HYPERPLANE ARRANGEMENTS OF TORELLI TYPE

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Abstract. We give a necessary and sufficient condition in order for a hyperplane arrangement to be of Torelli type, namely that it is recovered as the set of unstable hyperplanes of its Dolgachev sheaf of logarithmic differentials. Decompositions and semistability of non-Torelli arrangements are investigated.

Introduction

Let X be a smooth complex algebraic variety and D a reduced divisor on X. The sheaf $\Omega_X(\log D)$ of logarithmic differential forms on X with poles along D was defined by Deligne in [Del70] in case D has normal crossings, and by Saito in [Sai80] for more general D. The sheaf $\Omega_X(\log D)$ consists of the meromorphic differential forms ω on X such that both ω and $d\omega$ have at most a first-order pole along D. Deligne's $\Omega_X(\log D)$ is a locally free sheaf, whereas Saito's is only reflexive in general, hence locally free when X is a surface. However we have the residue exact sequence (see [Sai80, (2.5)]):

$$0 \to \Omega_X \to \Omega_X(\log D) \xrightarrow{\text{res}} \nu_*(\mathcal{M}_{\widetilde{D}}),$$

where $\mathcal{M}_{\widetilde{D}}$ is the sheaf of meromorphic functions on $\widetilde{D}$ and $\nu : \widetilde{D} \to D$ is a resolution of singularities of D. When D has normal crossings, the image of res is $\nu_*(\mathscr{O}_{\tilde{D}})$, otherwise it only contains $\nu_*(\mathscr{O}_{\tilde{D}})$ (see [Sai80, (2.8)]).

Dolgachev in [Dol07] introduced a sub-sheaf $\widetilde{\Omega}_X(\log D)$ of $\Omega_X(\log D)$ (see also Catanese-Hosten-Khetan-Sturmfels, [CHKS06] for a related sheaf). Although this sheaf may fail to be reflexive in general, it always fits into the residue exact sequence:

$$0 \to \Omega_X \to \widetilde{\Omega}_X(\log D) \xrightarrow{\text{res}} \nu_*(\mathscr{O}_{\widetilde{D}}) \to 0.$$

Let us now study these sheaves in the framework of hyperplane arrangements. Namely, we focus on the case where $X = \mathbb{P}^n$, and D is the union of ℓ distinct hyperplanes $H_1, \ldots, H_\ell$ of $\mathbb{P}^n$, so $H_i = \{f_i = 0\}$, where f_i is a linear form. The collection of the H_j's is the hyperplane arrangement, and D is the hyperplane arrangement divisor. In this case, the residue exact sequence reads:

$$0 \to \Omega_{\mathbb{P}^n} \to \widetilde{\Omega}_{\mathbb{P}^n}(\log D) \xrightarrow{\text{res}} \bigoplus_{i=1,\ldots,\ell} \mathscr{O}_{H_i} \to 0.$$

The topology, the geometry, and the combinatorial properties of the pair $(\mathbb{P}^n, D)$ are interesting from many points of view, we refer to [OT92] for a comprehensive treatment. Let us only mention that Arnold, in his paper [Arn69], first used the

2000 *Mathematics Subject Classification.* 14F05, 14C34, 52C35, 32S22.

Key words and phrases. Hyperplane arrangements, Torelli theorem, unstable hyperplanes, sheaf of logarithmic differentials.

All authors partially supported by ANR-09-JCJC-0097-0 INTERLOW and by ANR GEOLMI. D.M. has been partially supported by grant CNCSIS PNII-IDEI 1189/2008.

algebra generated by the logarithmic forms df_i/f_i, to give an explicit description of the cohomology ring of $\mathbb{P}^n \setminus D$, an approach generalized by Brieskorn, see [Bri73].

Let D be a hyperplane arrangement divisor with normal crossings, so D corresponds to a *generic* arrangement $\{H_1, \dots, H_\ell\}$, namely D is such that any k distinct hyperplanes among the H_i's meet along a $\mathbb{P}^{n-k}$. The sheaf $\Omega_{\mathbb{P}^n}(\log(D))$ is then associated with the arrangement. The main question asked (and solved) by Dolgachev and Kapranov in [DK93], is whether one can reconstruct D from $\Omega_{\mathbb{P}^n}(\log(D))$. We say that $\{H_1, \dots, H_\ell\}$ is a *Torelli arrangement* in this case (or simply D is Torelli). They proved that if $\deg(D) \geq 2n+3$, then D is Torelli if and only if D does not osculate a rational normal curve. The result was extended to the range $\deg(D) \geq n+2$ in [Val00].

However this result only covers generic arrangement, while the most interesting arrangements are far from being so. For example, if D consists of two generic sets of three concurrent lines, then $\Omega_{\mathbb{P}^2}(\log D) \cong \mathscr{O}_{\mathbb{P}^2}(1) \oplus \mathscr{O}_{\mathbb{P}^2}(2)$, and it is clearly impossible to reconstruct D from $\Omega_{\mathbb{P}^2}(\log D)$.

Dolgachev in [Dol07] studied the sheaf $\tilde{\Omega}_{\mathbb{P}^n}(\log(D))$, for hyperplane arrangements. It turns out that $\tilde{\Omega}_{\mathbb{P}^n}(\log(D))$ is locally free if and only if D has normal crossings, and that it agrees with $\Omega_{\mathbb{P}^n}(\log(D))$ if D has normal crossings in codimension 2. Moreover, the sheaf $\tilde{\Omega}_{\mathbb{P}^n}(\log(D))$ is a Steiner sheaf having a resolution of the form:

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-(n+1)} \to \mathscr{O}_{\mathbb{P}^n}^{\ell-1} \to \tilde{\Omega}_{\mathbb{P}^n}(\log(D)) \to 0,$$

in particular its Chern polynomial depends only on n and ℓ. Further, Dolgachev took up the study of the Torelli problem for the sheaf $\tilde{\Omega}_{\mathbb{P}^n}(\log(D))$. He proposed the following conjecture:

Conjecture. Assume $\tilde{\Omega}_{\mathbb{P}^n}(\log(D))$ is a semi-stable sheaf in the sense of Gieseker. Then D is Torelli if and only if the points given by the H_i's in the dual projective space $\mathbb{P}_n$ do not belong to a stable rational curve of degree n.

A *stable rational curve* here means a connected curve of arithmetic genus 0 which is the union of s smooth rational curves $C_1, \dots, C_s$, with $\deg(C_i) = d_i$ and $d_1 + \cdots + d_s = n$, each C_i spanning a $\mathbb{P}^{d_i}$, and the union of the $\mathbb{P}^{d_i}$'s spanning the dual space $\mathbb{P}_n$. In [Dol07], the conjecture is proved for arrangements of up to 6 lines.

In this paper we study in detail the Torelli problem for the sheaf $\tilde{\Omega}_{\mathbb{P}^n}(\log(D))$. We denote by Z a finite set of points, say ℓ points $y_1, \dots, y_\ell$, lying in the dual projective space $\mathbb{P}_n$ of $\mathbb{P}^n$, and by D_Z the union of the corresponding hyperplanes $H_{y_1}, \dots, H_{y_\ell}$. We say that $Z \subset \mathbb{P}_n$ is Torelli according to whether D_Z is Torelli or not. In order to state our result, we need to introduce what we call *Kronecker-Weierstrass varieties* (a reason for this name will be apparent later on). For $s \geq 0$, and given a string $(d, n_1, \dots, n_s)$ of $s+1$ positive integers such that $n = d + n_1 + \cdots + n_s$, we say that $Y \subset \mathbb{P}_n$ is a *Kronecker-Weierstrass (KW) variety of type* $(d; s)$ if $Y = C \cup L_1 \cup \cdots \cup L_s \subset \mathbb{P}_n$, where the L_i's are linear subspaces of dimension n_i and C is a smooth rational curve of degree d spanning a linear space L of dimension d such that:

i) for all i, $L \cap L_i$ is a single point which lies in C;
ii) the spaces L_i's are mutually disjoint.

For $s \geq 2$, we will also call Y a KW variety of type $(0; s)$ if $Y = L_1 \cup \cdots \cup L_s \subset \mathbb{P}_n$ where the L_i's are linear spaces of dimension $n_i \geq 1$, with $n = n_1 \cdots + n_s$, with the property that $\exists y \in \mathbb{P}_n$, such that $L_i \cap L_j = y$ for all $i \neq j$. The point y will be called the *distinguished point* of Y.

We formulate now our main result.

Theorem 1. *Assume that $Z = \{y_1, \dots, y_\ell\} \subset \mathbb{P}_n$ is contained in no hyperplane. Then Z fails to be Torelli if and only if Z is contained in a KW variety $Y \subset \mathbb{P}_n$ of type $(d;s)$ such that either $d > 0$, $s \geq 0$, or $d = 0$, $s \geq 2$, and the distinguished point of Y does not lie in Z.*

The main ingredient that we bring in the proof is a functorial definition of $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ as the dualized direct image of the sheaf of linear forms vanishing at Z in $\mathbb{P}_n$, under the natural point-hyperplane incidence variety. More precisely, if p, q are the maps from this incidence variety to $\mathbb{P}^n$ and to the dual space $\mathbb{P}_n$, we first consider the complex $\mathbf{R}p_*(q^*\mathcal{I}_Z(1))$, and then take its derived dual, twisted by $\mathscr{O}_{\mathbb{P}^n}(-1)$. This process can be thought of as an integral transform of Penrose-Radon, or Fourier-Mukai type of the ideal sheaf $\mathcal{I}_Z(1)$. In fact the main result will be obtained from a slightly more general one (Theorem 2) addressing non-reduced subschemes $Z \subset \mathbb{P}_n$; we will see shortly how to make sense of this.

As a corollary of the theorem above, we get that if Z is contained in a stable rational curve in $\mathbb{P}_n$, then Z is not Torelli, as conjectured by Dolgachev.

As another corollary, we will see that the converse implication holds on $\mathbb{P}^2$, even without the assumption that $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ is semistable. In higher dimension, this implication no longer holds, regardless of $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ being semistable or not. To understand why, one first remarks that in many examples Z is contained in a KW variety Y without lying on a stable rational curve. Yet one has to prove semistability of $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ for some of these examples. One way to do this is to provide a filtration of $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ associated with the decomposition of Y into irreducible components. This is the content of Theorem 4. Some exceptions to the "if" direction of Dolgachev's conjecture are Example 3.5 and 3.6. These are examples of plane arrangements D_Z in $\mathbb{P}^3$ that fail to be Torelli, and such that the points $Z \subset \mathbb{P}_3$ lie in no stable rational curve, while $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ is semistable in the sense of Gieseker.

0.1. **Structure of the paper.** In the next section we set up our framework for dealing with logarithmic sheaves, based on direct images of ideal sheaves. In section 2 we prove our main theorem, already stated above. This section also contains a result on the maximal number of unstable hyperplanes of a Steiner sheaf, see Theorem 3. Section 3 is devoted to build a decomposition tool for non-Torelli arrangements. In this last section we will outline some examples with interesting non-Torelli phenomena.

0.2. **Notations.** We refer to [OT92] for basic notions on hyperplane arrangements. As a matter of notation, we let $\mathbb{P}^n$ be the space of 1-dimensional quotients of a $\mathbf{k}$-vector space V of dimension $n+1$ over a field $\mathbf{k}$, and we write $\mathbb{P}^n = \mathbb{P}(V)$. We let $\mathbb{P}_n = \mathbb{P}(V^*)$ be the dual of $\mathbb{P}^n$, namely the space of hyperplanes of $\mathbb{P}^n$. Given a point $y \in \mathbb{P}_n$, we let H_y be the hyperplane of $\mathbb{P}^n$ given by y. We use the variables $x_0, \dots, x_n$ for the polynomial ring of $\mathbb{P}^n$, and the variables $z_0, \dots, z_n$ for the polynomial ring R of $\mathbb{P}_n$.

If Z is a finite set of distinct points $\{y_1, \dots, y_\ell\}$ in $\mathbb{P}_n$, then we have $\{H_{y_1}, \dots, H_{y_\ell}\}$, a collection of ℓ hyperplanes in $\mathbb{P}^n$. This collection, as well as Z, will be called a *hyperplane arrangement*. More generally, let Z be a finite length subscheme of the dual space $\mathbb{P}_n$ of $\mathbb{P}^n$. The scheme Z consists of finitely many points $y_1, \dots, y_s$, each y_i supporting a subscheme of length m_i. Then Z defines the divisor D_Z in $\mathbb{P}^n$, namely the set $H_{y_1}, \dots, H_{y_s}$ of hyperplanes of $\mathbb{P}^n$, each H_{y_i} counted with multiplicity m_i. Namely:

$$D_Z = m_1 H_{y_1} + \cdots + m_s H_{y_s}.$$

The divisor D_Z is called the *hyperplane arrangement divisor* associated with Z. Note that, if Z is not reduced, D_Z does not depend on the scheme structure of Z, rather only on the support of Z and on the length of Z at each point. We will define later on the sheaves of logarithmic derivations and tangent fields associated with Z, regardless on whether Z is reduced or not. These, on the other hand, will depend on the scheme structure of Z.

We will have to deal with complexes of coherent sheaves on a smooth projective variety X (in fact almost only on $\mathbb{P}^n$). A natural framework for them is the derived category $\mathbf{D}^b(X)$ of complexes of coherent sheaves with bounded cohomology. We refer to [GM96] for a comprehensive treatment. We will denote by $[i]$ the i-th shift to the left of a complex in the derived category. In particular if E, F are coherent sheaves on X, we have $\mathrm{Hom}_{\mathbf{D}^b(X)}(E, F[i]) \cong \mathrm{Ext}^i_X(E,F)$. To shorten notations, we will denote by $(a \to b \to c \xrightarrow{[1]})$ the exact triangle $(a \to b \to c \to a[1])$. We will write $\mathbf{R}F$ for the right derived functor of a functor F, with image in the derived category.

We write the first Chern class $c_1(E)$ of a coherent sheaf E on $\mathbb{P}^n$ as an integer, meaning the corresponding multiple of $c_1(\mathscr{O}_{\mathbb{P}^n}(1))$.

1. The Steiner sheaf associated with a hyperplane arrangement

We consider the incidence variety $\mathbb{F}^n_n$ of pairs $(x,y) \in \mathbb{P}^n \times \mathbb{P}_n$ where x lies in H_y. We let p and q be the projections from $\mathbb{F}^n_n$ respectively to $\mathbb{P}^n$ and to $\mathbb{P}_n$. These projections are $\mathbb{P}^{n-1}$-bundles. We have the natural exact sequence:

$$0 \to \mathscr{O}_{\mathbb{P}^n \times \mathbb{P}_n}(-1,-1) \to \mathscr{O}_{\mathbb{P}^n \times \mathbb{P}_n} \to \mathscr{O}_{\mathbb{F}^n_n} \to 0. \tag{1.1}$$

We regard the complex $\mathbf{R}p_*((q^*(\mathcal{I}_Z(1))))$ as an element of the derived category of complexes of coherent sheaves on $\mathbb{P}^n$. We give the definition of a sheaf $\mathscr{F}_Z$ on $\mathbb{P}^n$ associated with Z, although it will turn out (Proposition 1.3) that $\mathscr{F}_Z$ is in fact isomorphic to the sheaf $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ introduced by Dolgachev. However we will stick to the shorter notation $\mathscr{F}_Z$ all over the paper.

Definition 1.1. Given a finite length subscheme Z of $\mathbb{P}_n$ we define the following object of the derived category $\mathbf{D}^b(\mathbb{P}^n)$:

$$\mathscr{F}_Z = \mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(\mathbf{R}p_*(q^*(\mathcal{I}_Z(1))), \mathscr{O}_{\mathbb{P}^n}(-1)).$$

Whenever the vector space V underlying $\mathbb{P}^n$ is unclear, we will rather write $\mathscr{F}^V_Z$. According to this definition, $\mathscr{F}_Z$ is a double complex which represents an element in the derived category $\mathbf{D}^b(\mathbb{P}^n)$, and many of its cohomology sheaves $\mathcal{H}^k(\mathscr{F}_Z)$ can be non-zero. However, as we will see in the next proposition, $\mathscr{F}_Z$ is often concentrated in degree zero (i. e. $\mathcal{H}^k(\mathscr{F}_Z) = 0$ for $k \neq 0$) namely $\mathscr{F}_Z$ is isomorphic to a coherent sheaf, in which case we regard it as such.

Proposition 1.2. *Let $Z \subset \mathbb{P}_n$ be a (schematically) non-degenerate subscheme of length ℓ. Then $\mathscr{F}_Z$ is concentrated in degree zero. In this case $\mathscr{F}_Z$, regarded* *as a coherent sheaf, is a Steiner sheaf (see Definition 2.1), having the following resolution:*

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-(n+1)} \to \mathscr{O}_{\mathbb{P}^n}^{\ell-1} \to \mathscr{F}_Z \to 0.$$

In this setting, $\mathscr{F}_Z$ is a torsion-free sheaf if, locally around any point $z \in Z$, we have $\mathcal{I}_z^2 \subset \mathcal{I}_Z$.

Proof. Working on the product $\mathbb{P}^n \times \mathbb{P}_n$, we tensor (1.1) with $q^*(\mathcal{I}_Z(1))$, obtaining thus the exact sequence:

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1) \boxtimes \mathcal{I}_Z \to \mathscr{O}_{\mathbb{P}^n} \boxtimes \mathcal{I}_Z(1) \to q^*(\mathcal{I}_Z(1)) \to 0. \tag{1.2}$$

Since Z has finite length, we have $\mathrm{H}^k(\mathbb{P}_n, \mathcal{I}_Z(t)) = 0$ for all $k > 1$ and for all $t \in \mathbb{Z}$. Further, we have $\mathrm{H}^0(\mathbb{P}_n, \mathcal{I}_Z) = 0$ for Z is not empty and $\mathrm{H}^0(\mathbb{P}_n, \mathcal{I}_Z(1)) = 0$ since Z is non-degenerate; note also that Z being non-degenerate implies $\ell \geq n+1$. Therefore, taking direct image onto $\mathbb{P}^n$, we get the following distinguished triangle:

$$\mathbf{R}p_*(q^*(\mathcal{I}_Z(1))) \to \mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-1} \xrightarrow{M_Z} \mathscr{O}_{\mathbb{P}^n}^{\ell-(n+1)} \to \mathbf{R}p_*(q^*(\mathcal{I}_Z(1)))[1] \tag{1.3}$$

where M_Z is obtained applying $\mathbf{R}p_*(-)$ to the inclusion appearing in (1.2). Therefore $\mathbf{R}p_*(q^*(\mathcal{I}_Z(1)))$ has cohomology only in degree 0 and 1, and is isomorphic to the shift by one to the left of the cone of:

$$\mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-1} \xrightarrow{M_Z} \mathscr{O}_{\mathbb{P}^n}^{\ell-(n+1)}.$$

Taking $\mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(-, \mathscr{O}_{\mathbb{P}^n}(-1))$, we get that $\mathscr{F}_Z$ is isomorphic to the cone of:

$$\mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-(n+1)} \xrightarrow{M_Z^t} \mathscr{O}_{\mathbb{P}^n}^{\ell-1}.$$

Further, the sheaf $\mathbf{R}^1 p_*(q^*(\mathcal{I}_Z(1)))$ is supported at the points x of $\mathbb{P}^n$ such that $\mathrm{H}^1(H_x, \mathcal{I}_{Z \cap H_x}(1)) \neq 0$. In particular, it is a torsion sheaf. Therefore, the map M_Z^t is injective, hence $\mathscr{F}_Z$ is concentrated in degree zero, and we have the exact sequence:

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-(n+1)} \xrightarrow{M_Z^t} \mathscr{O}_{\mathbb{P}^n}^{\ell-1} \to \mathscr{F}_Z \to 0. \tag{1.4}$$

It remains to prove that $\mathscr{F}_Z$ is torsion-free under our assumptions. To unwind the double complex $\mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(\mathbf{R}p_*(q^*(\mathcal{I}_Z(1))), \mathscr{O}_{\mathbb{P}^n}(-1))$, we write the cohomology of (1.3) as the pair of exact sequences:

$$0 \to p_*(q^*(\mathcal{I}_Z(1)) \to \mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-1} \to \mathrm{Im}(M_Z) \to 0, \tag{1.5}$$

$$0 \to \mathrm{Im}(M_Z) \to \mathscr{O}_{\mathbb{P}^n}^{\ell-n-1} \to \mathbf{R}^1 p_*(q^*(\mathcal{I}_Z(1)) \to 0. \tag{1.6}$$

We apply the functor $\mathcal{H}om_{\mathbb{P}^n}(-, \mathscr{O}_{\mathbb{P}^n})$ to these sequences. We recall that $\mathcal{H}om_{\mathbb{P}^n}(\mathbf{R}^1 p_*(q^*(\mathcal{I}_Z(1))), \mathscr{O}_{\mathbb{P}^n}(-1)) = 0$, and we note that

$\mathcal{E}xt^1_{\mathbb{P}^n}(\mathrm{Im}(M_Z), \mathscr{O}_{\mathbb{P}^n}(-1)) \cong \mathcal{E}xt^2_{\mathbb{P}^n}(\mathbf{R}^1 p_*(q^*(\mathcal{I}_Z(1))), \mathscr{O}_{\mathbb{P}^n}(-1))$. We obtain an exact commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \to \mathscr{O}_{\mathbb{P}^n}^{\ell-1}(-1) & \longrightarrow & \mathcal{H}om_{\mathbb{P}^n}(\mathrm{Im}(M_Z), \mathscr{O}_{\mathbb{P}^n}(-1)) & \longrightarrow & \mathcal{E}xt^1_{\mathbb{P}^n}(\mathbf{R}^1 p_*(q^*(\mathcal{I}_Z(1))), \mathscr{O}_{\mathbb{P}^n}(-1)) & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
0 \to \mathscr{O}_{\mathbb{P}^n}^{\ell-1}(-1) & \xrightarrow{M_Z^t} & \mathscr{O}_{\mathbb{P}^n}^{\ell-n-1} & \longrightarrow & \mathscr{F}_Z & \longrightarrow & 0 \\
 & & \downarrow & & & & \\
 & & \mathcal{H}om_{\mathbb{P}^n}(p_*(q^*(\mathcal{I}_Z(1))), \mathscr{O}_{\mathbb{P}^n}(-1)) & & & & \\
 & & \downarrow & & & & \\
 & & \mathcal{E}xt^2_{\mathbb{P}^n}(\mathbf{R}^1 p_*(q^*(\mathcal{I}_Z(1))), \mathscr{O}_{\mathbb{P}^n}(-1)) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

We let $\mathscr{K}$ be the cokernel of the rightmost vertical arrow, and we get the two short exact sequences:

$$0 \to \mathcal{E}xt^1_{\mathbb{P}^n}(\mathbf{R}^1 p_*(q^*(\mathcal{I}_Z(1))), \mathscr{O}_{\mathbb{P}^n}(-1)) \to \mathscr{F}_Z \to \mathscr{K} \to 0, \tag{1.7}$$

$$\mathscr{K} \hookrightarrow \mathcal{H}om_{\mathbb{P}^n}(p_*(q^*(\mathcal{I}_Z(1))), \mathscr{O}_{\mathbb{P}^n}(-1)) \to \mathcal{E}xt^2_{\mathbb{P}^n}(\mathbf{R}^1 p_*(q^*(\mathcal{I}_Z(1))), \mathscr{O}_{\mathbb{P}^n}(-1)) \to 0. \tag{1.8}$$

The coherent sheaf $\mathscr{K}$ is always torsion-free, and it differs from $\mathscr{F}_Z$ if and only if $\mathbf{R}^1 p_*(q^*(\mathcal{I}_Z(1)))$ is supported in codimension 1. A necessary and sufficient condition for $\mathbf{R}^1 p_*(q^*(\mathcal{I}_Z(1)))$ to be supported in codimension 1, is that there is $z \in Z$ such that, for all $x \in H_z$, we have $\mathrm{H}^1(H_x, \mathcal{I}_{Z \cap H_x}(1)) \neq 0$. This is equivalent to say that, given any linear form f vanishing at z, the ideal of Z modulo f contains all the quadrics of R/f.

In order to check the above condition, we can assume that the reduced support of Z is a single point, for H_x generically avoids all other points. Working locally around this point $z \in Z$, our hypothesis is thus that all quadrics vanishing at z are in the ideal of Z. Therefore, the same thing takes place modulo f, and we are done. □

We will record the notation of the above proposition, so given Z we have the matrix M_Z:

$$\mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-1} \xrightarrow{M_Z} \mathscr{O}_{\mathbb{P}^n}^{\ell-(n+1)}, \qquad \mathscr{F}_Z \cong \mathrm{Cok}(M_Z^t). \tag{1.9}$$

Let us describe the relationship between our sheaf $\mathscr{F}_Z$ and the sheaves $\Omega_{\mathbb{P}^n}(\log D_Z)$ and $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$. We will assume that Z is reduced, because the latter two sheaves are defined only in this case. First, recall the definition of $\Omega_{\mathbb{P}^n}(\log D_Z)$. Let $f = \prod_{i=1}^{\ell} f_i$ be a polynomial defining D_Z, so D_Z is the hyperplane arrangement divisor associated to a set Z of ℓ points in $\mathbb{P}_n$. We consider the sheafified derivation module, or sheaf of *logarithmic tangent fields* $\mathscr{D}_0(Z)$. This is defined by the exact sequence:

$$0 \to \mathscr{D}_0(Z) \to \mathscr{O}_{\mathbb{P}^n}^{n+1} \xrightarrow{(\partial_0 f, \ldots, \partial_n f)} \mathscr{O}_{\mathbb{P}^n}(\ell - 1). \tag{1.10}$$

We refer for instance to [Sch03] for the study of this sheaf (which is called there the syzygy sheaf and denoted by $\mathcal{D}$). Then the sheaf $\Omega_{\mathbb{P}^n}(\log D_Z)$ can be defined (see for instance [MS01]) by:

$$\Omega_{\mathbb{P}^n}(\log D_Z) = \mathcal{H}om_{\mathbb{P}^n}(\mathscr{D}_0(Z), \mathscr{O}_{\mathbb{P}^n}(-1)).$$

Here is the result describing the relationship among these sheaves.

Proposition 1.3. *Assume that Z is reduced and non-degenerate. Then $\mathscr{F}_Z$ is isomorphic to Dolgachev's sheaf $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$. Moreover, we have:*

$$\Omega_{\mathbb{P}^n}(\log D_Z) \cong \mathcal{H}om_{\mathbb{P}^n}(p_*(q^*(\mathcal{I}_Z(1))), \mathscr{O}_{\mathbb{P}^n}(-1)) \cong \mathscr{F}_Z^{**}. \tag{1.11}$$

Let us start with the following useful result.

Claim 1.4. *We have a natural isomorphism:*

$$\mathrm{Ext}^1_{\mathbb{P}^n}(p_*(q^*(\mathscr{O}_Z)), \Omega_{\mathbb{P}^n}) \cong \mathrm{Hom}_{\mathbb{P}_n}(\mathscr{O}_{\mathbb{P}_n}, \mathscr{O}_Z)^*.$$

Proof. We have the natural isomorphisms:

$$\begin{aligned}\mathrm{Ext}^1_{\mathbb{P}^n}(p_*(q^*(\mathscr{O}_Z)), \Omega_{\mathbb{P}^n}) &\cong \mathrm{Ext}^{n-1}_{\mathbb{P}^n}(\Omega_{\mathbb{P}^n}(n+1), p_*(q^*(\mathscr{O}_Z)))^* \cong \\ &\cong \mathrm{Ext}^{n-1}_{\mathbb{F}^n_n}(p^*(\Omega_{\mathbb{P}^n}(n+1)), q^*(\mathscr{O}_Z))^* \cong \\ &\cong \mathrm{Hom}_{\mathbf{D}^b(\mathbb{F}^n_n)}(p^*(\Omega_{\mathbb{P}^n}(n+1)), q^*(\mathscr{O}_Z)[n-1])^*,\end{aligned}$$

where the first one is Serre duality and the second one is given by the fact that p^* is a left adjoint functor of p_*. Now we use the left adjoint functor to q^*, namely the functor $\mathbf{R}q_*(-\otimes \mathscr{O}_{\mathbb{F}^n_n}(-n,1))[n-1]$. Thus the latter Ext group above is isomorphic to:

$$\begin{aligned}&\cong \mathrm{Hom}_{\mathbf{D}^b(\mathbb{P}_n)}(\mathbf{R}q_*(p^*(\Omega_{\mathbb{P}^n}(1))) \otimes \mathscr{O}_{\mathbb{P}^n}(1), \mathscr{O}_Z)^* \cong \\ &\cong \mathrm{Hom}_{\mathbb{P}_n}(\mathscr{O}_{\mathbb{P}_n}, \mathscr{O}_Z)^*,\end{aligned}$$

where the last isomorphism is obtained using $\mathbf{R}q_*(p^*(\Omega_{\mathbb{P}^n}(1))) \cong \mathscr{O}_{\mathbb{P}_n}(-1)$. □

Proof of Proposition 1.3. Let us first prove the claim regarding $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$. We apply the functor $\mathbf{R}p_*(q^*(-))$ to the exact sequence:

$$0 \to \mathcal{I}_Z(1) \to \mathscr{O}_{\mathbb{P}_n}(1) \to \mathscr{O}_Z \to 0.$$

Using (1.2), we obtain the distinguished triangle:

$$\mathbf{R}p_*(q^*(\mathcal{I}_Z(1))) \to \mathcal{T}_{\mathbb{P}^n}(-1) \to \mathbf{R}p_*(q^*(\mathscr{O}_Z)) \xrightarrow{[1]} \tag{1.12}$$

Now, as Z is reduced, we have $Z = \{y_1, \ldots, y_\ell\}$. Note that:

$$\begin{aligned}q^*(\mathscr{O}_Z) &\cong \mathscr{O}_{q^{-1}(Z)} \cong \mathscr{O}_{\cup_{j=1,\ldots,\ell} H_{y_j}}, \\ p_*(q^*(\mathscr{O}_Z)) &\cong \bigoplus_{j=1,\ldots,\ell} \mathscr{O}_{H_{y_j}}.\end{aligned}$$

The sheaf $q^*(\mathscr{O}_Z)$ lies above the divisor $D_Z = \cup_{j=1,\ldots,\ell} H_{y_j}$, and $p : q^{-1}(Z) \to D_Z$ is a resolution of singularities of D_Z. For each j we have $\mathcal{E}xt^k_{\mathbb{P}^n}(\mathscr{O}_{H_{y_j}}, \mathscr{O}_{\mathbb{P}^n}) = 0$ if $k \neq 1$ and $\mathcal{E}xt^1_{\mathbb{P}^n}(\mathscr{O}_{H_{y_j}}, \mathscr{O}_{\mathbb{P}^n}) \cong \mathscr{O}_{H_{y_j}}(1)$, hence:

$$\mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(p_*(q^*(\mathscr{O}_Z)), \mathscr{O}_{\mathbb{P}^n})[1] \cong p_*(q^*(\mathscr{O}_Z))(1). \tag{1.13}$$

Therefore, taking $\mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(-, \mathscr{O}_{\mathbb{P}^n}(-1))$ of the triangle (1.12), we have the exact sequence:

$$0 \to \Omega_{\mathbb{P}^n} \to \mathscr{F}_Z \to p_*(q^*(\mathscr{O}_Z)) \to 0. \tag{1.14}$$

We will be done if we can prove that this is the residue exact sequence defining $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ according to [Dol07]. Again $p_*(\mathscr{O}_{q^{-1}(Z)}) \cong p_*(q^*(\mathscr{O}_Z)) \cong \bigoplus_{i=1,\dots,\ell} \mathscr{O}_{H_{y_i}}$. So the exact sequence above is given by an element of $\mathrm{Ext}^1_{\mathbb{P}^n}(p_*(q^*(\mathscr{O}_Z)), \Omega_{\mathbb{P}^n})$, and by Claim 1.4 this is identified with element of $\mathrm{Hom}_{\mathbb{P}_n}(\mathscr{O}_{\mathbb{P}_n}, \mathscr{O}_Z)^*$. So $\mathscr{F}_Z$ is given, up to isomorphism, as the extension of $\bigoplus_{i=1,\dots,\ell} \mathscr{O}_{H_{y_i}}$ by $\Omega_{\mathbb{P}^n}$ dual to the canonical surjection $\mathscr{O}_{\mathbb{P}_n} \to \mathscr{O}_Z$. Indeed, on one hand it is easy to see that any surjective map $\mathscr{O}_{\mathbb{P}_n} \to \mathscr{O}_Z$ gives the same extension up to isomorphism. On the other hand, consider an extension of $\bigoplus_{i=1,\dots,\ell} \mathscr{O}_{H_{y_i}}$ by $\Omega_{\mathbb{P}^n}$ dual to a map $\mathscr{O}_{\mathbb{P}_n} \to \mathscr{O}_Z$ which is not surjective, say $\mathscr{O}_{y_j}$ is not in the image. Such extension contains $\mathscr{O}_{H_{y_j}}$ as a direct summand, which contradicts $\mathscr{F}_Z$ being torsion-free, in contrast with Proposition 1.2.

Let us now turn to $\Omega_{\mathbb{P}^n}(\log D_Z)$. Let again $f = \prod_{i=1}^{\ell} f_i$ be an equation defining D_Z. Recall that the image of the rightmost map in (1.10) (the gradient map) is the Jacobian ideal $\mathscr{J}$ of D_Z. Denote by $\mathscr{J}_{D_Z}$ the image of $\mathscr{J}$ in $\mathscr{O}_{D_Z}$ (so $\mathscr{J}_{D_Z} = \mathscr{J} \cdot \mathscr{O}_{D_Z}$). Recall the natural exact sequence relating $\mathscr{J}_{D_Z}$ and the sheaf $\mathscr{D}_0(Z)$ (see e.g. [Dol07, Section 2]):

$$0 \longrightarrow \mathscr{D}_0(Z) \longrightarrow \mathcal{T}_{\mathbb{P}^n}(-1) \longrightarrow \mathscr{J}_{D_Z}(\ell-1) \longrightarrow 0. \tag{1.15}$$

Note also that we have:

$$\begin{aligned}\mathrm{Hom}_{\mathbb{P}_n}(\mathscr{O}_{\mathbb{P}_n}, \mathscr{O}_Z)^* &\cong \mathrm{Ext}^1_{\mathbb{P}^n}(p_*(q^*(\mathscr{O}_Z)), \Omega_{\mathbb{P}^n}) \cong \\ &\cong \mathrm{Hom}_{\mathbf{D}^b(\mathbb{P}^n)}(\mathcal{T}_{\mathbb{P}^n}, \mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(p_*(q^*(\mathscr{O}_Z)), \mathscr{O}_{\mathbb{P}^n})[1]) \cong \\ &\cong \mathrm{Hom}_{\mathbb{P}^n}(\mathcal{T}_{\mathbb{P}^n}, p_*(q^*(\mathscr{O}_Z))(1)),\end{aligned}$$

where we used again (1.13). Further, from [Dol07, Proposition 2.4] we get an inclusion of $\mathscr{J}_{D_Z}(\ell)$ into $p_*(\omega_{q^{-1}Z} \otimes \omega^*_{\mathbb{P}^n}) \cong p_*(q^*(\mathscr{O}_Z))(1)$.

Therefore, both $\mathscr{D}_0(Z)$ (by (1.15)) and $p_*(q^*(\mathcal{I}_Z(1)))$ (by the cohomology sequence of (1.12)) are defined as kernel of some map $\mathcal{T}_{\mathbb{P}^n}(-1) \to p_*(q^*(\mathscr{O}_Z))$ that is dual to a map $\mathscr{O}_{\mathbb{P}_n} \to \mathscr{O}_Z$. We claim that this gives an isomorphism:

$$p_*(q^*(\mathcal{I}_Z(1))) \cong \mathscr{D}_0(Z). \tag{1.16}$$

Indeed, on one hand it is easy to see that the duals of any two surjective maps $\mathscr{O}_{\mathbb{P}_n} \to \mathscr{O}_Z$ give the same kernel of $\mathcal{T}_{\mathbb{P}^n}(-1) \to p_*(q^*(\mathscr{O}_Z))$. On the other hand, recall again $p_*(q^*(\mathscr{O}_Z)) \cong \bigoplus_{i=1,\dots,\ell} \mathscr{O}_{H_{y_i}}$ and consider a map $\mathscr{O}_{\mathbb{P}_n} \to \mathscr{O}_Z$ which is not surjective, say $\mathscr{O}_{y_j}$ is not in the image. Then the dual map factors through $\mathcal{T}_{\mathbb{P}^n}(-1) \to \bigoplus_{i \neq j} \mathscr{O}_{H_{y_i}}$. Therefore the first Chern class of the kernel of such map is strictly greater than $1-\ell$. But, looking at the exact sequences (1.5), (1.6), (1.15) and since $\mathbf{R}^1 p_*(q^*(\mathcal{I}_Z(1))$ is supported in codimension 2, we know that $c_1(p_*(q^*(\mathcal{I}_Z(1)))) = c_1(\mathscr{D}_0(Z)) = 1-\ell$, so (1.16) is proved.

Let us now conclude the proof. Note that from the above discussion we get an exact sequence:

$$0 \to \mathscr{F}_Z \to \Omega_{\mathbb{P}^n}(\log D_Z) \to \mathcal{E}xt^2_{\mathbb{P}^n}(\mathbf{R}^1 p_* q^*(\mathcal{I}_Z(1)), \mathscr{O}_{\mathbb{P}^n}(-1)) \to 0.$$

The desired isomorphisms (1.11) easily follow from the above sequence and (1.16). $\square$

Remark 1.5. The support of the cokernel sheaf $\mathcal{E}xt^2_{\mathbb{P}^n}(\mathbf{R}^1 p_*(q^*(\mathcal{I}_Z(1))), \mathscr{O}_{\mathbb{P}^n}(-1))$ sits in codimension $k > 1$ if and only Z contains a subscheme of length $(n+1)$, spanning a linear subspace $\mathbb{P}_{k-1}$. Further, this shows again that $\mathscr{F}_Z$ and $\Omega_{\mathbb{P}^n}(\log D_Z)$ agree if D_Z is normal crossing in codimension 2, see [Dol07, Corollary 2.8].

Example 1.6. Consider the ideal $(z_0 z_2^2, (z_1 + z_1) z_1 z_2, z_0 z_1 z_2, z_0 z_1^2)$. This defines a subscheme $Z \subset \mathbb{P}_2$, which is the union of the first infinitesimal neighbourhood of $y_1 = (1 : 0 : 0)$ and the three collinear points $y_2 = (0 : 1 : 0)$, $y_3 = (0 : 0 : 1)$, $y_4 = (0 : 1 : -1)$. In turn, the divisor associated to Z is $D_Z = 3H_{y_1} + H_{y_2} + H_{y_3} + H_{y_4}$. The matrix M_Z of (1.9) reads:

$$M_Z = \begin{pmatrix} -x_0 & 0 & x_1 & 0 & 0 \\ x_0 & 0 & 0 & x_1 - x_2 & -x_2 \\ 0 & x_0 & 0 & 0 & x_2 \end{pmatrix}.$$

In this case $\mathscr{F}_Z$ is still torsion-free and we have:

$$0 \to \mathscr{F}_Z \to \Omega_{\mathbb{P}^n}(\log D_Z) \to \mathscr{O}_{p_1,\ldots,p_4} \to 0, \qquad \Omega_{\mathbb{P}^n}(\log D_Z) \cong \mathscr{O}_{\mathbb{P}^2}(2) \oplus \mathscr{O}_{\mathbb{P}^2}(1),$$

where $p_1, \ldots, p_4$ are $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$, $(0 : 1 : 1)$, the 4 points corresponding to the 4 lines in $\mathbb{P}_2$ which are 3-secant to Z. In this case we have $\Omega_{\mathbb{P}^n}(\log D_Z)$ splits as a direct sum of line bundles, namely $\Omega_{\mathbb{P}^n}(\log D_Z) \cong \mathscr{D}_0(Z)^*(-1) \cong \mathscr{O}_{\mathbb{P}^2}(2) \oplus \mathscr{O}_{\mathbb{P}^2}(1)$.

Example 1.7. Consider the scheme Z defined as the union of the second infinitesimal neighbourhood of $y_1 = (0 : 1 : 0)$ and the two points $y_2 = (1 : 0 : 0)$, $y_3 = (0 : 0 : 1)$. Namely, the ideal of Z is $(z_0 z_2^2, z_0^2 z_2, z_1 z_2^3, z_0^3 z_1)$. Accordingly, we have $D_Z = 6H_{y_1} + H_{y_2} + H_{y_3}$. In this case, the matrix M_Z of (1.9) can be written as:

$$M_Z = \begin{pmatrix} 0 & 0 & -x_1 & 0 & 0 & 0 & x_2 \\ x_0 & 0 & 0 & x_1 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & x_1 & 0 & 0 \\ -x_1 & x_0 & 0 & 0 & 0 & 0 & 0 \\ -x_2 & 0 & x_0 & 0 & 0 & x_1 & 0 \end{pmatrix}.$$

Here we get the line L defined as $\{x_1 = 0\}$ as support of the torsion part of $\mathscr{F}_Z$. We have $\Omega_{\mathbb{P}^n}(\log D_Z) \cong \mathscr{O}_{\mathbb{P}^2}(2) \oplus \mathscr{O}_{\mathbb{P}^2}(2)$ (and we say that Z free). The exact sequences (1.7) and (1.8) become:

$$0 \to \mathscr{O}_L(-2) \to \mathscr{F}_Z \to \mathscr{O}_{\mathbb{P}^2}(2)^2 \to \mathscr{O}_{Z_1 \cup Z_2} \to 0,$$

where Z_1, Z_2 are two length-2 subschemes, supported at the points $(1 : 0 : 0)$ and $(0 : 0 : 1)$, accounting for the two 4-secant lines to Z in $\mathbb{P}_2$, namely $\{z_0 = 0\}$ and $\{z_2 = 0\}$.

2. Unstable hyperplanes of logarithmic sheaves

The goal of this section is to prove our main result on Torelli arrangements, from which Theorem 1 from the introduction will immediately follow. We will first need some definitions.

Definition 2.1. Let $\mathscr{E}$ be a Steiner sheaf on $\mathbb{P}^n$, namely a sheaf $\mathscr{E}$ fitting into an exact sequence of the form:

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1)^a \to \mathscr{O}_{\mathbb{P}^n}^b \to \mathscr{E} \to 0,$$

for some integers a, b. Then a hyperplane H is *unstable* for $\mathscr{E}$ if:

$$\mathrm{H}^{n-1}(H, \mathscr{E}_{|H}(-n)) \neq 0.$$

A point y of $\mathbb{P}_n$ is unstable for $\mathscr{E}$ if the hyperplane H_y is unstable for $\mathscr{E}$.

We can give a scheme structure to the set $\mathrm{W}(\mathscr{E})$ of unstable hyperplanes of $\mathscr{E}$, considering them as the scheme-theoretic support of the sheaf $\mathbf{R}^{n-1}q_*(p^*(\mathscr{E}(-n)))$.

Definition 2.2. A finite length subscheme Z of $\mathbb{P}_n$ is said to be *Torelli* if Z can be recovered from $\mathscr{F}_Z$, namely if the *set* of unstable hyperplanes of $\mathscr{F}_Z$ is the support of Z, i. e. if we have a set-theoretic equality:

$$\mathrm{W}(\mathscr{F}_Z) = Z.$$

Lemma 2.3. *Let Z be a finite length non-degenerate subscheme of $\mathbb{P}_n$. Then we have a scheme-theoretic inclusion:*

$$Z \subset \mathrm{W}(\mathscr{F}_Z).$$

Proof. By Grothendieck duality, we have:

$$\mathscr{F}_Z(-n) \cong \mathbf{R}p_*(\mathbf{R}\mathcal{H}om_{\mathbb{F}_n^n}(q^*(\mathcal{I}_Z(1)), \mathscr{O}_{\mathbb{F}_n^n}(-n,-n)))[n-1],$$

from which we get an epimorphism:

$$\mathbf{R}q_*(p^*(\mathscr{F}_Z(-n)))[n-1] \to \mathbf{R}q_*(\mathbf{R}\mathcal{H}om_{\mathbb{F}_n^n}(q^*(\mathcal{I}_Z(1)), \mathscr{O}_{\mathbb{F}_n^n}(-n,-n)))[n-1].$$

Applying again Grothendieck duality, we get an isomorphism of the right hand side above and:

$$\mathbf{R}\mathcal{H}om_{\mathbb{P}_n}(\mathbf{R}q_*(q^*(\mathcal{I}_Z(1))), \mathscr{O}_{\mathbb{P}_n}(-n)),$$

which projects onto:

$$\mathbf{R}\mathcal{H}om_{\mathbb{P}_n}(\mathcal{I}_Z(1), \mathscr{O}_{\mathbb{P}_n}(-n)).$$

Summing up, we have an epimorphism:

$$\mathbf{R}q_*(p^*(\mathscr{F}_Z(-n)))[n-1] \twoheadrightarrow \mathbf{R}\mathcal{H}om_{\mathbb{P}_n}(\mathcal{I}_Z(1), \mathscr{O}_{\mathbb{P}_n}(-n)),$$

and taking cohomology in degree $n-1$ we get:

$$\mathbf{R}^{n-1}q_*(p^*(\mathscr{F}_Z(-n))) \twoheadrightarrow \mathcal{E}xt^{n-1}_{\mathbb{P}_n}(\mathcal{I}_Z, \mathscr{O}_{\mathbb{P}_n}(-n-1)) \cong \mathscr{O}_Z,$$

which proves our claim. $\square$

Remark 2.4. It was already proved in [Dol07] that any $z \in Z$ is unstable for $\mathscr{F}_Z$, hence Z is not Torelli if and only if the *set* of unstable hyperplanes of $\mathscr{F}_Z$ strictly contains Z. One could say that Z is *scheme-theoretically Torelli* if the subscheme of unstable hyperplanes is Z itself. A criterion analogous to Theorem 1 for Z to be scheme-theoretically Torelli is lacking at the time being.

Remark 2.5. We point out that $\mathrm{W}(\mathscr{F}_Z) = \mathrm{W}(\Omega_{\mathbb{P}^n}(\log D_Z))$ if and only if Z does not possess a subscheme of length $(n+1)$ contained in a line, as explained in Remark 1.5. This remark makes more precise Proposition 3.2 of [Dol07].

2.1. **Kronecker-Weierstrass varieties and unstable hyperplanes.** In order to prove Theorem 2, we introduce some geometric objects that we call Kronecker-Weierstrass varieties. The name is inspired by the tool that classifies them. Indeed, the isomorphism classes of these varieties are given by the standard Kronecker-Weierstrass forms of a matrix of homogeneous linear forms in two variables. We recall the definition given in the introduction.

Definition 2.6. Let $s \geq 0$ and $(d, n_1, \ldots, n_s)$ be a string of $s+1$ positive integers such that $n = d + n_1 \cdots + n_s$. Then $Y \subset \mathbb{P}_n$ is a *Kronecker-Weierstrass (KW) variety of type* $(d; s)$ if $Y = C \cup L_1 \cup \cdots \cup L_s \subset \mathbb{P}_n$, where the L_i's are linear subspaces of dimension n_i and C is a smooth rational curve of degree d (called the *curve part* of Y) spanning a linear space L of dimension d such that:

i) for all i, $L \cap L_i$ is a single point which lies in C;
ii) the spaces L_i's are mutually disjoint.

If $d = 0$ and $s \geq 2$ a KW variety of type $(0; s)$ is defined as $Y = L_1 \cup \cdots \cup L_s \subset \mathbb{P}_n$, where the L_i's are linear subspaces of dimension n_i with $n = n_1 + \cdots + n_s$ and all the linear spaces L_i meet only at a point y, which is called *the distinguished point of* Y.

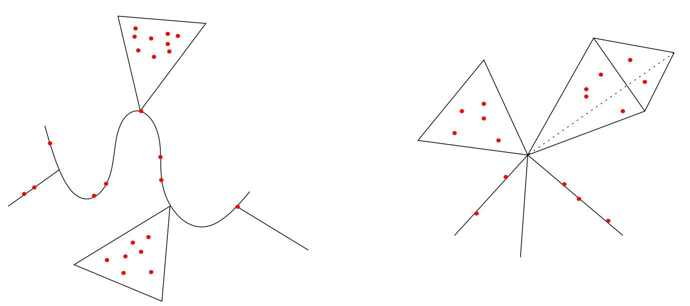


FIGURE 1. Points contained in a Kronecker Weierstrass variety.

Example 2.7. We give below a few examples of KW varieties.

1) A rational normal curve is a KW variety of type $(n; 0)$.
2) A union of two lines in $\mathbb{P}^2$ is a KW variety in three ways, two of them of type $(1; 1)$, and one of type $(0; 2)$ (the intersection point is the distinguished point).

Having this setup, we can now state our main result.

Theorem 2. *Let $Z \subset \mathbb{P}_n$ be a finite-length, set-theoretically non-degenerate subscheme. Then Z fails to be Torelli if and only if Z is contained in a KW variety $Y \subset \mathbb{P}_n$ of type $(d; s)$ such that either $d > 0$, $s \geq 0$, or $d = 0$, $s \geq 2$, and the distinguished point of Y does not lie in Z.*

We can now move towards the proof of this theorem. We need a series of lemmas and the following construction.

Given a point y of $\mathbb{P}_n$, we consider the Koszul complex resolving the ideal sheaf $\mathcal{I}_y$, namely a long exact sequence:

$$0 \to \mathscr{O}_{\mathbb{P}_n}(-n) \xrightarrow{d_n} \mathscr{O}_{\mathbb{P}_n}^n(-n+1) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_3} \mathscr{O}_{\mathbb{P}_n}^{\binom{n}{2}}(-2) \xrightarrow{d_2} \mathscr{O}_{\mathbb{P}_n}^n(-1) \xrightarrow{d_1} \mathcal{I}_y \to 0.$$

We let $\mathcal{S}_y$ be the sheaf $\mathrm{Im}(d_{n-1})$, twisted by $\mathscr{O}_{\mathbb{P}_n}(n)$. We have:

$$0 \to \mathscr{O}_{\mathbb{P}_n} \xrightarrow{(h_1,\dots,h_n)} \mathscr{O}_{\mathbb{P}_n}^n(1) \to \mathcal{S}_y \to 0, \tag{2.1}$$

where the h_i's are linear forms on $\mathbb{P}_n$ and y is defined by $\{h_1 = \cdots = h_n = 0\}$.

The following lemma is the key to our argument. It is inspired on a generalization of [Val11, Proposition 6.1]

Lemma 2.8. *Let y be a point of $\mathbb{P}_n$, and let Z be a finite length subscheme of $\mathbb{P}_n$ not containing y. Then y is unstable for $\mathscr{F}_Z$ if and only if $\mathrm{H}^0(\mathbb{P}_n, \mathcal{S}_y \otimes \mathcal{I}_Z) \neq 0$.*

Proof. By definition y is unstable for $\mathscr{F}_Z$ if and only if $\mathrm{H}^{n-1}(\mathbb{P}^n, \mathscr{O}_{H_y} \otimes \mathscr{F}_Z(-n)) \neq 0$. By the proof of Proposition 1.2, we have that the sheaf $\mathscr{F}_Z$ is not annihilated by the linear form f_y defining $H_y \subset \mathbb{P}^n$ because y lies away from Z. Therefore we get $\mathcal{T}or_j(\mathscr{O}_{H_y}, \mathscr{F}_Z) = 0$ for $j \neq 0$. Then, we have the natural isomorphisms:

$$\begin{aligned} \mathrm{H}^{n-1}(\mathbb{P}^n, \mathscr{O}_{H_y} \otimes \mathscr{F}_Z(-n)) &\cong \mathrm{H}^{n-1}(\mathbb{P}^n, \mathscr{O}_{H_y} \overset{\mathbf{L}}{\otimes} \mathscr{F}_Z(-n)) \cong \\ &\cong \mathrm{Hom}_{\mathbf{D}^b(\mathbb{P}^n)}(\mathbf{R}p_*(q^*(\mathcal{I}_Z(1))), \mathscr{O}_{H_y}(-n-1)[n-1]) \cong \\ &\cong \mathrm{Hom}_{\mathbf{D}^b(\mathbb{P}^n)}(\mathscr{O}_{H_y}, \mathbf{R}p_*(q^*(\mathcal{I}_Z(1)))[1])^*, \\ &\cong \mathrm{Hom}_{\mathbf{D}^b(\mathbb{F}^n_n)}(p^*(\mathscr{O}_{H_y}), q^*(\mathcal{I}_Z(1))[1])^*, \end{aligned}$$

where we used the definition of $\mathscr{F}_Z$ as twisted derived dual of $\mathbf{R}p_*(q^*(\mathcal{I}_Z(1)))$, Serre duality, and that p^* is a left adjoint functor of $\mathbf{R}p_*$. We use now the fact that the functor $\mathbf{R}q_*(- \otimes \mathscr{O}_{\mathbb{F}^n_n}(-n,1))[n-1]$ is a left adjoint functor of q^*. Therefore, the above group is naturally isomorphic to:

$$\mathrm{Hom}_{\mathbf{D}^b(\mathbb{P}_n)}(\mathbf{R}q_*(p^*(\mathscr{O}_{H_y}(-n))), \mathcal{I}_Z[2-n])^*. \tag{2.2}$$

We can compute this as:

$$\mathrm{H}^0(\mathbb{P}_n, \mathbf{R}\mathcal{H}om_{\mathbb{P}_n}\left(\mathbf{R}q_*(p^*(\mathscr{O}_{H_y}(-n))[2-n]), \mathscr{O}_{\mathbb{P}_n}\right) \overset{\mathbf{L}}{\otimes} \mathcal{I}_Z)^*. \tag{2.3}$$

where the $\mathbf{L}$ above $\otimes$ here stands for left-derived tensor product.

To compute this, we first look at $\mathbf{R}q_*(p^*(\mathscr{O}_{H_y}(-n)))$. Making use of (1.1), we get a distinguished triangle:

$$\mathbf{R}q_*(p^*(\mathscr{O}_{H_y}(-n))) \to \mathscr{O}_{\mathbb{P}_n}(-1)^n[-n+2] \xrightarrow{P_y} \mathscr{O}_{\mathbb{P}_n}[-n+2] \xrightarrow{[1]}$$

Here, it is easy to see that P_y is a matrix of linear forms defining y in $\mathbb{P}_n$. Dualizing the above diagram, we get an exact sequence (of sheaves):

$$0 \to \mathscr{O}_{\mathbb{P}_n} \xrightarrow{P_y^t} \mathscr{O}_{\mathbb{P}_n}(1)^n \to \mathbf{R}\mathcal{H}om_{\mathbb{P}_n}\left(\mathbf{R}q_*(p^*(\mathscr{O}_{H_y}(-n))), \mathscr{O}_{\mathbb{P}_n}\right)[-n+2] \to 0.$$

By the definition of the sheaf $\mathcal{S}_y$, we have thus an isomorphism:

$$\mathcal{S}_y \cong \mathbf{R}\mathcal{H}om_{\mathbb{P}_n}\left(\mathbf{R}q_*(p^*(\mathscr{O}_{H_y}(-n))), \mathscr{O}_{\mathbb{P}_n}\right)[-n+2].$$

Then the space appearing in (2.3) is non-zero if and only if

$$\mathrm{H}^0(\mathbb{P}_n, \mathcal{S}_y \overset{\mathbf{L}}{\otimes} \mathcal{I}_Z) \neq 0.$$

But one easily proves that $\mathcal{T}or_j(\mathcal{S}_y, \mathcal{I}_Z) = 0$ for $j > 0$, so (2.3) is non-zero if and only if:

$$\mathrm{H}^0(\mathbb{P}_n, \mathcal{S}_y \otimes \mathcal{I}_Z) \neq 0.$$

So y is unstable if and only if the above vector space is not zero, and the lemma is proved. □

Lemma 2.9. *Let y be a point and Z be a finite-length, non-degenerate subscheme of $\mathbb{P}_n$, not containing y. Then $\mathrm{H}^0(\mathbb{P}_n, \mathcal{S}_y \otimes \mathcal{I}_Z) \neq 0$ if and only if Z is contained in the rank-1 locus of a $2 \times n$ matrix M of linear forms having non-proportional rows, with one row defining y.*

Proof. Recalling the exact sequence (2.1) defining $\mathcal{S}_y$, we let $h_1, \ldots, h_n$ be a regular sequence defining $y \in \mathbb{P}_n$, and we note that a section in $\mathrm{H}^0(\mathbb{P}_n, \mathcal{S}_y \otimes \mathcal{I}_Z)$ is given by a global section s of $\mathcal{S}_y$ such that s vanishes along Z. In turn, s lifts to $\tilde{s}$ as in the diagram:

$$\begin{array}{ccccccccc} & & & & & & \mathscr{O}_{\mathbb{P}_n} & & \\ & & & & & \overset{\tilde{s}}{\swarrow} & \downarrow s & & \\ 0 & \longrightarrow & \mathscr{O}_{\mathbb{P}_n} & \xrightarrow{(h_1,\ldots,h_n)} & \mathscr{O}_{\mathbb{P}_n}^n(1) & \longrightarrow & \mathcal{S}_y & \longrightarrow & 0. \end{array}$$

Now $\tilde{s}$ is given by $(g_1, \ldots, g_n)$, where the g_i's are linear forms and the row $(g_1, \ldots, g_n)$ is not proportional to $(h_1, \ldots, h_n)$. Then in order for s to vanish on Z, we must have that Z is contained in the locus Y cut by the 2×2 minors of the matrix:

$$M = \begin{pmatrix} h_1 & \cdots & h_n \\ g_1 & \cdots & g_n \end{pmatrix},$$

Note that Y is not all of $\mathbb{P}_n$, because the two rows of M are not proportional. Since all the construction is reversible, the lemma is proved. □

Lemma 2.10. *Let Z be a finite-length, set-theoretically non-degenerate subscheme of $\mathbb{P}^n$ and $y \in \mathbb{P}_n$. Then the equivalent conditions of the previous lemma are satisfied if and only Z is contained in a KW variety Y of type $(d; s)$ with either $d > 0$ and y is in the curve part of Y, or $d = 0$, and y is the distinguished point of Y.*

Proof. Let us assume that the conditions of the previous lemma are satisfied, and look for the KW variety Y. So let us consider the matrix M given by the above lemma as a morphism of sheaves:

$$\mathscr{O}_{\mathbb{P}_n}(-1)^n \to \mathscr{O}_{\mathbb{P}_n}^2.$$

We have that Z is contained in the rank-1 locus of M, hence in the support of the cokernel sheaf $\mathscr{T}$ of the above map, hence in the image in $\mathbb{P}_n$ of the natural map $\mathbb{P}(\mathscr{T}) \to \mathbb{P}_n$.

The matrix M can be written in coordinates as $M_{i,j} = \sum_{k=0}^n a_{i,j,k} z_k$ for some scalars $a_{i,j,k}$, with $i = 0, 1$ and $j = 0, \ldots, n-1$. This gives a matrix N of size $n \times (n+1)$, this time over $\mathbf{k}[\xi_0, \xi_1]$, by:

$$N_{j,k} = \sum_{i=0,1} a_{i,j,k} \xi_i. \tag{2.4}$$

Therefore, we think of the above matrix N as a map:

$$N : \mathscr{O}_{\mathbb{P}^1}(-1)^n \to \mathscr{O}_{\mathbb{P}^1}^{n+1},$$

where the target space is identified with $V \otimes \mathscr{O}_{\mathbb{P}^1}$, with $V = \mathrm{H}^0(\mathbb{P}_n, \mathscr{O}_{\mathbb{P}_n}(1))$.

Note that this map is injective. Indeed, if y is defined by the forms $h_1, \ldots, h_n$, up to a change of basis we may assume $h_i = z_i$, so that the identity matrix of size n is a submatrix of N evaluated at $(1 : 0)$. The sheaf $\mathscr{L} = \mathrm{Cok}(N)$ decomposes as:

$$\mathscr{L} \cong \mathscr{O}_{\mathbb{P}^1}(d) \oplus \mathscr{O}_{\mathbb{P}^1, p_1}^{n_1} \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1, p_s}^{n_s},$$

for some distinct points $p_i \in \mathbb{P}^1$, and some integers $d, n_1, \dots, n_s \in [0, n]$. Since the sheaf $\mathscr{L}$ has degree n, we must have $d + n_1 + \cdots + n_s = n$.

The matrix N is classified by its standard Kronecker-Weierstrass (KW) form (hence the name of Y); we refer for this standard form for instance to [BCS97, Chapter 19]. This means that N can be written, in an appropriate basis, in block form like:

$$N = \left(\begin{array}{c|c|c|c} N_0 & 0 & \cdots & 0 \\ \hline 0 & N_1 & & 0 \\ \hline \vdots & & \ddots & \\ \hline 0 & 0 & & N_s \end{array}\right). \tag{2.5}$$

Here, N_0 is of size $d \times (d+1)$, with $\mathrm{Cok}(N_0) \cong \mathscr{O}_{\mathbb{P}^1}(d)$ and N_i is a square matrix of size n_i that degenerates on p_i only. For $i > 0$, each N_i can be further decomposed into its normal Jordan blocks, which are all of size one if and only if N_i is diagonal. Note also that N_0 can be written as:

$$N_0 = \begin{pmatrix} \xi_0 & 0 & \cdots & 0 \\ \xi_1 & \xi_0 & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \xi_1 & \xi_0 \\ 0 & \cdots & 0 & \xi_1 \end{pmatrix}. \tag{2.6}$$

Let us show that, with these elements, one can define Y.

Case $d > 0$**:** In this case, since $d + n_1 + \cdots + n_s = n$, we have $1 \leq n_j \leq n - 1$ for all j. We define then the curve C as the image of $\mathbb{P}(\mathscr{L})$ in $\mathbb{P}_n$ obtained by taking global sections of the quotient $\mathscr{O}_{\mathbb{P}^1}(d)$ of $\mathscr{L}$. Namely, C is just $\mathbb{P}^1$ mapped to $\mathbb{P}_n$ by $\mathscr{O}_{\mathbb{P}^1}(d)$, and spans the d-dimensional linear subspace $L = \mathbb{P}(\mathrm{H}^0(\mathbb{P}^1, \mathscr{L}))$ corresponding to the projection $\mathrm{H}^0(\mathbb{P}^1, \mathscr{L}) \to \mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d))$. In an appropriate basis, the curve C is cut in the space $L = \{z_{d+1} = \cdots = z_n = 0\}$ as the rank-1 locus of:

$$\begin{pmatrix} z_1 & \cdots & z_d \\ z_0 & \cdots & z_{d-1} \end{pmatrix}.$$

We define then L_j as the cone over the image in $\mathbb{P}_n$ of p_j and the space given by the projection $\mathrm{H}^0(\mathbb{P}^1, \mathscr{L}) \to \mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1, p_j}^{n_j})$. Each L_j meets L only at p_j, and the p_j's are all distinct if $d > 0$. Since L_i meets L_j only along C, all linear spaces L_j's are mutually disjoint for $d > 0$. This defines the KW variety $Y = C \cup L_1 \cup \cdots \cup L_s$.

Note that y belongs to C. Indeed, in the basis under consideration, we have that $y = (1 : 0 : \ldots : 0)$, and C goes through this point. Note also that Y clearly contains the image of $\mathbb{P}(\mathscr{L}) \cong \mathbb{P}(\mathscr{T})$ in $\mathbb{P}_n$ under the natural map $\mathbb{P}(\mathscr{L}) \to \mathbb{P}(\mathrm{H}^0(\mathbb{P}^1, \mathscr{L}))$. But this image is the rank-1 locus of M, which contains Z. So Y contains Z.

Case $d = 0$**:** In this case, under the decomposition (2.5), we have $N_0 = 0$. The sheaf $\mathscr{L}$ defines a projection of $\mathbb{P}^1$ to a point of $\mathbb{P}_n$, which in the basis under consideration has coordinates $(1 : 0 : \ldots : 0)$, i. e. this point is y. In this case, each linear space L_j is a cone over y and $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1, p_j}^{n_j}))$, hence all the L_j's meet only at y. Once we prove that $1 \le n_j \le n-1$ for all j, we can define $Y = L_1 \cup \cdots \cup L_s$, and clearly Z is contained in Y.

So let us show $1 \le n_j \le n-1$ for all j, in other words let us prove $s \ge 2$. Assume thus $s = 1$, and note that $\mathscr{L} \cong \mathscr{O}^n_{\mathbb{P}^1,p}$, with $p_1 = p = (a:b)$, so that N_1 degenerates on $(a:b)$ only. Note that the standard KW form of N_1 cannot be a multiple of the $n \times n$ identity matrix, times $b\xi_0 - a\xi_1$, for the two rows of the corresponding matrix M would be proportional. Hence the KW form of N_1 has at least one non-trivial Jordan block (i. e. of size at least 2). Then, the corresponding rank-1 locus of M is a multiple structure over a linear space of dimension at most $n-1$. But then Z is contained in a multiple structure over a hyperplane, a contradiction, since Z is set-theoretically non-degenerate.

To prove the converse implication, let us be given a KW variety Y of type $(d;s)$ containing Z, with $d > 0$, let L_0 be the span of the curve part C of Y and let $L_1, \dots, L_s$ be the linear spaces of Y. For each L_i, we choose a basis of an $(n_i - 1)$-dimensional linear subspace disjoint from L_0, and we complete this to a basis of V by stacking a basis of L_0. We take N_0 as in (2.6), and, for $i = 1, \dots, s$, we let (a_i, b_i) be the points on $\mathbb{P}^1$ corresponding to the intersection $C \cap L_i$ under the parametrization $\mathbb{P}^1 \to C$. We define N_i as a square matrix of size n_i having $b_i\xi_0 - a_i\xi_1$ on the diagonal and zero anywhere else. We have thus presented the matrix as is (2.4), hence we have a $2 \times n$ of the form $M_{i,j} = \sum_{k=0}^n a_{i,j,k} z_k$ in the coordinates given by the chosen basis. The first row of M thus defines y, and the rank-1 locus of M is Y.

If $d = 0$ we choose a projection $\mathbb{P}^1 \to \{y\}$, and we choose s distinct points $(a_i : b_i)$ in $\mathbb{P}^1$. We still have the matrices N_i, and the matrix N_0 is the zero matrix with one row. Constructing N as in (2.5), the same choice of basis for V allows to write the matrix M, whose first row defines y and whose rank-1 locus is Y. □

We can now prove our main results, Theorem 1 and 2. Clearly the first one follows from the second. To prove Theorem 2, let $Z \subset \mathbb{P}_n$ be a finite-length, set-theoretically non-degenerate subscheme. Then we have to show that the set of unstable hyperplanes $\mathrm{W}(\mathscr{F}_Z)$ contains at least another point $y \notin Z$ if and only if Z is contained in a KW variety Y of type $(d;s)$ whose distinguished point (if $d = 0$) does not lie in Z.

Proof of Theorem 2. Let us assume that Z is not Torelli, and prove that Z is contained in a KW variety. Since Z is not Torelli, there is a point $y \in \mathbb{P}_n$, not belonging to Z, unstable for $\mathscr{F}_Z$. We can apply Lemmas 2.8, 2.9, 2.10 since Z is set-theoretically non-degenerate. Then, there is a KW variety Y containing Z, and we are done.

Conversely given a KW variety Y of type $(d;s)$ containing Z, we look at two cases. If $d = 0$, then by assumption Z does not contain the distinguished point y of Y. But by Lemmas 2.8, 2.9, 2.10, the point y is unstable for $\mathscr{F}_Z$, so Z is not Torelli. If $d > 0$, we let y be any point of the curve part C of Y. By Lemmas 2.8, 2.9, 2.10, y is unstable for Z. But Z is of finite length, so there is $y \in C \setminus Z$ and Z is not Torelli. □

Recall Dolgachev's conjecture from the introduction (see [Dol07, Conjecture 5.8]). It states that a semi-stable arrangement of hyperplanes Z (i. e. such that

$\mathscr{F}_Z$ is a semi-stable sheaf) is Torelli if and only if Z belongs to no stable rational curve of degree n.

Corollary 2.11. *The "only if" implication of Dolgachev's conjecture is true. Namely, a reduced finite set $Z \subset \mathbb{P}_n$ which is contained is a stable rational curve gives a non-Torelli arrangement.*

Proof. If Z belongs to a curve $C = C_0 \cup \cdots \cup C_s$ as above, then we fix one component $C = C_0$ and we define L_i as the span of C_i, for $i > 0$. The variety $Y = C \cup L_1 \cup \cdots \cup L_s$ is a KW variety containing Z, so Z is not Torelli. □

Corollary 2.12. *A finite length subscheme Z of $\mathbb{P}^2$, whose set-theoretic support is not contained in a line, is Torelli if and only if it is not contained in a conic.*

Hence Dolgachev's conjecture holds on $\mathbb{P}^2$. In fact something stronger is true, for no stability condition is required in our result. Moreover, in this statement we allow Z to be non-reduced, and $\mathscr{F}_Z$ needs not even be torsion-free, however Torelli property still can hold.

We note in the next corollary that, for generic arrangements, our approach allows to recover some of the main results of [DK93] and [Val00]. Also, we note some simple examples of non-generic Torelli arrangements.

Corollary 2.13. *Let Z be a subscheme of length $\ell < \infty$ of $\mathbb{P}_n$.*

i) If the subscheme Z is contained in no quadric, then Z is Torelli;
ii) assume that Z is in general linear position and $\ell \geq n+3$. Then Z is contained in a smooth rational normal curve of degree n if and only if Z is not Torelli.

Proof. The statement (i) is clear, since all 2×2 minors of the matrix M of the previous lemma are quadrics.

For (ii), we want to show that, if $\ell \geq n+3$ and Z is in general linear position, then Z is contained in a KW variety Y if and only if it is contained in a rational normal curve of degree n. One direction is clear, so we assume that there are C, $L_1, \ldots, L_s$ as in Theorem 2, such that $Y = C \cup L_1 \cup \cdots \cup L_s$ contains Z, with $s \geq 1$. Note that the span L' of $C \cup L_1 \cup \cdots \cup L_{s-1}$ has dimension $d + a_1 + \cdots + a_{s-1}$, hence there are at most $d + a_1 + \cdots + a_{s-1} + 1$ points of Z in L'. Also, L_s contains at most $a_s + 1$ points of Z. Hence Y contains at most $d + a_1 + \cdots + a_s + 2 = n+2$ points of Z, which contradicts the fact that Z is contained in Y, as $\ell \geq n+3$. □

2.2. **Maximal number of unstable hyperplanes.** One can ask, given a Steiner sheaf $\mathscr{E}$, how to recognize if $\mathscr{E}$ is isomorphic to $\mathscr{F}_Z$, for some Z in $\mathbb{P}_n$. The next theorem gives an answer to this question. We note that the same answer was given in the case of arrangements with normal crossing in [AO01, Corollary 5.11].

Theorem 3. *Let $\mathscr{E}$ be a Steiner sheaf having resolution:*

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-n-1} \to \mathscr{O}_{\mathbb{P}^n}^{\ell-1} \to \mathscr{E} \to 0.$$

Assume that $\mathrm{W}(\mathscr{E})$ contains ℓ distinct points $\{y_1, \ldots, y_\ell\} = Z$, and that $\mathscr{O}_{H_{y_i}}$ is not a direct summand of $\mathscr{E}$, for any j. Then $\mathscr{E}$ is isomorphic to $\mathscr{F}_Z$.

Proof. Let H be an unstable hyperplane of $\mathscr{E}$, hence assume $\mathrm{H}^{n-1}(H, \mathscr{E}_{|H}(-n)) \neq 0$, i. e. $\mathrm{H}^{n-1}(\mathbb{P}^n, \mathscr{E} \otimes \mathscr{O}_H(-n)) \neq 0$. We have:

$$\begin{aligned}\mathrm{H}^{n-1}(\mathbb{P}^n, \mathscr{E} \otimes \mathscr{O}_H(-n)) &\cong \mathrm{Hom}_{\mathbf{D}^b(\mathbb{P}^n)}(\mathscr{O}_{\mathbb{P}^n}, \mathscr{E} \overset{\mathbf{L}}{\otimes} \mathscr{O}_H(-n)[n-1]) \cong \\ &\cong \mathrm{Hom}_{\mathbf{D}^b(\mathbb{P}^n)}(\mathscr{O}_{\mathbb{P}^n}, \mathscr{E} \overset{\mathbf{L}}{\otimes} \mathbf{R}\mathcal{H}om(\mathscr{O}_H, \mathscr{O}_{\mathbb{P}^n}(-1-n))[n]) \cong \\ &\cong \mathrm{Hom}_{\mathbf{D}^b(\mathbb{P}^n)}(\mathscr{O}_H, \mathscr{E}(-1-n)[n]) \cong \\ &\cong \mathrm{Hom}_{\mathbb{P}^n}(\mathscr{E}, \mathscr{O}_H)^*,\end{aligned}$$

where we used $\mathbf{R}\mathrm{Hom}_{\mathbb{P}^n}(\mathscr{O}_H, \mathscr{O}_{\mathbb{P}^n}(-1-n))[1] \cong \mathscr{O}_H(-n)$.

Looking at the resolutions of $\mathscr{E}$ and $\mathscr{O}_H$, one sees that any non-zero map $\mathscr{E} \to \mathscr{O}_H$ is surjective, and that the kernel $\mathscr{E}'$ of such a map is again a Steiner sheaf.

Let now $H' \neq H$ be another unstable hyperplane of $\mathscr{E}$. By the induced map $\mathrm{H}^{n-1}(H', \mathscr{E}'_{|H'}(-n)) \twoheadrightarrow \mathrm{H}^{n-1}(H', \mathscr{E}_{|H'}(-n))$ we see that H' is unstable for $\mathscr{E}'$ as well. Let $\mathscr{K}$ by the kernel of the (surjective) map $\mathscr{E}' \to \mathscr{O}_{H'}$. Then $\mathscr{K}$ injects in $\mathscr{E}$, and we let $\mathscr{C}$ be $\mathscr{E}/\mathscr{K}$. We claim that $\mathscr{C}$ is isomorphic to $\mathscr{O}_H \oplus \mathscr{O}_{H'}$. Indeed, we have $\mathscr{E}'/\mathscr{K} \cong \mathscr{O}_{H'}$, hence we get an exact sequence:

$$0 \to \mathscr{O}_{H'} \to \mathscr{C} \to \mathscr{O}_H \to 0.$$

Switching the roles of H and H' provides a splitting of the above sequence, so that $\mathscr{C} \cong \mathscr{O}_H \oplus \mathscr{O}_{H'}$.

Iterating this procedure, we find a surjective map:

$$\mathscr{E} \twoheadrightarrow \bigoplus_{i=1,\ldots,\ell} \mathscr{O}_{H_{y_i}}.$$

Note that the kernel of this map is $\Omega_{\mathbb{P}^n}$. Indeed, by diagram chasing, it is the kernel of a surjective map $\mathscr{O}_{\mathbb{P}^n}(-1)^{n+1} \twoheadrightarrow \mathscr{O}_{\mathbb{P}^n}$. Therefore we have an exact sequence:

$$0 \to \Omega_{\mathbb{P}^n} \to \mathscr{E} \to \bigoplus_{i=1,\ldots,\ell} \mathscr{O}_{H_{y_i}} \to 0.$$

Now, we note that $\bigoplus_{i=1,\ldots,\ell} \mathscr{O}_{H_{y_i}}$ is naturally isomorphic to $p_*(q^*(\mathscr{O}_Z))$. Having this set up, to conclude we can use Claim 1.4, by the same argument used in Proposition 1.3. Namely, $\mathscr{F}_Z$ is given, up to isomorphism, as the only extension of $\bigoplus_{i=1,\ldots,\ell} \mathscr{O}_{H_{y_i}}$ by $\Omega_{\mathbb{P}^n}$ associated by Claim 1.4 with the canonical surjection $\mathscr{O}_{\mathbb{P}_n} \to \mathscr{O}_Z$. An extension of $\bigoplus_{i=1,\ldots,\ell} \mathscr{O}_{H_{y_i}}$ by $\Omega_{\mathbb{P}^n}$ not isomorphic to $\mathscr{F}_Z$ corresponds then to a map $\mathscr{O}_{\mathbb{P}_n} \to \mathscr{O}_Z$ which is not surjective, say $\mathscr{O}_{y_j}$ is not in the image. Such extension contains $\mathscr{O}_{H_{y_j}}$ as a direct summand, which contradicts our hypothesis on $\mathscr{E}$. □

We get the following bound on the number of unstable hyperplanes of a Steiner sheaf.

Corollary 2.14. *Let $\mathscr{E}$ be a torsion-free Steiner sheaf with resolution:*

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-n-1} \to \mathscr{O}_{\mathbb{P}^n}^{\ell-1} \to \mathscr{E} \to 0.$$

Assume that $\mathrm{W}(\mathscr{E})$ contains ℓ distinct points $\{y_1, \ldots, y_\ell\} = Z$ not contained in a KW variety in $\mathbb{P}_n$. Then $\mathrm{W}(\mathscr{E}) = Z$.

The following proposition gives an elementary way to write down the matrix M_Z of (1.9). See also [AO01, Proposition 3.11] for the case of arrangements with normal crossings.

Proposition 2.15. *Let $Z = \{y_1, \dots, y_\ell\}$ be a non-degenerate Torelli arrangement, and consider the equations $f_1, \dots, f_\ell$ of the ℓ hyperplanes of $\mathbb{P}^n$. Then, up to permutation of $1, \dots, \ell$, there are constants $\alpha_{i,j}$ such that:*

$$f_\ell = \sum_{i=1,\dots,\ell-1} \alpha_{i,j} f_i, \tag{2.7}$$

for all $j = 1, \dots, \ell - n - 1$, and the matrix M_Z can be written as:

$$M = \begin{pmatrix} \alpha_{1,1} f_1 & \cdots & \alpha_{\ell,1} f_{\ell-1} \\ \vdots & & \vdots \\ \alpha_{1,\ell-n-1} f_1 & \cdots & \alpha_{\ell,\ell-n-1} f_{\ell-1} \end{pmatrix}.$$

Proof. The ℓ forms $f_1, \dots, f_\ell$ span the space V that has dimension $n+1$, hence up to reordering there are $\ell - n - 1$ linearly independent ways of writing f_ℓ as combination of $f_1, \dots, f_{\ell-1}$, and we have the constants $\alpha_{i,j}$.

Now, the i-th column of the matrix M above vanishes identically on the hyperplane H_i, which implies that H_i is unstable for the cokernel $\mathscr{E}$ of M^{t} for $i = 1, \dots, \ell - 1$. Further, in view of (2.7), we have that H_ℓ is also unstable for $\mathscr{E}$. Therefore, since Z is Torelli we conclude that $\mathrm{W}(\mathscr{E}) = Z$, hence, by the previous theorem, M_Z can be taken to be precisely M. □

3. Decomposition of logarithmic sheaves

Here we develop a tool for studying semistability of non-Torelli arrangements. This tool will take the form of a filtration associated with any non-Torelli arrangement. We will use this to provide some exceptions to Dolgachev's conjecture.

3.1. **Blowing up a linear subspace.** Let U be a $k+1$-dimensional subspace of V, with $1 \le k \le n-1$, and consider the subspace $\mathbb{P}_k = \mathbb{P}(U^*)$ of $\mathbb{P}_n = \mathbb{P}(V^*)$, embedded by $i : \mathbb{P}(U^*) \hookrightarrow \mathbb{P}_n$. Define $U^\perp$ as the kernel of the projection $V^* \to U^*$, and note that $U^\perp \cong (V/U)^*$. Denote by $\tilde{\mathbb{P}}^n_U$ the blowing up of $\mathbb{P}^n$ along $\mathbb{P}^{n-k-1} = \mathbb{P}(V/U) \subset \mathbb{P}^n$, and write $\pi_U : \tilde{\mathbb{P}}^n \to \mathbb{P}^k$ and $\sigma_U : \tilde{\mathbb{P}}^n \to \mathbb{P}^n$ for the two natural projections (we will drop this index U whenever possible). In our convention, points of $\mathbb{P}(V)$ and $\mathbb{P}(U)$ are quotients of V and U, so one can write:

$$\tilde{\mathbb{P}}^n = \{(x,u) \in \mathbb{P}^n \times \mathbb{P}^k \mid x_{|U} = u\}.$$

We consider $\mathbb{F}^k_k = \{(u,v) \in \mathbb{P}^k \times \mathbb{P}_k \mid u \in H_v\}$ and p_U and q_U are the natural projections to $\mathbb{P}^k$ and $\mathbb{P}_k$. In order to compare the incidence varieties $\mathbb{F}^n_n$ over $\mathbb{P}^n$ and $\mathbb{F}^k_k$ over $\mathbb{P}^k$, we consider the blown-up flag:

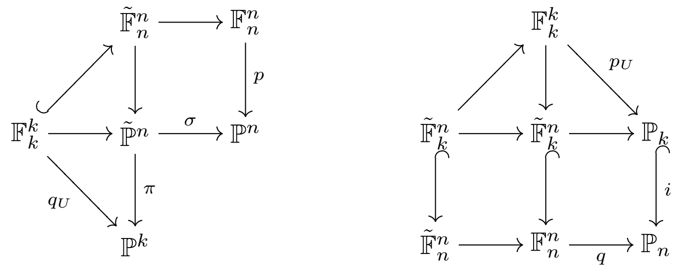
$$\tilde{\mathbb{F}}^n_n = \{(x,u,y) \in \mathbb{P}^n \times \mathbb{P}^k \times \mathbb{P}_n \mid x_{|U} = u, x \in H_y\}.$$

This blown-up flag contains the relative blown-up flag:

$$\tilde{\mathbb{F}}^n_k = \{(x,u,v) \in \mathbb{P}^n \times \mathbb{P}^k \times \mathbb{P}_k \mid x_{|U} = u, x \in H_v\}.$$

Projecting onto the different coordinates we get the commutative diagrams:

(3.1)



Let us analyze the sheaf $\mathscr{F}_Z$ when Z is degenerate, namely Z spans a proper subspace $\mathbb{P}(U^*) = \mathbb{P}_k \subset \mathbb{P}_n$. We may assume that the last $n-k$ coordinates in $\mathbb{P}_n$ vanish on $\mathbb{P}_k$. This amounts to ask that the equations of the hyperplanes of Z only depend on the variables $x_0, \ldots, x_k$. The same happens to the matrix M_Z, that now naturally defines the Steiner sheaf $\mathscr{F}^U_Z$ over $\mathbb{P}^k$ associated with $Z \subset \mathbb{P}_k$. Note that we have the rational map:

$$\rho : \mathbb{P}^n \dashrightarrow \mathbb{P}^k$$

It is tempting to look at $\rho^*(\mathscr{F}^U_Z)$ as a piece of $\mathscr{F}_Z$, defined by the same matrix M_Z, pulled back on $\mathbb{P}^n$ by ρ. The following lemma proves that this can be done (up to resolving the indeterminacy of ρ), and that the remaining piece consists of $(n-k)$ copies of $\mathscr{O}_{\mathbb{P}^n}(-1)$.

Lemma 3.1. *Let Z be a finite length subscheme of $\mathbb{P}_n$, assume that Z spans a $\mathbb{P}_k = \mathbb{P}(U^*)$ with $1 \le k \le n-1$, and let $\sigma = \sigma_U, \pi = \pi_U$. Then we have:*

$$\mathscr{F}_Z \cong V/U \otimes \mathscr{O}_{\mathbb{P}^n}(-1) \oplus \sigma_*(\pi^*(\mathscr{F}^U_Z)).$$

Proof. Assume that Z is contained in $\mathbb{P}_k = \mathbb{P}(U^*)$ and consider the exact sequence:

$$0 \to \mathcal{I}_{\mathbb{P}_k,\mathbb{P}_n}(1) \to \mathcal{I}_{Z,\mathbb{P}_n}(1) \to i_*(\mathcal{I}_{Z,\mathbb{P}_k}(1)) \to 0,$$

and the Koszul complex resolving $\mathcal{I}_{\mathbb{P}_k,\mathbb{P}_n}(1)$, namely:

$$0 \to \mathscr{O}_{\mathbb{P}_n}(k-n+1) \to \cdots \to \wedge^2 U^\perp \otimes \mathscr{O}_{\mathbb{P}_n}(-1) \to U^\perp \otimes \mathscr{O}_{\mathbb{P}_n} \to \mathcal{I}_{\mathbb{P}_k,\mathbb{P}_n}(1) \to 0.$$

Applying $\mathbf{R}p_*(q^*(-))$ to these exact sequences, in view of the vanishing $\mathbf{R}p_*(q^*(\mathscr{O}_{\mathbb{P}_n}(t)))$ for $2-n \le t \le -1$, we get a distinguished triangle:

$$U^\perp \otimes \mathscr{O}_{\mathbb{P}^n} \to \mathbf{R}p_*(q^*(\mathcal{I}_Z(1))) \to \mathbf{R}p_*(q^*(i_*(\mathcal{I}_{Z,\mathbb{P}_k}(1)))) \xrightarrow{[1]}$$

Taking $\mathbf{R}\mathcal{H}om_{\mathbb{P}_n}(-, \mathscr{O}_{\mathbb{P}_n}(-1))$, we obtain the distinguished triangle:

$$\mathbf{R}\mathcal{H}om_{\mathbb{P}_n}(\mathbf{R}p_*(q^*(i_*(\mathcal{I}_{Z,\mathbb{P}_k}(1)))), \mathscr{O}_{\mathbb{P}_n}(-1)) \to \mathscr{F}_Z \to V/U \otimes \mathscr{O}_{\mathbb{P}^n}(-1) \xrightarrow{[1]}$$

Our task is thus to prove that the leftmost complex in the triangle above is a sheaf isomorphic to $\sigma_*(\pi^*(\mathscr{F}^U_Z))$. Let $\mathscr{E}_Z$ be this complex, for the remaining part of the proof.

Using repeatedly commutativity of the diagrams (3.1) together with projection formula, it is easy to get a natural transformation:

$$\mathbf{R}\sigma_* \circ (\mathbf{R}\tilde{p}_U)_* \circ \alpha^* \circ q_U^* \cong \mathbf{R}p_* \circ q^* \circ i_*,$$

where α is the projection $\tilde{\mathbb{F}}^n_k \to \mathbb{F}^k_k$. By smooth base change, we also have:

$$(\mathbf{R}\tilde{p}_U)_* \circ \alpha^* \cong \pi^* \circ (\mathbf{R}p_U)_*,$$

where $\tilde{p}_U$ is the projection $\tilde{\mathbb{F}}^n_n \to \tilde{\mathbb{P}}^n$. This gives at once the natural isomorphism:

$$\mathbf{R}\sigma_*(\pi^*(\mathbf{R}p_U)_* q_U^*(\mathcal{I}_{Z,\mathbb{P}_k}(1))) \cong \mathbf{R}p_*(q^*(i_*(\mathcal{I}_{Z,\mathbb{P}_k}(1)))).$$

Therefore, in order to compute $\mathscr{E}_Z$, we have to apply $\mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(-, \mathscr{O}_{\mathbb{P}^n}(-1))$ to the left hand side. But we have seen that this simply amounts to transpose a matrix of linear forms of size $(\ell-1)\times(\ell-k-1)$, just as well as transposition is needed to define $\mathscr{F}_Z^U$ from $\mathbf{R}(p_U)_*(q_U^*(\mathcal{I}_{Z,\mathbb{P}_k}(1)))$ on $\mathbb{P}^k$, so that dualization of these complexes commutes with taking $\mathbf{R}\sigma_*(\pi^*(-))$. Hence we have shown that $\mathscr{E}_Z$ is isomorphic to $\mathbf{R}\sigma_*(\pi^*(\mathscr{F}_Z^U))$, and therefore to $\sigma_*(\pi^*(\mathscr{F}_Z^U))$.

This provides a short exact sequence:

$$0 \to \sigma_*(\pi^*(\mathscr{F}_Z^U)) \to \mathscr{F}_Z \to V/U \otimes \mathscr{O}_{\mathbb{P}^n}(-1) \to 0.$$

We will be done once this sequence splits, which in turn would be ensured by the vanishing:

$$\operatorname{Ext}^1_{\mathbb{P}^n}(\mathscr{O}_{\mathbb{P}^n}(-1), \sigma_*(\pi^*(\mathscr{F}_Z^U))) = 0.$$

But this vanishing is clear since $\sigma_*(\pi^*(\mathscr{F}_Z^U))$ is a Steiner sheaf. □

In the above situation, we set:

$$\mathscr{E}_Z^U = \sigma_*(\pi^*(\mathscr{F}_Z^U)).$$

3.2. **Decomposing non-Torelli arrangements.** Let us borrow the notations from the previous paragraph. In particular, recall that, given a $(k+1)$-dimensional subspace U of V, and Z in $\mathbb{P}(U^*)$, we have a sheaf $\mathscr{F}_Z^U$ over $\mathbb{P}(U)$, and hence a sheaf $\sigma_*(\pi^*(\mathscr{F}_Z^U))$ over $\mathbb{P}^n = \mathbb{P}(V)$, where $\sigma = \sigma^U$ and $\pi = \pi_U$ are the natural projections to $\mathbb{P}^n$ and $\mathbb{P}(U)$ from the blow-up $\tilde{\mathbb{P}}^n$ of $\mathbb{P}^n$ along $\mathbb{P}(V/U)$.

Lemma 3.2. *Assume that Z is contained in a rational normal curve C of degree k spanning $\mathbb{P}(U^*) \subset \mathbb{P}_n$. Then $\mathscr{F}_Z^U$ is isomorphic to $\mathscr{F}_{Z'}^U$, for any other subscheme Z' contained in C having the same length as Z.*

Proof. Let ℓ be the length of Z. We consider the exact sequence:

$$0 \to \mathcal{I}_{C,\mathbb{P}(U^*)}(1) \to \mathcal{I}_{Z,\mathbb{P}(U^*)}(1) \to \mathscr{O}_C((d-\ell)p) \to 0,$$

where, given an integer a, we write $\mathscr{O}_C(ap)$ for a divisor of degree a in C, namely a times a point $p \in C \cong \mathbb{P}^1$. The the sheafified minimal graded free resolution of $\mathcal{I}_{C,\mathbb{P}(U^*)}(1)$ over $\mathbb{P}(U^*)$ is the Eagon-Northcott complex (see for instance [Eis95]):

$$0 \to \mathscr{O}_{\mathbb{P}(U^*)}(1-k)^{k-1} \to \cdots \to \mathscr{O}_{\mathbb{P}(U^*)}(-j)^{j\binom{k}{j+1}} \to \cdots \to \mathscr{O}_{\mathbb{P}(U^*)}(-1)^{\binom{k}{2}} \to \mathcal{I}_{C,\mathbb{P}(U^*)}(1) \to 0.$$

So, we easily get:

$$\mathbf{R}(p_U)_*(q_U^*(\mathcal{I}_{C,\mathbb{P}(U^*)}(1))) = 0.$$

Therefore the complex $\mathbf{R}(p_U)_*(q_U^*(\mathcal{I}_{Z,\mathbb{P}(U^*)}(1)))$ only depends on the value ℓ, hence so does $\mathscr{F}_Z^U$. □

By the previous lemma, if C_d is a rational normal curve of degree d spanning a subspace $\mathbb{P}_d = \mathbb{P}(U^*)$ in $\mathbb{P}_n$ we can set:

$$\mathscr{E}_\ell^{C_d} = \sigma_*(\pi^*(\mathscr{F}_Z^U)),$$

for any subscheme Z of length ℓ of C_d.

The next result gives a decomposition tool for an arrangement Z which is contained in a KW-variety Y. So, let $Y = C \cup L_1 \cup \cdots \cup L_s$, where $L_i = \mathbb{P}(U_i) = \mathbb{P}_{n_i}$

and C is a smooth rational curve of degree $d > 0$, and the conditions (i) and (ii) of the introduction are satisfied. Let $y_i = C \cap L_i$.

Theorem 4. *Let $Z = Z_0 \cup \cdots \cup Z_s \subset \mathbb{P}_n$ be a subscheme of length ℓ, smooth at y_i for all i. Assume that L_i is the span of Z_i, and that $Z_0 \subset C \setminus \{y_1, \ldots, y_s\}$. Set ℓ_i for the length of Z_i. Then:*

i) we have a natural exact sequence:

$$0 \to \bigoplus_{i=1,\ldots,s} \mathscr{E}_{Z_i}^{U_i} \to \mathscr{F}_Z \to \mathscr{E}_{\ell_0+s}^{C_d} \to 0; \tag{3.2}$$

ii) we have the resolutions:

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1)^{\ell_i-n_i-1} \to \mathscr{O}_{\mathbb{P}^n}^{\ell_i-1} \to \mathscr{E}_{Z_i}^{U_i} \to 0,$$
$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1)^{\ell_0+s-d-1} \to \mathscr{O}_{\mathbb{P}^n}^{\ell_0+s-1} \to \mathscr{E}_{\ell_0+s}^{C_d} \to 0.$$

Proof. Since Z lies in $Y = C \cup L_1 \cup \cdots \cup L_s$, we have the sequences:

$$0 \to \mathcal{I}_{Y,\mathbb{P}_n}(1) \to \mathcal{I}_{Z,\mathbb{P}_n}(1) \to \mathcal{I}_{Z,Y}(1) \to 0. \tag{3.3}$$

The following claim ensures that $\mathcal{I}_{Y,\mathbb{P}_n}(1)$ does not contribute to $\mathscr{F}_Z$.

Claim 3.3. *Given Y as above, we have $\mathbf{R}p_*(q^*(\mathcal{I}_{Y,\mathbb{P}_n}(1))) = 0$.*

Let us postpone the proof of the claim above, and assume it holds for the time being. Set $\mathbb{L} = L_1 \cup \cdots \cup L_s$, $Z' = Z_1 \cup \cdots \cup Z_s$ and $Z_0' = Z_0 \cup y_1 \cup \cdots \cup y_s$.

By the definition of Y and the hypothesis on Z we deduce the following commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{Z_0',C}(1) & \longrightarrow & \mathscr{O}_C((d-s)p) & \longrightarrow & \mathscr{O}_{Z_0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{Z,Y}(1) & \longrightarrow & \mathscr{O}_Y(1) & \longrightarrow & \mathscr{O}_Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{Z',\mathbb{L}}(1) & \longrightarrow & \mathscr{O}_{\mathbb{L}}(1) & \longrightarrow & \mathscr{O}_{Z'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.4}$$

Here, p is a point in $C \cong \mathbb{P}^1$. Moreover, clearly we have:

$$\mathcal{I}_{Z',\mathbb{L}}(1) \cong \bigoplus_{i=1,\ldots,s} \mathcal{I}_{Z_i,L_i}(1). \tag{3.5}$$

Hence, we may rewrite the leftmost column of the above diagram as:

$$0 \to \mathscr{O}_C((-s-\ell_0+d)p) \to \mathcal{I}_{Z,Y}(1) \to \bigoplus_{i=1,\ldots,s} \mathcal{I}_{Z_i,L_i}(1) \to 0. \tag{3.6}$$

Notice also that we can switch the roles of C and $\mathbb{L}$, to obtain:

$$0 \to \bigoplus_{i=1,\ldots,s} \mathcal{I}_{y_i,L_i}(1) \to \mathscr{O}_Y(1) \to \mathscr{O}_C(1) \to 0. \tag{3.7}$$

Applying the functor $\mathbf{R}p_*(q^*(-))$ to (3.3) and dualizing, we have, in view of Claim 3.3:

$$\mathscr{F}_Z \cong \mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(\mathbf{R}p_*(q^*(\mathcal{I}_{Z,Y}(1))), \mathscr{O}_{\mathbb{P}^n}(-1)).$$

Applying $\mathbf{R}p_*(q^*(-))$ and $\mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(-, \mathscr{O}_{\mathbb{P}^n}(-1))$ to (3.6) gives the desired exact sequence (3.2). Indeed, for each of the terms $\mathcal{I}_{y_i,L_i}(1)$ appearing in the isomorphisms (3.5), we can use the argument used in Lemma 3.1, that gives:

$$\mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(\mathbf{R}p_*(q^*(\mathcal{I}_{y_i,L_i}(1))), \mathscr{O}_{\mathbb{P}^n}(-1)) \cong \sigma_*^{U_i}(\pi_{U_i}^*(\mathscr{F}_{Z_i}^{U_i})) = \mathscr{E}_{Z_i}^{U_i}.$$

For $\mathscr{O}_C(d-\ell_0-s)$ we use the same argument and Lemma 3.2 to obtain:

$$\mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(\mathbf{R}p_*(q^*(\mathscr{O}_C(d-\ell_0-s))), \mathscr{O}_{\mathbb{P}^n}(-1)) \cong \sigma_*^{U_0}(\pi_{U_0}^*(\mathscr{F}_{Z_0'}^{U_0})) = \mathscr{E}_{\ell_0+s}^{C_d}.$$

Summing up, (i) is now proved. The resolutions required for (ii) are provided by Lemma 3.1. It remains to prove Claim 3.3. □

Proof of Claim 3.3. Using the description of the incidence variety given by (1.1), we see that the claim follows if we prove that $\mathcal{I}_Y(1)$ is the cohomology of a complex where only the sheaves $\mathscr{O}_{\mathbb{P}_n}(1-n), \dots, \mathscr{O}_{\mathbb{P}_n}(-1)$ appear.

We can use Beilinson's theorem to prove that this is the case (we refer for instance [OSS80, Theorem 3.1.4]). Indeed, by Beilinson's theorem the sheaf $\mathcal{I}_Y(1)$ is the cohomology of a complex whose terms are of the form $\mathscr{O}_{\mathbb{P}_n}(-h) \otimes \mathrm{H}^k(\mathbb{P}_n, \mathcal{I}_Y(1) \otimes \Omega_{\mathbb{P}_n}^h(h))$, for $0 \le h \le n$. Therefore, in order to show that only the terms with $1 \le h \le n-1$ appear (so that we exclude $\mathscr{O}_{\mathbb{P}_n}$ and $\mathscr{O}_{\mathbb{P}_n}(-n)$), since $\Omega_{\mathbb{P}_n}^n(n) \cong \mathscr{O}_{\mathbb{P}_n}(-1)$, we merely have to prove the following vanishing results:

$$\mathrm{H}^k(\mathbb{P}_n, \mathcal{I}_Y(t)) = 0, \qquad \text{for all } k, \text{ and for } t = 0, 1. \tag{3.8}$$

To show this, we take cohomology of (3.7). Note that $\mathrm{H}^k(L_i, \mathcal{I}_{y_i,L_i}(1)) = 0$ for $i, k > 0$, and that $\mathrm{h}^0(L_i, \mathcal{I}_{y_i,L_i}(1)) = n_i$, $\mathrm{h}^0(C, \mathscr{O}_C(1)) = d$. Since $d + n_1 + \dots + n_s = n$, we get:

$$\mathrm{H}^k(\mathbb{P}_n, \mathscr{O}_Y(1)) = 0, \qquad \text{for all } k > 0, \qquad \mathrm{h}^0(\mathbb{P}_n, \mathscr{O}_Y(1)) = n+1.$$

Since Y is non-degenerate, we have $\mathrm{H}^0(\mathbb{P}_n, \mathcal{I}_Y(1)) = 0$ so from the previous display we get (3.8) for $t = 1$.

We take now cohomology of (3.7), twisted by $\mathscr{O}_{\mathbb{P}_n}(-1)$. Note that $\mathrm{H}^k(L_i, \mathcal{I}_{y_i,L_i}) = 0$ for all i, k and $\mathrm{h}^0(C, \mathscr{O}_C) = 1$. We easily deduce (3.8) for $t = 0$. This finishes the proof is of the claim. □

Corollary 3.4. *With the notations of the previous theorem, $\mathscr{E}_{Z_i}^{U_i}$ is a direct summand of $\mathscr{F}_Z$ if y_i belongs to Z.*

Proof. Order $1, \dots, s$ so that $y_1, \dots, y_r$ belong to Z and $y_{r+1}, \dots, y_s$ do not. Using (3.7) and a diagram similar to (3.4), we get an exact sequence:

$$0 \to \bigoplus_{i=1,\dots,r} \mathcal{I}_{Z_i,L_i}(1) \oplus \bigoplus_{i=r+1,\dots,s} \mathcal{I}_{Z_i \cup y_i, L_i}(1) \to \mathcal{I}_{Z,Y}(1) \to \mathscr{O}_C((d-r-\ell_0)p) \to 0.$$

Comparing with (3.6), we see that, for $i = 1, \dots, r$, $\mathcal{I}_{Z_i,L_i}(1)$ is a direct summand of $\mathcal{I}_{Z,Y}(1)$, so that $\mathscr{E}_{Z_i}^{U_i}$ is a direct summand of $\mathscr{F}_Z$. □

3.3. **Exceptions to Dolgachev's conjecture.** We conclude the paper with some examples of hyperplane arrangements having interesting unstable loci, giving some counterexamples to the "only if" implication of Dolgachev's conjecture. Namely, we describe finite sets Z in $\mathbb{P}_n$ such that $\mathrm{W}(\mathscr{F}_Z)$ is the union of Z and a line in $\mathbb{P}_3$, or Z and a plane in $\mathbb{P}_4$, or Z and a point in $\mathbb{P}_4$. The results of this section are used to prove semistability in some cases.

Example 3.5. We consider the union Z_1 of 5 points on a unique conic, spanning a plane L_1 in $\mathbb{P}_3$, and the union Z_0 of 2 more points on a line L_0. We assume that L_0 does not meet the conic $D \subset L_1$ passing through Z_1, and that $Z_0 \cap L_1 = \emptyset$. We let $Z = Z_0 \cup Z_1$.

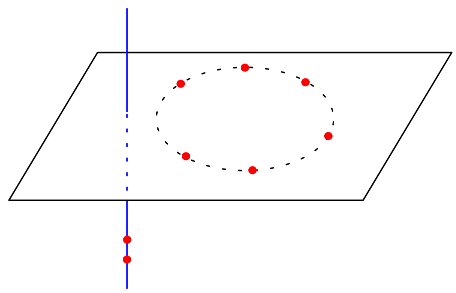


FIGURE 2. Seven points in $\mathbb{P}_3$ with an unstable line.

Consider a point y of L_0. Then there are a rational normal curve through y (take L_0) and a plane (take L_1) such that $L_0 \cup L_1$ contains Z, and satisfying (i) and (ii). Thus all points of L_0 are unstable, and Z is not Torelli.

On the other hand, if $y \notin Z$ does not lie in L_0, then y is not unstable for $\mathscr{F}_Z$. Indeed, any subvariety $Y \subset \mathbb{P}_n$ through y and Z as in Theorem 1 would have to contain Z_1 and L, hence be $L_0 \cup L_1$. So y has to lie in L_1. But even the points of $L_1 \setminus Z$ are not unstable, for we should have a conic in L_1 through y and Z_1 (hence the conic is D) and a line through Z_1 (hence the line is L_0) meeting at a single point; but D does not pass through $L_0 \cap L_1$.

Finally, note that $\mathscr{F}_Z$ is a semistable sheaf (not a stable one though), at least for most choices of the 5 points of Z_1. In fact, let us prove it under the assumption that $Z_1 = \{\zeta_1, \dots, \zeta_5\}$ is such that ζ_3 lies in intersection of the lines N_1 and N_2 through ζ_1, ζ_2 and ζ_4, ζ_5 (still $D = N_1 \cup N_2$ disjoint from L_0). In this case, Theorem 4 applies to give a short exact sequence:

$$0 \to \mathscr{F}_1 \to \mathscr{F}_Z \to \mathscr{F}_0 \to 0,$$

where $\mathscr{F}_1$ is $\mathscr{E}_{Z_1}^{U_1}$ (we set $L_i = \mathbb{P}(U_i)$) and $\mathscr{F}_0$ is $\mathscr{E}_{-3}^{L_0}$, which in this case is isomorphic to $\mathcal{I}_{M_0}(1)$, where M_0 is the line dual to L_0. Here $\mathscr{F}_1$ splits, in view of Corollary 3.4, as $\mathcal{I}_{M_1}(1) \oplus \mathcal{I}_{M_2}(1)$, where the M_i's are the lines dual to the N_i's. Then, it is straightforward to check that $\mathscr{F}_Z$ is strictly semistable, for the graded object associated with the above filtration of $\mathscr{F}_Z$ is $\mathcal{I}_{M_0}(1) \oplus \mathcal{I}_{M_1}(1) \oplus \mathcal{I}_{M_2}(1)$.

In coordinates, we could take L_0 as $\{z_2 = z_3 = 0\}$ and L_1 as $\{z_1 = 0\}$. Further, N_1 and N_2 can be taken as $\{z_0 - z_2 = z_1 = 0\}$ and $\{z_0 - z_3 = z_1 = 0\}$, so that

$\zeta_3 = (1:0:1:1)$. The matrix M_Z in this case is:

$$M_Z = \begin{pmatrix} x_0+x_1 & -x_1 & 0 & x_3 & 0 & x_2 \\ 0 & 0 & x_0+x_2 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0+x_3 & x_2 \end{pmatrix},$$

Example 3.6. With a little more work one can modify the above example so that $\mathscr{F}_Z$ is even stable. This can be achieved adding a point on L_0 and a further point on L_1, outside $N_1 \cup N_2$.

In coordinates, we can add $(1:2:0:0)$ and $(0:0:1:1)$. This gives rise (up to permutation) to the matrix M_Z:

$$\begin{pmatrix} x_0+x_1 & 0 & -x_1 & 0 & x_3 & 0 & x_2 & 0 \\ 0 & x_0+2x_1 & -2x_1 & 0 & x_3 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_0+x_2 & x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0+x_3 & x_2 & 0 \\ x_0+x_1 & 0 & -x_1 & 0 & 0 & 0 & 0 & x_2+x_3 \end{pmatrix}$$

Stability of $\mathscr{F}_Z$ can be deduced by the following resolutions:

$$0 \to \mathscr{O}_{\mathbb{P}^3}(-3) \oplus \mathscr{O}_{\mathbb{P}^3}(-2) \to \mathscr{O}_{\mathbb{P}^3}(-2) \oplus \mathscr{O}_{\mathbb{P}^3}(-1)^4 \to \mathscr{F}_Z^{**}(-2) \to 0,$$
$$0 \to \mathscr{O}_{\mathbb{P}^3}(-4) \to \mathscr{O}_{\mathbb{P}^3}(-3) \oplus \mathscr{O}_{\mathbb{P}^3}(-1)^3 \to \mathscr{F}_Z^{*}(1) \to 0.$$

Example 3.7. Let L_1 and L_2 be two planes in $\mathbb{P}_4$, meeting at a single point y. Then y is the distinguished point of the KW variety $L_1 \cup L_2$. Let $Z_1 \subset L_1$ and $Z_2 \subset L_2$ be subschemes of length $\ell_1, \ell_2 < \infty$, both disjoint from y. Then $Z = Z_1 \cup Z_2$ cannot be Torelli, for y is always an unstable hyperplane of $\mathscr{F}_Z$.

If there is no conic through Z_1 and y nor through Z_2 and y, then y is the *only point of* $\mathbb{P}_4$ *outside* Z *giving an unstable hyperplane for* $\mathscr{F}_Z$. If Z_1 consists of 3 points such that $Z_1 \cup y$ is in general linear position, then for a general point z of L_1, there is a conic C through $z \cup y \cup Z_1$, and Z is contained in the KW variety $C \cup L_2$. Hence any point of C is unstable. So *all the points of* L_1 *give unstable hyperplanes* in this case.

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