FREENESS OF LINE ARRANGEMENTS WITH MANY CONCURRENT LINES

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Abstract. We propose here a new approach in order to study line arrangements on the projective plane. We use this approach to prove Terao's conjecture when many lines of the arrangement are concurrent.

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A line arrangement in $\mathbb{P}^2 = \mathbb{P}(\mathbb{C}[x_0, x_1, x_2])$ is a finite collection of lines, say $\{l_1, \cdots, l_s\}$. The union of these lines is a reduced divisor denoted by $D = \{f = 0\}$, where f is the product of the s linear forms defining the l_i's. Saito (see [Sai80]) associates to D the bundle $T(\log D)$ of vector fields with logarithmic poles along D. This is a vector bundle of rank 2, defined by the following exact sequence of sheaves:

$$0 \longrightarrow T(\log D) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^3 \xrightarrow{(\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2})} \mathcal{O}_{\mathbb{P}^2}(s-1). \tag{1}$$

We say that the arrangement is *free* when $T(\log D)$ splits as a sum of two line bundles and more precisely we will say that it is *free of type* (a, b), with $0 \leq a \leq b$, when $T(\log D) \simeq \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b)$.

The main open question about these bundles (also valid on $\mathbb{P}^n$, for $n \geq 2$) is the so-called Terao's conjecture (see [OT92]):

Conjecture 1 (Terao)**.** Freeness of D depends only on its combinatorial type.

By combinatorial type here we mean the intersection lattice associated to the arrangement. Its incidence graph has one vertex v_i for each line l_i of the arrangement and one vertex $u_{i,j}$ an intersection point $l_i \cap l_j$. The vertex $u_{i,j}$ is linked by an edge to all vertices v_k such that $l_i \cap l_j$ lies in l_k. So two line arrangements are said to have the same combinatorial type if these graphs are isomorphic.

We propose a new approach to Terao's conjecture, based on projective duality. Any line of the divisor D corresponds to a point in $\mathbb{P}^{2\vee}$. This way we associate to D a finite set Z of points in $\mathbb{P}^{2\vee}$. From now, in order to insist on the correspondence, we will denote by $Z \subset \mathbb{P}^{2\vee}$ the finite set of points and by $D_Z \subset \mathbb{P}^2$ the corresponding divisor.

Let us now introduce the variety $\mathbb{F} \subset \mathbb{P}^2 \times \mathbb{P}^{2\vee}$. which is the incidence variety *point-line* in $\mathbb{P}^2$, and the projections p and q on $\mathbb{P}^2$ and $\mathbb{P}^{2\vee}$.

$$\begin{array}{ccc} \mathbb{F} & \xrightarrow{q} & \mathbb{P}^{2\vee} \\ {\scriptstyle p}\downarrow & & \\ \mathbb{P}^2 & & \end{array}$$

Let $\mathcal{J}_Z$ be the ideal sheaf of Z in $\mathbb{P}^{2\vee}$. We show first that the Saito vector bundle $T(\log D)$ is obtained by looking at $\mathcal{J}_Z(1)$ on $\mathbb{P}^2$. More precisely, we prove:

Theorem 0.1. *$p_*q^*\mathcal{J}_Z(1) \simeq T(\log D)$.*

Proof. Let us consider the canonical exact sequence:

$$0 \longrightarrow \mathcal{J}_Z(1) \longrightarrow \mathcal{O}_{\mathbb{P}^{2\vee}}(1) \longrightarrow \mathcal{O}_Z(1) \longrightarrow 0.$$

Looking at the above exact sequence over $\mathbb{P}^2$ means applying the functor p_*q^*. Then, denoting by $T_{\mathbb{P}^2}$ the tangent bundle to $\mathbb{P}^2$, we have:

$$0 \longrightarrow p_*q^*\mathcal{J}_Z(1) \longrightarrow T_{\mathbb{P}^2}(-1) \longrightarrow \oplus_{l\in Z}\mathcal{O}_l.$$

Now, the equation f of D_Z provides a non zero global section of the ideal sheaf generated by the partial derivatives of f, namely the Jacobian ideal $\mathcal{J}_f(s)$. This amounts to an injective morphism of sheaves of the form $\mathcal{O}_{\mathbb{P}^2} \to \mathcal{J}_f(s)$. This morphism induces a commutative diagram:

$$\begin{tikzcd}
 & & \mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r, equal] \arrow[d] & \mathcal{O}_{\mathbb{P}^2}(-1) \arrow[d, "f"'] & \\
0 \arrow[r] & T(\log D) \arrow[r] \arrow[d, equal] & \mathcal{O}^3_{\mathbb{P}^2} \arrow[r] \arrow[d] & \mathcal{J}_f(s-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T(\log D) \arrow[r] & T_{\mathbb{P}^2}(-1) \arrow[r] & \mathcal{C} \arrow[r] & 0,
\end{tikzcd}$$

where the middle row is the exact sequence (1) defining $T(\log D)$. The sheaf $\mathcal{C}$ is the ideal sheaf of the singular locus of the hypersurface $\{f=0\}$ considered on the hypersurface. We have a natural inclusion $\mathcal{C} \subset \oplus_{l\in Z}\mathcal{O}_l$ by desingularization. Then, since the homomorphism $T_{\mathbb{P}^2}(-1) \to \oplus_{l\in Z}\mathcal{O}_l$ is essentially unique (see [Val07]) this proves that both kernels $p_*q^*\mathcal{J}_Z(1)$ and $T(\log D)$ coincide. □

In order to show that this approach is relevant we prove here a special case of Terao's conjecture, without using any further material.

Theorem 0.2. *Terao's conjecture is true for a free divisor D_Z of type $(n, n+r)$, with $r \geq 0$, as soon as $(n+2)$ points of Z are collinear.*

Saying that $(n+2)$ points of Z are collinear amounts to require that $(n+2)$ lines of D_Z are concurrent, hence we may say that freeness of arrangements with many concurrent lines is combinatorial.

The first step to prove the theorem is the following lemma relating sections on one side to decomposition over lines on the dual side.

Lemma 0.3. *Let $Z \subset \mathbb{P}^{2\vee}$ be a set of points and x be a general point in $\mathbb{P}^{2\vee}$. Assume that $T(\log D_Z) \otimes \mathcal{O}_{x^\vee} = \mathcal{O}_{x^\vee}(-n) \oplus \mathcal{O}_{x^\vee}(-n-r)$ with $r \geq 0$. Then $H^0((\mathcal{J}_Z \otimes \mathcal{J}_x^n)(n+1)) \neq 0$.*

Proof. Let us denote by $\widehat{\mathbb{P}}$ the blowing up of $\mathbb{P}^{2\vee}$ along the point x. We recall that $\widehat{\mathbb{P}} \simeq p^{-1}(x^\vee) \subset \mathbb{F}$ and we consider the induced incidence diagram:

$$\begin{tikzcd}
\widehat{\mathbb{P}} \arrow[r, "\widehat{q}"] \arrow[d, "\widehat{p}"'] & \mathbb{P}^{2\vee} \\
x^\vee &
\end{tikzcd}$$

Moreover we have the following resolution of $\widehat{\mathbb{P}}$ in $\mathbb{F}$:

$$0 \longrightarrow p^*\mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow \mathcal{O}_{\mathbb{F}} \longrightarrow \mathcal{O}_{\widehat{\mathbb{P}}} \longrightarrow 0.$$

Tensoring the exact sequence above by $q^*\mathcal{J}_Z(1)$ we get:

$$0 \to q^*(\mathcal{J}_Z(1)) \otimes p^*\mathcal{O}_{\mathbb{P}^2}(-1) \to q^*(\mathcal{J}_Z(1)) \to \hat{q}^*(\mathcal{J}_Z(1)) \to 0.$$

Now we apply the functors $R^i p_*$ to the above sequence (see for instance [Har77, Chapter III]). Let us describe the effect of applying p_* (i.e. $R^i p_*$ for $i = 0$) to the above sequence. For the middle term, the result is computed by Theorem 1 and agrees with $T(\log D)$. For the leftmost term, we get $T(\log D)(-1)$ by Theorem 1 and projection formula (see again [Har77, Chapter III]). For the rightmost term, we get $\widehat{p}_*\widehat{q}^*\mathcal{J}_Z(1)$ for $\widehat{p}$ and $\widehat{q}$ are the restrictions of p and q to $\widehat{\mathbb{P}}$. Denote by $R^1T(\log D_Z)$ the sheaf $R^1p_*q^*\mathcal{J}_Z(1)$. We can now write the long exact sequence obtained applying R^ip_* for $i = 0, 1$ the exact sequence above.

$$0 \to T(\log D_Z)(-1) \xrightarrow{x^\vee} T(\log D_Z) \longrightarrow \widehat{p}_*\widehat{q}^*\mathcal{J}_Z(1) \to$$

$$\to R^1T(\log D_Z)(-1) \xrightarrow{x^\vee} R^1T(\log D_Z) \longrightarrow R^1\widehat{p}_*\widehat{q}^*\mathcal{J}_Z(1) \to 0.$$

Since x is general, any line through x is at most 1-secant to Z. Then the support of the sheaf $R^1\widehat{p}_*\widehat{q}^*\mathcal{J}_Z(1)$, which is the locus of 3-secant lines to Z through x, is empty. We have proved $\widehat{p}_*\widehat{q}^*\mathcal{J}_Z(1) = T(\log D_Z) \otimes \mathcal{O}_{x^\vee}$.

Then the decomposition $T(\log D_Z) \otimes \mathcal{O}_{x^\vee} = \mathcal{O}_{x^\vee}(-n) \oplus \mathcal{O}_{x^\vee}(-n-r)$ gives an injective homomorphism:

$$\mathcal{O}_{x^\vee}(-n) \hookrightarrow \widehat{p}_*\widehat{q}^*\mathcal{J}_Z(1).$$

This means that we have a non zero map on $\widehat{\mathbb{P}}$:

$$\widehat{p}^*\mathcal{O}_{x^\vee}(-n) \hookrightarrow \widehat{q}^*\mathcal{J}_Z(1),$$

that we can write also as:

$$\mathcal{O}_{\widehat{\mathbb{P}}} \hookrightarrow \widehat{q}^*\mathcal{J}_Z(1) \otimes \widehat{p}^*\mathcal{O}_{x^\vee}(n).$$

This last map is equivalent to a non zero map on $\mathbb{P}^{2\vee}$:

$$\mathcal{O}_{\mathbb{P}^{2\vee}} \hookrightarrow \mathcal{J}_Z(1) \otimes \mathcal{J}_x^n(n) = (\mathcal{J}_Z \otimes \mathcal{J}_x^n)(n+1).$$

$\square$

Proof of the main theorem. We first describe the combinatorial type according to the given data. By hypothesis, there exists a line L such that $|L \cap Z| \geq n+2$. This line is a fixed component in the linear system of curves of degree $n+1$ passing through Z. Since x is general, a curve of degree $n+1$ passing through Z and having multiplicity n at x is the union of L and of n lines through x. Then there are at most n points of Z that do not lie in L. Since the length of Z is $2n+r+1$, there are at least $n+r+1$ points on L. In fact, according to the decomposition, L is exactly $(n+r+1)$ secant to Z. Indeed, if there were strictly more than $n+r+1$ points on L, then one could find, for a general x, a curve of degree n passing through Z and having multiplicity $n-1$ at x (take the union of L and of the $n-1$ lines through x and the remaining points) and this contradicts the decomposition.

Assume now that Z_0 has the same combinatorial type than Z. Then, according to Yoshinaga ([Yos04, Thm. 2.2]) the splitting of $T(\log D_{Z_0})$ on the general line $l = x^\vee$ (where x is a general point) is of type $\mathcal{O}_{x^\vee}(-n+t) \oplus \mathcal{O}_{x^\vee}(-n-r-t)$ with

$t \geq 0$. This means that there is a curve of degree $n-t+1$ passing through Z_0 and having multiplicity $n-t$ at x. Then this curve is the union of L and $n-t$ lines through x. But since there are n points outside L the number t cannot be positive. So the arrangement D_{Z_0} is free of type $(n, n+r)$. $\square$

Remark. We can say more about the combinatorial type of Z, assuming that it is free of type $(n, n+r)$, and that it admits a $(n+r+1)$-secant line. Let us write the reduction exact sequence. Set $Z_1 = Z \setminus Z \cap L$. Then we have:

$$0 \longrightarrow \mathcal{J}_{Z_1} \longrightarrow \mathcal{J}_Z(1) \longrightarrow \mathcal{O}_L(-n-r) \longrightarrow 0.$$

We apply the functor $p_* q^*$ to obtain the following long exact sequence:

$$\begin{gathered} 0 \to \mathcal{O}_{\mathbb{P}^2}(-n) \longrightarrow p_* q^* \mathcal{J}_Z(1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-n-r) \to \\ \to R^1 p_* q^* \mathcal{J}_{Z_1} \longrightarrow R^1 p_* q^* \mathcal{J}_Z(1) \longrightarrow R^1 p_* q^* \mathcal{O}_L(-n-r) \to 0. \end{gathered}$$

Since $p_* q^* \mathcal{J}_Z(1) \cong \mathcal{O}_{\mathbb{P}^2}(-n) \oplus \mathcal{O}_{\mathbb{P}^2}(-n-r)$, we have a short exact sequence relating the locus of 2-secant lines to Z_1 to the locus of 3-secant lines to Z:

$$0 \to R^1 p_* q^* \mathcal{J}_{Z_1} \longrightarrow R^1 p_* q^* \mathcal{J}_Z(1) \longrightarrow R^1 p_* q^* \mathcal{O}_L(-n-r) \to 0.$$

The last sheaf is the structure sheaf of the fat point of length $\binom{n+r}{2}$ supported on $L^\vee$. So any 2-secant line to Z_1 must correspond to a 3-secant line to Z, i.e. any line passing through $r+2$ points ($r \geq 0$) of Z_1 must further pass through a point of Z lying on L.

References

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[OT92] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[Sai80] Kyoji Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):265–291, 1980.

[Val07] Jean Vallès. Fibrés logarithmiques sur le plan projectif. *Ann. Fac. Sci. Toulouse Math. (6)*, 16(2):385–395, 2007.

[Yos04] Masahiko Yoshinaga. Characterization of a free arrangement and conjecture of Edelman and Reiner. *Invent. Math.*, 157(2):449–454, 2004.

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