcal{F}(1)) = 12, \quad h^0(\mathbb{P}^2, S^3\mathcal{F}(-1)) = 12. \tag{57}$$

Indeed, the first equality in (57) is just obtained twisting (42) by $\mathcal{O}_{\mathbb{P}^2}(1)$ and taking global sections. For the second equality, we tensor the third symmetric product of the exact sequence (57) with $\mathcal{O}_{\mathbb{P}^2}(-1)$, obtaining

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-4) \to \mathcal{O}_{\mathbb{P}^2}(-3)^{15} \to \mathcal{O}_{\mathbb{P}^2}(-2)^{45} \xrightarrow{r_1} \mathcal{O}_{\mathbb{P}^2}(-1)^{35} \xrightarrow{r_0} S^3\mathcal{F}(-1) \to 0.$$

Taking cohomology, we get

$$H^i(\mathbb{P}^2, S^3\mathcal{F}(-1)) \simeq H^{i+1}(\mathbb{P}^2, \ker r_0) \simeq H^{i+2}(\mathbb{P}^2, \ker r_1)$$

for all i, which implies $H^i(\mathbb{P}^2, S^3\mathcal{F}(-1)) = 0$ for $i > 0$. Then

$$h^0(\mathbb{P}^2, S^3\mathcal{F}(-1)) = \chi(\mathbb{P}^2, S^3\mathcal{F}(-1)) = 12.$$

By (55) and (57) it follows that the inclusion in (56) is actually an equality. Geometrically, this means that all the global sections of $S^3\mathcal{F}(-1)$ vanish along the relative quadric Q', that is the surface X is the union of a relative quadric and a relative plane. Hence we cannot have an irreducible triple cover with Tschirnhausen bundle $\mathcal{F}$, and this case must be excluded.

□

Remark 3.15. Another way to describe triple planes of type VI is the following. Let X' be the blow-up of $\mathbb{P}^2$ at 10 points, embedded in $\mathrm{Gr}(1, \mathbb{P}^3)$ as a surface of bidegree (3, 6) via the complete linear system $|7L - \sum_{i=1}^{10} 2E_i|$, see [**G93**, Theorem 4.2 (i)]. There is a family of 6-secant planes to X'; projecting from one of these planes, we obtain a birational model of the triple cover $f: X \to \mathbb{P}^2$ (in fact, X is the blow-up of X' at six points).

Remark 3.16. Triple planes of types VI were previously considered (using completely different, "classical" methods) by Du Val in [**DuVal35**].

3.7 Triple planes of type VII

In this case we have

$$K_X^2 = -7, \quad b = 8, \quad h = 22, \quad g(H) = 6.$$

and by Theorem 2.12 the twisted Tschirnhausen bundle $\mathcal{F}$ has a resolution of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^4 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}^6 \longrightarrow \mathcal{F} \longrightarrow 0. \tag{58}$$

By Remark 2.18, we can start the adjunction process on X by using the first adjoint divisor $K_X + H$. According to Subsection 1.3, we denote by α_n the number of exceptional curves contracted by the n-th adjunction map $\varphi_n : X_{n-1} \to X_n$. Recall that α_1 is precisely the number of unstable lines of the twisted Tschirnhausen bundle $\mathcal{F}$, see §2.3.2.

3.7.1 The cases for type VII

Proposition 3.17. *If $f: X \longrightarrow \mathbb{P}^2$ is a triple plane of type* VII*, then X belongs to the following list. The cases marked with $(*)$ do actually exist.*

(VII.1a) $\alpha_1 = 1$, $\alpha_2 = 14$: *X is the blow-up at* 15 *points of a Hirzebruch surface* $\mathbb{F}_n$, *with* $n \in \{0, 2\}$, *and*

$$H = 5\mathfrak{c}_0 + \left(\frac{5}{2}n + 6\right)\mathfrak{f} - \sum_{i=1}^{14} 2E_i - E_{15};$$

(VII.1b)(∗) $\alpha_1 = 1$, $\alpha_2 = 15$: *X is the blow-up of* $\mathbb{P}^2$ *at* 16 *points and*

$$H = 8L - \sum_{i=1}^{15} 2E_i - E_{16};$$

(VII.2)(∗) $\alpha_1 = 2$: *X is the blow-up of* $\mathbb{P}^2$ *at* 16 *points and*

$$H = 9L - \sum_{i=1}^{4} 3E_i - \sum_{j=5}^{14} 2E_j - \sum_{k=15}^{16} E_k;$$

(VII.3)(∗) $\alpha_1 = 3$: *X is the blow-up of* $\mathbb{P}^2$ *at* 16 *points and*

$$H = 10L - 4E_1 - \sum_{i=2}^{7} 3E_i - \sum_{j=8}^{13} 2E_j - \sum_{k=14}^{16} E_k;$$

(VII.4a) $\alpha_1 = 4$, $\alpha_2 = 2$: *X is the blow-up of* $\mathbb{F}_n$ (*with* $n \in \{0, 1, 2, 3\}$) *at* 15 *points and*

$$H = 6\mathfrak{c}_0 + (3n + 8)\mathfrak{f} - \sum_{i=1}^{9} 3E_i - \sum_{j=10}^{11} 2E_j - \sum_{k=12}^{15} E_k;$$

(VII.4b)(∗) $\alpha_1 = 4$, $\alpha_2 = 3$: *X is the blow-up of* $\mathbb{P}^2$ *at* 16 *points and*

$$H = 10L - \sum_{i=1}^{9} 3E_i - \sum_{j=10}^{12} 2E_j - \sum_{k=13}^{16} E_k;$$

(VII.4c) $\alpha_1 = 4$, $\alpha_2 = 4$: *X is the blow-up of* $\mathbb{P}^2$ *at* 16 *points and*

$$H = 12L - \sum_{i=1}^{7} 4E_i - 3E_8 - \sum_{j=9}^{12} 2E_j - \sum_{k=13}^{16} E_k;$$

(VII.5a) $\alpha_1 = 5$, $\alpha_2 = 0$: *X is the blow-up of* $\mathbb{P}^1 \times \mathbb{P}^1$ *at* 15 *points, and*

$$H = 7L_1 + 7L_2 - \sum_{i=1}^{10} 3E_i - \sum_{j=11}^{15} E_j;$$

(VII.5b)(∗) $\alpha_1 = 5$, $\alpha_2 = 1$: *X is the blow-up of* $\mathbb{P}^2$ *at* 16 *points and*

$$H = 12L - \sum_{i=1}^{6} 4E_i - \sum_{j=7}^{10} 3E_j - 2E_{11} - \sum_{k=12}^{16} E_k;$$

(VII.6)(∗) $\alpha_1 = 6$: *X is the blow-up of* $\mathbb{P}^2$ *at* 16 *points and*

$$H = 13L - \sum_{i=1}^{10} 4E_i - \sum_{j=11}^{16} E_j;$$

(VII.7)$(*)$ $\alpha_1 = 7$: *X is the blow-up of an Enriques surface at 7 points.*

Proof. We have a birational morphism

$$\varphi_{|K_X+H|} : X \to X_1 \subset \mathbb{P}^5$$

and an intersection matrix

$$\begin{pmatrix} (D_1)^2 & K_{X_1}D_1 \\ K_{X_1}D_1 & (K_{X_1})^2 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & -7+\alpha_1 \end{pmatrix}.$$

By Hodge Index Theorem we infer $0 \leq \alpha_1 \leq 7$. Let us consider separately the different cases.

• $\boldsymbol{\alpha_1 = 0}$. The second adjunction map gives a pair (X_2, D_2), such that the intersection matrix on the surface $X_2 \subset \mathbb{P}^5$ is

$$\begin{pmatrix} (D_2)^2 & K_{X_2}D_2 \\ K_{X_2}D_2 & (K_{X_2})^2 \end{pmatrix} = \begin{pmatrix} 3 & -7 \\ -7 & -7+\alpha_2 \end{pmatrix}.$$

This gives a contradiction, since a smooth surface of degree 3 in $\mathbb{P}^5$ is necessarily contained in a hyperplane. Hence the case $\alpha_1 = 0$ cannot occur.

• $\boldsymbol{\alpha_1 = 1}$. The second adjunction map gives a pair (X_2, D_2), such that the intersection matrix on the surface $X_2 \subset \mathbb{P}^5$ is

$$\begin{pmatrix} (D_2)^2 & K_{X_2}D_2 \\ K_{X_2}D_2 & (K_{X_2})^2 \end{pmatrix} = \begin{pmatrix} 4 & -6 \\ -6 & -6+\alpha_2 \end{pmatrix}.$$

A smooth, linearly normal surface of degree 4 in $\mathbb{P}^5$ is either a rational scroll or the Veronese surface. In the former case we have $(K_{X_2})^2 = 8$, hence $\alpha_2 = 14$ and, using the classification of rational scrolls in $\mathbb{P}^5$ (see the proof of Proposition 3.7), we get **(VII.1a)**. In the latter case we have $(K_{X_2})^2 = 9$, hence $\alpha_2 = 15$. This gives **(VII.1b)**.

• $\boldsymbol{\alpha_1 = 2}$. The second adjunction map gives a pair (X_2, D_2), such that the intersection matrix on the surface $X_2 \subset \mathbb{P}^5$ is

$$\begin{pmatrix} (D_2)^2 & K_{X_2}D_2 \\ K_{X_2}D_2 & (K_{X_2})^2 \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ -5 & -5+\alpha_2 \end{pmatrix}.$$

In particular X_2 has degree 5, hence it must be a Del Pezzo surface. So $(K_{X_2})^2 = 5$, that is $\alpha_2 = 10$. This gives **(VII.2)**.

• $\boldsymbol{\alpha_1 = 3}$. The second adjunction map gives a pair (X_2, D_2), such that the intersection matrix on the surface $X_2 \subset \mathbb{P}^5$ is

$$\begin{pmatrix} (D_2)^2 & K_{X_2}D_2 \\ K_{X_2}D_2 & (K_{X_2})^2 \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ -4 & -4+\alpha_2 \end{pmatrix}.$$

The Hodge Index Theorem implies $\alpha_2 \leq 6$. On the other hand, Theorem 1.4 implies $(K_{X_2}+D_2)^2 \geq 0$, hence $\alpha_2 \geq 6$. It follows $\alpha_2 = 6$, hence $(K_{X_2}+D_2)^2 = 0$. So X_2 is a conic bundle of degree 6 and sectional genus 2 in $\mathbb{P}^5$, containing precisely 6 reducible fibres because $(K_{X_2})^2 = 2$. It turns out that X_2 is the blow-up of $\mathbb{P}^2$ at 7 points, embedded in $\mathbb{P}^5$ via the linear system

$$D_2 = 4L - 2E_1 - \sum_{i=2}^{7} E_i,$$

see **[Io81]**. This is case **(VII.3)**.

• $\alpha_1 = 4$. The second adjunction map gives a pair (X_2, D_2), such that the intersection matrix on the surface $X_2 \subset \mathbb{P}^5$ is

$$\begin{pmatrix} (D_2)^2 & K_{X_2}D_2 \\ K_{X_2}D_2 & (K_{X_2})^2 \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ -3 & -3+\alpha_2 \end{pmatrix}.$$

The Hodge Index Theorem implies $\alpha_2 \leq 4$, whereas the condition $(K_{X_2} + D_2)^2 \geq 0$ gives $\alpha_2 \geq 2$; then $2 \leq \alpha_2 \leq 4$.

⋄ If $\alpha_2 = 2$ then by [**Io84**, p. 148] it follows that X_2 is the blow-up at 9 points of $\mathbb{F}_n$, with $n \in \{0, 1, 2, 3\}$), and that

$$D_2 = 2\mathfrak{c}_0 + (n+4)\mathfrak{f} - \sum_{i=1}^{9} E_i.$$

This is case (**VII.4a**).

⋄ If $\alpha_2 = 3$ then the third adjunction map gives a pair (X_3, D_3) whose intersection matrix is

$$\begin{pmatrix} (D_3)^2 & K_{X_3}D_3 \\ K_{X_3}D_3 & (K_{X_3})^2 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -3 & \alpha_3 \end{pmatrix}.$$

This implies $(X_3, D_3) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, so $\alpha_3 = 9$. This is case (**VII.4b**).

⋄ If $\alpha_2 = 4$ then (X_2, D_2) is as in case (6) of Theorem 1.4. This is (**VII.4c**).

• $\alpha_1 = 5$. The second adjunction map gives a pair (X_2, D_2), such that the intersection matrix on the surface $X_2 \subset \mathbb{P}^5$ is

$$\begin{pmatrix} (D_2)^2 & K_{X_2}D_2 \\ K_{X_2}D_2 & (K_{X_2})^2 \end{pmatrix} = \begin{pmatrix} 8 & -2 \\ -2 & -2+\alpha_2 \end{pmatrix}.$$

Then the Hodge Index Theorem implies $0 \leq \alpha_2 \leq 2$.

⋄ If $\alpha_2 = 0$ then the third adjunction map gives a pair (X_3, D_3), where $X_3 \subset \mathbb{P}^3$ and whose intersection matrix is

$$\begin{pmatrix} (D_3)^2 & K_{X_3}D_3 \\ K_{X_3}D_3 & (K_{X_3})^2 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -4 & -2+\alpha_3 \end{pmatrix}.$$

Hence $(X_3, D_3) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1))$, so in particular $\alpha_3 = 10$. This is case (**VII.5a**).

⋄ If $\alpha_2 = 1$ then the third adjunction map gives a pair (X_3, D_3), with $X_3 \subset \mathbb{P}^3$ and whose intersection matrix is

$$\begin{pmatrix} (D_3)^2 & K_{X_3}D_3 \\ K_{X_3}D_3 & (K_{X_3})^2 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -3 & -1+\alpha_3 \end{pmatrix}.$$

Therefore $X_3 = \mathbb{P}^2(p_1, \ldots, p_6)$ is a smooth cubic surface, in particular $\alpha_3 = 4$ and $D_3 = 3L - \sum_{i=1}^{6} E_i$. This is case (**VII.5b**).

⋄ If $\alpha_2 = 2$ then the third adjunction map gives a pair (X_3, D_3), with $X_3 \subset \mathbb{P}^3$ and whose intersection matrix is

$$\begin{pmatrix} (D_3)^2 & K_{X_3}D_3 \\ K_{X_3}D_3 & (K_{X_3})^2 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -2 & \alpha_3 \end{pmatrix}.$$

Therefore X_3 is a smooth quartic surface, a contradiction because we are assuming $p_g(X) = 0$. This case cannot occur.

• $\alpha_1 = 6$. The second adjunction map gives a pair (X_2, D_2), such that the intersection matrix on the surface $X_2 \subset \mathbb{P}^5$ is

$$\begin{pmatrix} (D_2)^2 & K_{X_2}D_2 \\ K_{X_2}D_2 & (K_{X_2})^2 \end{pmatrix} = \begin{pmatrix} 9 & -1 \\ -1 & -1+\alpha_2 \end{pmatrix}.$$

Then the Hodge Index Theorem implies $0 \leq \alpha_2 \leq 1$.

⋄ If $\alpha_2 = 0$ then the third adjunction map gives a pair (X_3, D_3), with $X_3 \subset \mathbb{P}^4$ and whose intersection matrix is

$$\begin{pmatrix} (D_3)^2 & K_{X_3}D_3 \\ K_{X_3}D_3 & (K_{X_3})^2 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ -2 & -1+\alpha_3 \end{pmatrix}.$$

Then X_3 is a smooth surface of degree 6 and sectional genus 3 in $\mathbb{P}^4$. Looking at the classification given in [**Io81**] we see that X_3 is a Bordiga surface, see Remark 3.10, so $\alpha_3 = 0$ and

$$D_3 = 4L - \sum_{i=1}^{10} E_i.$$

This gives case **(VII.6)**.

⋄ If $\alpha_2 = 1$ then the third adjunction map gives a pair (X_3, D_3), with $X_3 \subset \mathbb{P}^4$ and whose intersection matrix is

$$\begin{pmatrix} (D_3)^2 & K_{X_3}D_3 \\ K_{X_3}D_3 & (K_{X_3})^2 \end{pmatrix} = \begin{pmatrix} 7 & -1 \\ -1 & \alpha_3 \end{pmatrix}.$$

By Hodge Index Theorem we obtain $\alpha_3 = 0$, hence $(K_{X_3})^2 = 0$. This is a contradiction, because the unique non-degenerate, smooth rational surface of degree 7 in $\mathbb{P}^4$ has $K^2 = -2$, see Remark 3.12. So this case does not occur.

• $\boldsymbol{\alpha_1 = 7}$. In this case the intersection matrix on the surface $X_1 \subset \mathbb{P}^5$ is

$$\begin{pmatrix} (D_1)^2 & K_{X_1}D_1 \\ K_{X_1}D_1 & (K_{X_1})^2 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 0 \end{pmatrix}.$$

The Hodge Index Theorem implies that K_{X_1} is numerically trivial. So X_1 is a minimal Enriques surface, and X is the blow-up of X_1 at 7 points. This yields **(VII.7)**.

The proof of the existence for the cases marked with $(*)$ goes as follows. We first choose $\alpha_1 \in \{1, \dots, 7\}$. According to Proposition 2.17, we need a rank two Steiner bundle $\mathcal{F}$ on $\mathbb{P}^2$ with a resolution like (58) and α_1 distinct unstable lines. There is a well-known explicit construction of these bundles, described in Proposition 1.10.

Then, we take $\mathbb{P}(\mathcal{F})$ and we choose a sufficiently general global section η of $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(3\xi - 2\ell)$. We do this by looking directly at the image Y of $q \colon \mathbb{P}(\mathcal{F}) \to \mathbb{P}^5$, namely we consider η as a global section of $\mathcal{R}(3)$, cf. §2.3. In this setting, Y is a Bordiga scroll of degree 6 in $\mathbb{P}^5$ defined by the minors of order 3 of the 3×4 matrix of linear forms N over $\mathbb{P}^5$ obtained via the construction of §2.3, i.e.

$$\mathcal{O}_{\mathbb{P}^5}(-1)^4 \xrightarrow{N} \mathcal{O}^3_{\mathbb{P}^5},$$

and the zero locus of η is a cubic hypersurface of $\mathbb{P}^5$ containing the union of two surfaces S_1 and S_2 in Y, both obtained as the image via q of a divisor belonging to $|\mathcal{O}_{\mathbb{P}(\mathcal{F})}(\ell)|$.

Concretely, S_1 and S_2 lie in the net generated by the rows of N, i. e. they can be defined by the 2×2 minors of 4×2 matrices obtained taking random linear combinations of these rows.

Now we compute the resolution of the homogeneous ideal defining $S_1 \cup S_2$ in $\mathbb{P}^5$, we take a general cubic in this ideal and we consider the residual surface X_1 in Y. The image of the first adjunction map

$$\varphi_{|K_X + H|} \colon X \to \mathbb{P}^5$$

is precisely X_1, so that X is the blow-up of X_1 at α_1 points.

It remains to compute α_2, or equivalently $(K_{X_2})^2$. In order to do this, we observe that the second adjunction map of X is defined by the restriction to X_1 of the linear system $|\mathcal{O}_Y(2\xi - \ell)|$, and this in turn coincides with the restriction to X_1 of the linear system generated by the six quadrics in the ideal defining S_1.

The image of X_1 via this linear system is the surface X_2, hence we compute $(K_{X_2})^2$ by taking the dual of the resolution of the homogeneous ideal of X_2 in the target $\mathbb{P}^5$. All this, together with the verification that X_1 (and hence X) is smooth, is done with the help of the Computer Algebra System `Macaulay2`. In the Appendix at the end of the paper we carry out in detail the computation in the case $\alpha_1 = 1$, whereas the complete scripts can be downloaded from the web-page of the second author. □

Remark 3.18. In [**Al88**], Alexander showed the existence of a non-special, linearly normal surface of degree 9 in $\mathbb{P}^4$, obtained by embedding the blow-up of $\mathbb{P}^2$ at 10 general points via the very ample complete linear system

$$\left|13L - \sum_{i=1}^{10} 4E_i\right|.$$

By using LeBarz formula, see [**LeB90**], we can see that Alexander surface has precisely one 6-secant line. Projecting from this line to $\mathbb{P}^2$, one obtains a birational model of a general triple cover; it is immediate to see that this corresponds to case **(VII.6)** in Proposition 3.17.

Remark 3.19. Let us say something more about case (VII.7). Since $\alpha_1 = 7$, we deduce that $\mathcal{F}$ has 7 unstable lines, hence it is a logarithmic bundle (see Proposition 1.10). In this situation, the surface X_1 is a smooth Enriques surface of degree 10 and sectional genus 6 in $\mathbb{P}^5$, that is a so-called *Fano model*. Actually, one can check that X_1 is contained into the Grasmannian Gr(1, $\mathbb{P}^3$) as a *Reye congruence*, i.e. a 2-dimensional cycle of bidegree (3, 7), see [**G93**, Theorem 4.3]. In particular, X_1 admits a family of 7-secant planes, and the projection from one of these planes provides a birational model of the triple cover $f: X \to \mathbb{P}^2$ (in fact, X is the blow-up of X_1 at 7 points).

For more details about Fano and Reye models, see [**Cos83**, **CV93**].

3.7.2 Some further considerations on triple planes of type VII

We mentioned in the previous subsection that we are able to construct many, but not all cases of triple planes of type VII (see Proposition 3.17). We conjecture that the remaining cases do not exist. More precisely, our expectation is that the values of α_1 and α_2 should necessarily satisfy the rule:

$$\alpha_2 = \binom{7-\alpha_1}{2}.$$

Let us explain now what is the geometrical motivation beyond our conjecture.

The second adjunction map $\varphi_2 : X_1 \to X_2 \subset \mathbb{P}^5$ can be lifted to the map $\zeta : \mathbb{P}(\mathcal{F}) \to \mathbb{P}^5$ associated with the linear system $|\mathcal{O}_{\mathbb{P}(\mathcal{F})}(2\xi - \ell)|$. Note that

$$H^0(\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(2\xi - \ell)) \simeq H^0(\mathbb{P}^2, S^2\mathcal{F}(-1)) \simeq \wedge^2 W^*,$$

where the second isomorphism is obtained taking global sections in the second exterior power of the short exact sequence defining $\mathcal{F}$, namely

$$0 \to \wedge^2 W^* \otimes \mathcal{O}_{\mathbb{P}^2}(-3) \to W^* \otimes U \otimes \mathcal{O}_{\mathbb{P}^2}(-2) \to S^2 U \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to S^2\mathcal{F}(-1) \to 0.$$

One can show that the projective closure Y' of the image of the map $\zeta : \mathbb{P}(\mathcal{F}) \dashrightarrow \mathbb{P}(\wedge^2 W^*)$ is contained in the Plücker quadric $\mathbb{G} = \mathrm{Gr}(1, \mathbb{P}(W^*))$ and that Y' is the degeneracy locus of a map on $\mathbb{G}$ defined by the tensor $\phi \in U \otimes V \otimes W$ considered in §1.4.1. More precisely, denoting by $\mathcal{U}$ the tautological rank two subbundle on $\mathbb{G}$, once noted that $H^0(\mathbb{G}, \mathcal{U}^*) = W$ we see that ϕ gives a morphism

$$V^* \otimes \mathcal{U} \to U \otimes \mathcal{O}_{\mathbb{G}}.$$

The variety Y' is the vanishing locus of the determinant of this morphism, so that Y' can be expressed as a complete intersection of the Plücker quadric and a cubic hypersurface in $\mathbb{P}^5$.

The locus where this morphism has rank ≤ 4 is contained in the singular locus of Y' and coincides with it for a general choice of $\mathcal{F}$. By Porteous' formula, for such a general choice we expect that Y' has 21 singular points. One can see that these points are precisely the images of the sections of negative self-intersection of the Hirzebruch surfaces in $\mathbb{P}(\mathcal{F})$ lying above the smooth conics in $\mathbb{P}^2$ where $\mathcal{F}$ splits as $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(7)$, once chosen an isomorphism to $\mathbb{P}^1$ (it would be natural to call these conics *unstable conics*, and the argument above shows that generically one has 21 of them).

Also, the indeterminacy locus of ζ is exactly the union of the sections of negative self-intersection on the Hirzebruch surfaces above the unstable lines of $\mathcal{F}$. So, α_1 and α_2 should depend only on $\mathcal{F}$ and not on X, and moreover α_1 should determine α_2. However, it is not clear yet how the number of unstable lines determines the precise number of unstable conics. We plan to investigate this in the future.

4 Moduli spaces

In this section we will briefly describe some moduli problems related to our triple planes. For $b \in \{2, 3, 4\}$ we set

$$\mathcal{E}_b := \begin{cases} \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) & \text{if } b = 2 \\ \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) & \text{if } b = 3 \\ \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) & \text{if } b = 4, \end{cases}$$

whereas for $b \in \{5, 6, 7, 8\}$ we denote by $\mathcal{E}_b$ a rank 2 Steiner bundle on $\mathbb{P}^2$ having minimal free resolution of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1-b)^{b-4} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(2-b)^{b-2} \longrightarrow \mathcal{E}_b \longrightarrow 0.$$

Then, for any $b \in \{2, \dots, 8\}$, we define two spaces $\mathfrak{N}_b$ and $\mathfrak{M}_b$ as follows:

$$\mathfrak{N}_b = \left\{ (\mathcal{E}_b, \eta) \;\middle|\; \begin{array}{l} \eta \in \mathbb{P}H^0(\mathbb{P}^2, S^3\mathcal{E}_b^* \otimes \wedge^2\mathcal{E}_b) \text{ and } D_0(\eta) \text{ provides a general triple} \\ \text{plane with } p_g = q = 0 \text{ and Tschirnhausen bundle } \mathcal{E}_b \end{array} \right\} / \simeq$$

$$\mathfrak{M}_b = \left\{ (\mathcal{E}_b, \eta) \;\middle|\; \begin{array}{l} \eta \in \mathbb{P}H^0(\mathbb{P}^2, S^3\mathcal{E}_b^* \otimes \wedge^2\mathcal{E}_b) \text{ and } D_0(\eta) \text{ provides a general triple} \\ \text{plane with } p_g = q = 0 \text{ and Tschirnhausen bundle } \mathcal{E}_b \end{array} \right\} / \cong$$

Here we set $(\mathcal{E}_b, \eta) \simeq (\mathcal{E}'_b, \eta')$ if and only if there is an isomorphism $\Phi \colon \mathcal{E} \to \mathcal{E}'$ such that $\Phi^*\eta' = \eta$ and the following diagram commutes

$$\begin{array}{ccc} \mathcal{E}_b & \xrightarrow{\Phi} & \mathcal{E}'_b \\ \downarrow & & \downarrow \\ \mathbb{P}^2 & \xrightarrow{id} & \mathbb{P}^2. \end{array}$$

whereas $(\mathcal{E}_b, \eta) \cong (\mathcal{E}'_b, \eta')$ if and only if there is an isomorphism $\Phi \colon \mathcal{E} \to \mathcal{E}'$ and an automorphism $\lambda \colon \mathbb{P} \to \mathbb{P}$ such that $\Phi^*\eta' = \eta$ and the following diagram commutes

$$\begin{array}{ccc} \mathcal{E}_b & \xrightarrow{\Phi} & \mathcal{E}'_b \\ \downarrow & & \downarrow \\ \mathbb{P}^2 & \xrightarrow{\lambda} & \mathbb{P}^2. \end{array}$$

We have $\mathfrak{M}_b = \mathfrak{N}_b/\mathrm{PGL}_3(\mathbb{C})$, because the equivalence $(\mathcal{E}_b, \eta) \cong (\mathcal{E}'_b, \eta')$ is obtained from $(\mathcal{E}_b, \eta) \simeq (\mathcal{E}'_b, \eta')$ via the natural $\mathrm{PGL}_3(\mathbb{C})$-action on the base. Note that, with the terminology of [**HuyLehn10**, Chapter 4], the pair $(\mathcal{E}_b, \eta)$ is a *framed sheaf*.

Given a general triple plane $f : X \to \mathbb{P}^2$ branched over a curve of degree $2b$, by Theorems 1.2 and 2.12 we can functorially associate to (X, f) a framed sheaf $(\mathcal{E}_b, \eta)$, and conversely. In other words, considering the set of framed sheaves $(\mathcal{E}_b, \eta)$ up to the equivalence relation $\simeq$ or $\cong$ defined above is actually equivalent to considering the set of pairs (X, f) up to the corresponding, obvious equivalence relation.

Thus, from this point of view, $\mathfrak{M}_b$ can be identified with the moduli space of the pairs (X, f) up to isomorphisms, and $\mathfrak{N}_b$ with the moduli space of the pairs (X, f) up to *cover* isomorphisms.

In the sequel, we will use interchangeably the above notation $\mathfrak{N}_b$ and $\mathfrak{M}_b$, with $b \in \{2, \dots, 8\}$, and the notation $\mathfrak{N}_i$ and $\mathfrak{M}_i$, with $i \in \{\mathrm{I}, \dots, \mathrm{VII}\}$. In each case, the moduli space $\mathfrak{N}_b$ can be constructed as follows:

- take the versal deformation space $\mathrm{Def}(\mathcal{E}_b)$ of $\mathcal{E}_b$;
- stratify $\mathrm{Def}(\mathcal{E}_b)$ in such a way that $H^0(\mathbb{P}^2, S^3\mathcal{E}_b^* \otimes \wedge^2\mathcal{E}_b)$ has constant rank and consider the locally trivial projective bundle over each stratum whose fibres are given by $\mathbb{P}H^0(\mathbb{P}^2, S^3\mathcal{E}_b^* \otimes \wedge^2\mathcal{E}_b)$;
- consider the quotient of this projective bundle by the natural action of the group $\mathrm{Aut}(\mathcal{E}_b)$.

In order to obtain $\mathfrak{M}_b$, we must further take the quotient of the above moduli space by the natural action of $\mathrm{PGL}_3(\mathbb{C})$. In particular, the expected dimensions of $\mathfrak{N}_b$ and $\mathfrak{M}_b$ will be given by

$$\begin{aligned} \text{exp-dim}\,\mathfrak{N}_b &= \dim \mathrm{Def}(\mathcal{E}_b) + h^0(\mathbb{P}^2, S^3\mathcal{E}_b^* \otimes \wedge^2\mathcal{E}_b) - \dim \mathrm{Aut}(\mathcal{E}_b), \\ \text{exp-dim}\,\mathfrak{M}_b &= \dim \mathrm{Def}(\mathcal{E}_b) + h^0(\mathbb{P}^2, S^3\mathcal{E}_b^* \otimes \wedge^2\mathcal{E}_b) - \dim \mathrm{Aut}(\mathcal{E}_b) - 8. \end{aligned} \tag{59}$$

From now on, we will simply write $\mathcal{E}$ instead of $\mathcal{E}_b$ if no confusion can arise.

Let us first consider cases I, II, III. Here $\mathcal{E}$ splits as a sum of two line bundles and it is rigid, in fact

$$\dim \mathrm{Def}(\mathcal{E}) = \dim \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) = h^1(\mathbb{P}^2, \mathcal{E}^* \otimes \mathcal{E}) = 0.$$

Theorem 4.1. *The following holds*:

(1) *the moduli space $\mathfrak{M}_{\mathrm{I}}$ consists of a single point*;

(2) *the moduli space $\mathfrak{M}_{\mathrm{II}}$ is unirational of dimension* 7;

(3) *the moduli space $\mathfrak{M}_{\mathrm{III}}$ is unirational of dimension* 12.

Proof. **(1)** In case I we have $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$, so an automorphism of $\mathcal{E}$ is given by a 2×2 scalar invertible matrix and this shows that $\mathrm{Aut}(\mathcal{E}) = \mathrm{GL}_2(\mathbb{C})$. Moreover

$$h^0(\mathbb{P}^2, S^3\mathcal{E}^* \otimes \wedge^2\mathcal{E}) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^4) = 12,$$

hence (59) yields

$$\text{exp-dim}\,\mathfrak{M}_{\mathrm{I}} = 12 - 4 - 8 = 0.$$

This number coincides with the effective dimension $\dim \mathfrak{M}_{\mathrm{I}}$. In fact, the branch curve $B \subset \mathbb{P}^2$ is in this case a 3-cuspidal plane quartic curve, which is unique up to projective transformations. By a standard monodromy argument this implies that there exist at most finitely many triple planes of type I up to isomorphisms, and their number equals the number of group epimorphisms

$$\pi_1(\mathbb{P}^2 - B) \to S_3$$

up to conjugation. Now, it is well-known that

$$\pi_1(\mathbb{P}^2 - B) = \mathrm{B}_3(\mathbb{P}^2) = \langle \alpha, \beta \mid \alpha^3 = \beta^2 = (\beta\alpha)^2 \rangle,$$

see [**Dim92**, Chapter 4], and this group has a unique epimorhism to S_3 up to conjugation. Thus $\mathfrak{M}_{\mathrm{I}}$ consists of a single point.

(2) In case II, $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$. So an automorphism of $\mathcal{E}$ is given by a 2×2 matrix

$$A = \begin{pmatrix} \lambda_1 & L \\ 0 & \lambda_2 \end{pmatrix},$$

where λ_1, λ_2 are non-zero scalars and L is a linear form in the homogeneous coordinates x_0, x_1, x_2, and this yields $\dim \mathrm{Aut}(\mathcal{E}) = 5$. Moreover

$$h^0(\mathbb{P}^2, S^3\mathcal{E}^* \otimes \wedge^2\mathcal{E}) = h^0\Big(\mathbb{P}^2, \bigoplus_{k=0}^{3} \mathcal{O}_{\mathbb{P}^2}(k)\Big) = 20,$$

hence (59) implies

$$\text{exp-dim}\,\mathfrak{M}_{\mathrm{II}} = 20 - 5 - 8 = 7.$$

This numbers equals the effective dimension $\dim \mathfrak{M}_{\mathrm{II}}$. In order to see this, recall that in this case the branch locus $B \subset \mathbb{P}^2$ is a plane sextic curve with six cusps lying on the same conic. Each such a curve can be written as

$$(f_2)^3 + (f_3)^2 = 0, \tag{60}$$

where f_k denotes a homogeneous form of degree k, and the construction depends on

$$6+10-1-\dim \mathrm{PGL}_3(\mathbb{C})=7$$

parameters. The same monodromy argument used in part **(1)** shows that this also computes the effective dimension $\dim \mathfrak{M}_{\mathrm{II}}$. More precisely, we can see that every fixed curve B of equation (60) is the branch locus of a unique triple cover up to isomorphisms, namely the one whose birational model is provided by the hypersurface

$$z^3+bz+c=0,$$

where $b=-f_2/\sqrt[3]{4}$ and $c=f_3/\sqrt{-27}$. In fact, we have

$$\pi_1(\mathbb{P}^2-B)=(\mathbb{Z}/2\mathbb{Z})*(\mathbb{Z}/3\mathbb{Z})=\langle \alpha,\, \beta \mid \alpha^3=\beta^2=1\rangle,$$

see again [**Dim92**, Chapter 4], and this group has a unique epimorhism to S_3 up to conjugation.

(3) In case III we have $\mathcal{E}=\mathcal{O}_{\mathbb{P}^2}(-2)\oplus\mathcal{O}_{\mathbb{P}^2}(-2)$. So, as in case **(1)**, the automorphism group of $\mathcal{E}$ is isomorphic to $\mathrm{GL}_2(\mathbb{C})$. Moreover

$$h^0(\mathbb{P}^2,\, S^3\mathcal{E}^*\otimes\wedge^2\mathcal{E})=h^0\big(\mathbb{P}^2,\, \mathcal{O}_{\mathbb{P}^2}(2)^4\big)=24,$$

hence (59) implies

$$\text{exp-dim}\,\mathfrak{M}_{\mathrm{III}}=24-4-8=12.$$

This number coincides with the effective dimension $\dim \mathfrak{M}_{\mathrm{III}}$. In fact, in this case X is the blow-up at 9 points of $\mathbb{F}_n$, with $n\in\{0,\,1,\,2,\,3\}$. The stratum of maximal dimension corresponds to the value of n such that $\mathrm{Aut}(\mathbb{F}_n)=H^0(\mathbb{F}_n,\, T_{\mathbb{F}_n})$ has minimal dimension, namely to $n=0$ for which we have

$$\dim \mathfrak{M}_{\mathrm{III}}=2\cdot 9-\dim \mathrm{Aut}(\mathbb{F}_n)=18-6=12.$$

□

We now start the analysis of the cases IV, . . . , VII, where $\mathcal{E}$ is indecomposable. Using the notation introduced in Section 2, we will write $\mathcal{F}=\mathcal{E}(b-2)$, so that $\mathcal{F}$ fits into the short exact sequence

$$0\longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{b-4}\longrightarrow \mathcal{O}_{\mathbb{P}^2}^{b-2}\longrightarrow \mathcal{F}\longrightarrow 0.$$

Thus $\mathrm{Def}(\mathcal{E})=\mathrm{Def}(\mathcal{F})$ and

$$H^0(\mathbb{P}^2,\, S^3\mathcal{E}^*\otimes\wedge^2\mathcal{E})=H^0(\mathbb{P}^2,\, S^3\mathcal{F}(6-b)).$$

The vector bundle $\mathcal{F}$ is stable (Theorem 2.12), so $\mathrm{Aut}(\mathcal{F})=\mathbb{C}^*$; its deformation space $\mathrm{Def}(\mathcal{F})$ is described for instance in [**Ca02**], and we have

$$\dim \mathrm{Def}(\mathcal{F})=3(b-4)(b-2)-1=(b-1)(b-5).$$

Then (59) yields

$$\begin{aligned}\dim \mathfrak{N}_b=\text{exp-dim}\,\mathfrak{N}_b&=(b-1)(b-5)+h^0(\mathbb{P}^2,\, S^3\mathcal{F}(6-b))-1,\\ \text{exp-dim}\,\mathfrak{M}_b&=(b-1)(b-5)+h^0(\mathbb{P}^2,\, S^3\mathcal{F}(6-b))-9.\end{aligned}\tag{61}$$

Furthermore, the equality $\text{exp-dim}\,\mathfrak{M}_b=\dim \mathfrak{M}_b$ holds if and only if $\mathrm{PGL}_3(\mathbb{C})$ acts on $\mathfrak{N}_b$ with generically finite stabilizer.

Theorem 4.2. *For* $i\in\{\mathrm{IV},\, \mathrm{V},\, \mathrm{VI}\}$ *the moduli space* $\mathfrak{N}_i$ *is rational and irreducible, while* $\mathfrak{M}_i$ *is unirational of dimension* $\dim \mathfrak{N}_i-8$, *where*

- **(4)** $\dim \mathfrak{N}_{\mathrm{IV}}=23$;
- **(5)** $\dim \mathfrak{N}_{\mathrm{V}}=24$;

(6) $\dim \mathfrak{N}_{\mathrm{VI}} = 23$.

Moreover

(7) *the moduli space $\mathfrak{N}_{\mathrm{VII}}$ has at least seven connected components, all of dimension 20, that are distinguished by the number $\alpha_1 \in \{1, \dots, 7\}$ of unstable lines for $\mathcal{F}$.*

Proof. Although some arguments are uniform, we prefer to articulate the proof into a a case-by-case analysis in order to better describe the interesting geometrical details.

(4) In case IV, i.e. $b = 5$, we use Proposition 3.7 and the arguments exploited in its proof. We have $\mathcal{F} = T_{\mathbb{P}^2}(-1)$ and a natural identification

$$H^0(\mathbb{P}^2, S^3\mathcal{F}(1)) = H^0(\mathbb{P}^{2*}, T_{\mathbb{P}^{2*}}(2)) = \mathbb{C}^{24}. \tag{62}$$

Then (61) implies

$$\begin{aligned} \dim \mathfrak{N}_{\mathrm{IV}} = \text{exp-dim}\, \mathfrak{N}_{\mathrm{IV}} = 0 + 24 - 1 = 23, \\ \text{exp-dim}\, \mathfrak{M}_{\mathrm{IV}} = 0 + 24 - 9 = 15. \end{aligned} \tag{63}$$

In fact, the bundle $\mathcal{F}$ is rigid, stable and unobstructed, hence its moduli space is a single reduced point. Consequently, the triple cover $f : X \to \mathbb{P}^2$ only depends on the section η, so by (62) we deduce that $\mathfrak{N}_{\mathrm{IV}}$ is an open, dense subset of $\mathbb{P}^{23}$.

It remains to show that $\text{exp-dim}\, \mathfrak{M}_{\mathrm{IV}} = \dim \mathfrak{M}_{\mathrm{IV}}$, or equivalently that $\mathrm{PGL}_3(\mathbb{C})$ acts on $\mathbb{P}^{23} = \mathbb{P}H^0(\mathbb{P}^{2*}, T_{\mathbb{P}^{2*}}(2))$ with generically finite stabilizer. Take $\eta \in \mathbb{P}^{23}$ and let $G = G_\eta \subset \mathrm{PGL}_3(\mathbb{C})$ be its stabilizer. Every projectivity in G must preserve the 0-dimensional subscheme $Z(\eta) \subset \mathbb{P}^{2*}$, which generically consists of 13 reduced points, so we obtain a group homomorphism

$$\psi : G \to S_{13}.$$

Since $23 > 2 \cdot 4$, we see that (any subset of) four points of $\mathbb{P}^{2*}$ in general position can be contained in $Z(\eta)$ for a suitable choice of η. On the other hand, any projectivity of $\mathbb{P}^{2*}$ fixing four points in general position is necessarily the identity, and this in turn implies that ψ is injective for our choice of η. Thus G is generically a finite group and we are done.

(5) In case IV, i.e. $b = 5$, we use Proposition 3.11 and the arguments exploited in its proof.

We know that there is a conic $W(\mathcal{F}) \subset \mathbb{P}^{2*}$ of unstable lines for $\mathcal{F}$. Associating $\mathcal{F}$ with such a conic, we can identify the moduli space of $\mathcal{F}$ with the open subset $\mathcal{U} \subset \mathbb{P}^5$ consisting of smooth conics via the Veronese embedding. There is a natural identification

$$H^0(\mathbb{P}^2, S^3\mathcal{F}(6-b)) = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) = \mathbb{C}^{20},$$

so (61) implies

$$\begin{aligned} \dim \mathfrak{N}_{\mathrm{V}} = \text{exp-dim}\, \mathfrak{N}_{\mathrm{V}} = 5 \cdot 1 + 20 - 1 = 24, \\ \text{exp-dim}\, \mathfrak{M}_{\mathrm{V}} = 5 \cdot 1 + 20 - 9 = 16. \end{aligned} \tag{64}$$

In fact, $\mathfrak{N}_{\mathrm{V}}$ is a rational variety, because it can be identified with an open dense subset of a locally trivial $\mathbb{P}^{19}$-bundle over $\mathcal{U}$.

It remains to show that $\text{exp-dim}\, \mathfrak{M}_{\mathrm{V}} = \dim \mathfrak{M}_{\mathrm{V}}$, or equivalently that $\mathrm{PGL}_3(\mathbb{C})$ acts on the set of pairs $(\mathcal{F}, \eta)$ with generically finite stabilizer. Let $G = G_{(\mathcal{F}, \eta)} \subset \mathrm{PGL}_3(\mathbb{C})$ be the stabilizer of $(\mathcal{F}, \eta)$. Then every element $g \in G$ must fix $\mathcal{F}$, and hence the conic $W(\mathcal{F})$. By [**FulHa91**, p. 154], any automorphism of $\mathbb{P}^n$ that preserves a rational normal curve C_n is precisely $\mathrm{PGL}_2(\mathbb{C})$, so G is embedded into a copy of $\mathrm{PGL}_2(\mathbb{C})$ inside $\mathrm{PGL}_3(\mathbb{C})$. On the other hand, g fixes $\eta \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$, hence it fixes the cubic surface $S \subset \mathbb{P}^3$. Since the conic $W(\mathcal{F})$ corresponds to the twisted cubic C, it follows that g must preserve the nine points in the intersection $S \cap C$. This shows that there is a group homomorphism

$$\psi : G \to S_9,$$

that must be injective since an element of $\mathrm{PGL}_2(\mathbb{C})$ fixing more than 3 points in general position is necessarily the identity. So G is a subgroup of S_9, hence a finite group.

A related, but slightly different way to compute $\dim \mathfrak{M}_{\mathrm{V}}$ is the following. The surface X is the blow-up of a smooth cubic surface $S \subset \mathbb{P}^3$ at the nine points given by the intersection of S with the twisted cubic curve C. Now the group $\mathrm{PGL}_4(\mathbb{C})$ naturally acts on the product $\mathrm{Hilb}_{\mathbb{P}^3}(S) \times \mathrm{Hilb}_{\mathbb{P}^3}(C)$, and there is a group homomorphism from the stabilizer $\widetilde{G}$ of a point (S, C) to $\mathrm{Aut}(S) \times \mathrm{Aut}(C)$. If S is general then its automorphism group is trivial (and in any case it is finite, see for instance [**Ko88**, **Hos97**]), so again by [**FulHa91**, p. 154] we have an injective group homomorphism $\widetilde{G} \to \mathrm{Aut}(C) = \mathrm{PGL}_2(\mathbb{C})$. On the other hand, any element of $\widetilde{G}$ must preserve the nine intersection points of S and C, so we deduce as before that $\widetilde{G}$ is a finite group and we obtain

$$\dim \mathfrak{M}_{\mathrm{V}} = \dim \mathrm{Hilb}_{\mathbb{P}^3}(S) + \mathrm{Hilb}_{\mathbb{P}^3}(C) - \dim \mathrm{PGL}_4(\mathbb{C}) = 19 + 12 - 15 = 16.$$

(6) In case VI, i.e. $b = 7$, we use Proposition 3.14 and the arguments exploited in its proof.

In this case $\mathcal{F}$ is a logarithmic bundle associated with six lines of $\mathbb{P}^2$, and its moduli space is an open dense subset $\mathcal{U}$ of the Hilbert scheme of six points of $\mathbb{P}^{2*}$ (in particular, for the general bundle $\mathcal{F}$ such points are in general position). There is a natural identification

$$H^0(\mathbb{P}^2, S^3\mathcal{F}(6-b)) = H^0(\mathbb{P}^4, \mathcal{R}(3)) = \mathbb{C}^{12},$$

where the sheaf $\mathcal{R}(3)$ fits into the short exact sequence (50). Then (61) implies

$$\begin{aligned} \dim \mathfrak{N}_{\mathrm{VI}} &= \text{exp-dim}\, \mathfrak{N}_{\mathrm{VI}} = 6 \cdot 2 + 12 - 1 = 23, \\ \text{exp-dim}\, \mathfrak{M}_{\mathrm{VI}} &= 6 \cdot 2 + 12 - 9 = 15. \end{aligned} \tag{65}$$

In fact, $\mathfrak{N}_{\mathrm{VI}}$ is a rational variety of dimension 23, occurring as a dense subset of a locally trivial $\mathbb{P}^{11}$-bundle over $\mathcal{U}$.

We can give another geometrical interpretation of the equality $\text{exp-dim}\, \mathfrak{M}_{\mathrm{VI}} = 15$ as follows. The surface X is the blow-up of a Bordiga surface $X_1 \subset Y$, where Y is a determinantal cubic 3-fold in $\mathbb{P}^4$, at the six nodes of Y, see Proposition 3.14. Since Y is given by a 3×3 matrix of linear forms in the homogeneous coordinates $x_0, \ldots, x_4$, it depends on

$$(3 \cdot 3 \cdot 5 - 1) - 2 \cdot \dim \mathrm{PGL}_3(\mathbb{C}) = 28$$

parameters. On the other hand, X_1 moves into a 11-dimension linear system $|X_1|$ inside Y, so the construction depends on

$$28 + 11 - \dim \mathrm{PGL}_4(\mathbb{C}) = 15$$

parameters, which is consistent with (65).

It only remains to show that $\text{exp-dim}\, \mathfrak{M}_{\mathrm{VI}} = \dim \mathfrak{M}_{\mathrm{VI}}$, or equivalently that $\mathrm{PGL}_3(\mathbb{C})$ acts on the set of pairs $(\mathcal{F}, \eta)$ with generically finite stabilizer. Let $G = G_{(\mathcal{F}, \eta)} \subset \mathrm{PGL}_3(\mathbb{C})$ be the stabilizer of $(\mathcal{F}, \eta)$. Then every element $g \in G$ must fix $\mathcal{F}$, and hence the set of its six unstable lines. Consequently, g permutes the corresponding six points in $\mathbb{P}^{2*}$, so we have a group homomorphism

$$\psi \colon G \to S_6.$$

The kernel of ψ is given by those elements in G that fix the set of six points pointwise. On the other hand, G is a subgroup of $\mathrm{PGL}_2(\mathbb{C})$, and any element in the last group fixing more than four points in general position must be the identity. This means that, for a general choice of $\mathcal{F}$, the homomorphism ψ is injective, hence G is isomorphic to a subgroup of S_6 and our claim is proven.

(7) Let us finally consider case VII, i.e. $b = 8$. Proposition 3.17 shows the existence of seven families of triple planes, one for each value of the number $\alpha_1 \in \{1, \ldots, 7\}$ of unstable lines of $\mathcal{F}$. Such families necessarily belong to pairwise distinct connected components

$$\mathfrak{N}^1_{\mathrm{VII}}, \ldots, \mathfrak{N}^7_{\mathrm{VII}}$$

of $\mathfrak{N}_{\mathrm{VII}}$, because α_1 coincides with the number of lines contracted by the first adjunction map of X, and such a number is an invariant of the triple cover.

Let us consider the 21-dimensional (rational) moduli space $\mathrm{M}_{\mathbb{P}^2}(2, 4, 10)$ of rank-2 stable bundles on $\mathbb{P}^2$ with Chern classes (4, 10) and having a Steiner-type resolution, and let $\mathcal{U}^{\alpha_1} \subset \mathrm{M}_{\mathbb{P}^2}(2, 4, 10)$ be the stratum corresponding to vector bundles having α_1 unstable lines. The codimension of this stratum is precisely α_1.

Our computations with `Macaulay2` show that there exist examples of bundles $\mathcal{F}$ with α_1 unstable lines and satisfying

$$h^0(\mathbb{P}^2, S^3\mathcal{F}(-2)) = \alpha_1, \tag{66}$$

so, by Proposition 1.12 and semicontinuity, equality (66) holds for the general member of the stratum $\mathcal{U}^{\alpha_1}$. Thus, for every $\alpha_1 \in \{1, \dots, 7\}$ we obtain

$$\dim \mathfrak{N}_{\mathrm{VII}}^{\alpha_1} = \dim \mathcal{U}^{\alpha_1} + h^0(\mathbb{P}^2, S^3\mathcal{F}(-2)) - 1 = (21 - \alpha_1) + \alpha_1 - 1 = 20,$$

and this completes the proof. □

Remark 4.3. We could also given a geometric interpretation of the equality $\dim \mathfrak{N}_{\mathrm{VII}}^{\alpha_1} = 20$ by using in each case the explicit description of the surface X provided by Proposition 3.17. We will not develop this point here, and we limit ourselves to discussing as an example the case $\alpha_1 = 6$. In this situation, we know that X is isomorphic to the blow-up at six points of an Alexander surface of degree 9 in $\mathbb{P}^4$, see Remark 3.18. Such points are the intersection of the Alexander surface with its unique 6-secant line, and they completely determine the triple cover map $f : X \to \mathbb{P}^2$. So the dimension of the component $\mathfrak{N}_{\mathrm{VII}}^{6}$ equals the dimension of an open, dense subset of $S^{10}(\mathbb{P}^2)$, that is 20.

Appendix: an example of Macaulay2 script

In this appendix we explain how we used the Computer Algebra System `Macaulay2` in order to show the existence of triple planes in the cases marked with $(*)$ in Proposition 3.16. We explain in detail the script used in case **(VII.1b)**, where $\alpha_1 = 1$; all the others (easily obtained by modifying the one below) can be downloaded from the web-page of the second author, at the url

`https://sites.google.com/site/francescopolizzi/publications`.

In order to speed up the computations, we work over a finite field; however, we can also work over $\mathbb{Q}$, by replacing the first line with `k=QQ`. The lines written in black are input lines, the lines written in red are output lines.

We start by constructing the 6×4 matrix of linear forms M defining the Steiner bundle $\mathcal{F}$, that is

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^4 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}^6 \longrightarrow \mathcal{F} \to 0.$$

Note that we can impose the existence of a single unstable line for $\mathcal{F}$ by using Proposition 1.9. Then, transposing this resolution as explained in Subsection 1.4, we obtain the 3×4 matrix of linear forms defining the Steiner sheaf $\mathcal{G}$, namely

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-1)^4 \xrightarrow{N} \mathcal{O}_{\mathbb{P}^5}^3 \longrightarrow \mathcal{G} \to 0.$$

```
b=8;
k = ZZ/32003;
R2 = k[x_0..x_2]
h0 = matrix({{x_0}})
M0 = h0**random (R2^~{4:0},R2^{1:0});
M1 = random (R2^{4:0},R2^{5:-1});
M = matrix({{M0,M1}});
MM = map(R2^{b-4:1},R2^{b-2:0},M);
F = coker transpose MM;
HH^0 (sheaf F)
```

Now we construct the threefold Y, image of the morphism $q\colon \mathbb{P}(\mathcal{F}) \to \mathbb{P}^5$, and we check that it has degree 6 and precisely one singular point.

```
S = k[y_0..y_(b-3)]
R = R2 ** S
Q = substitute(vars S,R) * (substitute(transpose(M),R));
N = substitute((coefficients(Q,Variables=>{x_0,x_1,x_2}))_1,S);
IY = minors(rank target N,N);
degree IY
singY = ideal singularLocus IY;
IY2 = minors((rank target N)-1,N);
(dim singY, degree singY)
(dim IY2, degree IY2)
dim(singY:IY2), degree(singY:IY2)
isSubset((singY:IY2),IY2)
(singY:IY2):IY2
singY = radical ideal singularLocus Y;
dim variety singY, degree variety singY
```

Let us consider the union of two general divisors S_1, S_2 on Y belonging to the net generated by the rows of N, and compute the resolution of the corresponding homogeneous ideal in $\mathbb{P}^5$.

```
A = matrix random(S^{3:0},S^{3:0})
N = transpose(N)*A;

N1 = submatrix(N, {0,1});
N2 = submatrix(N, {0,2});
IS1 = minors(2, N1);
IS2= minors(2, N2);
I12 = intersect(IS1,IS2);

betti res I12
```

		0	1	2	3	4	5
total	:	1	20	53	56	27	5
0	:	1	.	.	.	.	.
1	:	.	.	.	.	.	.
2	:	.	5	3	.	.	.
3	:	.	15	50	56	27	5

The output shows that such a homogeneous ideal is generated by 5 cubics and 15 quartics. The image X_1 of X via the first adjunction map $\varphi_{|D|}\colon X \to \mathbb{P}^5$ can be obtained by taking a random cubic containing $S_1 \cup S_2$ and considering the residual surface in Y. It is a surface of degree 10.

```
G=submatrix(mingens I12, , {0..4});
IC3 = ideal(G*random(S^{5:0},S^{0}));
IX1 = ((IC3 + IY):I12);

X1 = variety(IX1);
dim X1, degree X1
```

(2, 10)

We check that X_1 has arithmetic genus 0 and sectional genus 6. Then we show that X_1 is smooth and irreducible, that it contains the singular locus of Y and that $p_g(X_1) = q(X_1) = 0$.

```
genera X1
```

(0, 6, 9)

```
dim singularLocus X1, degree singularLocus X1
```

(-infinity, 0)

```
isSubset(IX1, IY2)
```

true

```
X=X1
rank HH^0(OO_X), rank HH^1(OO_X), rank HH^2(OO_X)
```

(1, 0, 0)

Finally, we want to calculate α_2, i.e. the number of exceptional lines contracted by the second adjunction map $\varphi_2 : X_1 \to X_2 \subset \mathbb{P}^5$. This is equivalent to find $(K_{X_2})^2$, and the corresponding computation can be carried out as follows. First, we construct the homogeneous ideal of X_2 and we check that X_2 has degree 4.

```
RX = S/IX1;
scroll1 = variety IS1;
I1X = saturate substitute (ideal (scroll1),RX);
T = k[t_0..t_5];
IX2 = ker map(RX,T,gens I1X);
X2 = variety IX2;
degree IX2
```

4

Then we construct the dualizing sheaf $\omega_{X_2} = \mathcal{O}_{X_2}(K_{X_2})$, that allows us to compute by Riemann-Roch theorem $(K_{X_2})^2$ as $\chi(2K_{X_2}) - 1$.

```
omegaX2 = \Ext^2(IX2,T^{1:-6});
RX2 = T/IX2;
omega2X2 = omegaX2**RX2;
K2 = euler(dual omega2X2)-1
```

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This shows that $K_{X_2}^2 = 9$, hence $\alpha_2 = 15$ and X belongs to case **(VII.1b)**.

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