A SMOOTH SURFACE OF TAME REPRESENTATION TYPE

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Abstract. We show that the Segre product of a line and a smooth conic, naturally embedded in $\mathbb{P}^5$, is a smooth projective surface of tame representation type, namely all continuous families of indecomposable ACM bundles have dimension one. To our knowledge, this is the only example of smooth projective variety of this kind, besides the elliptic curve, which is of tame representation type according to Atiyah (1957).

1. Introduction

Let X be a smooth connected n-dimensional projective variety over an algebraically closed field k, with $n \geq 1$, embedded in $\mathbb{P}^N$ by a very ample divisor class, denoted by h. Let R_X be the homogeneous graded coordinate ring of X, and assume that R_X is a Cohen-Macaulay ring, so that X is an ACM variety. A rank-r vector bundle E on X is said to be *arithmetically Cohen-Macaulay (ACM)* if its module of global sections is a maximal Cohen-Macaulay module over R_X. This is equivalent to the condition:

$$H^i_*(X,E) := \bigoplus_{t \in \mathbb{Z}} H^i\big(X, E(th)\big) = 0, \qquad \text{for each } i = 1, \ldots, n-1.$$

The variety X is said to be of *finite representation type*, or *finite CM type* if it supports, up to twist by $\mathcal{O}_X(th)$ and isomorphism, only a finite number of indecomposable ACM bundles. One must be aware that varieties of finite CM type are classified, see [7]. Their list is: linear embeddings of projective spaces and of smooth quadrics, rational normal curves, a smooth cubic scroll in $\mathbb{P}^4$, and the Veronese surface in $\mathbb{P}^5$.

When X supports continuous families of indecomposable ACM bundles of a given rank r, all non-isomorphic to one another, we say that X is of *tame representation type* if, there is a finite number of such families, and each of them has dimension at most one (and this for all r). On the other hand, X is of *wild representation type* if there are ℓ-dimensional families of non-isomorphic indecomposable ACM sheaves, for arbitrarily large ℓ.

Here we announce that the surface $X = \mathbb{P}^1 \times \mathbb{P}^1$, embedded in $\mathbb{P}^5$ by the linear system $|\mathcal{O}_X(h)|$ of bidegree $(1,2)$ (the product of a line and a smooth conic) is of tame representation type.

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Theorem 1. *Up to a twist by $\mathcal{O}_X(th)$, any indecomposable non-zero ACM bundle on X is either a line bundle $\mathcal{O}_X$, $\mathcal{O}_X(-1,0)$ or $\mathcal{O}_X(-1,-1)$, or an extension of the form:*

$$0 \to \mathcal{O}_X(0,-1)^{\oplus a} \to E \to \mathcal{O}_X(-1,1)^{\oplus b} \to 0, \tag{1}$$

with either $|a-b| = 1$, in which case E is rigid, or $a = b \geq 1$, in which case the deformations of E are parametrized by a projective line.

To our knowledge, this is the first example of smooth variety of tame representation type besides the elliptic curve, which in turn is of this kind according to early work of Atiyah, [1], see also [6]. In [5] we prove that, besides this example and the well-known cases of finite type, all other embeddings of homogeneous spaces are of wild representation type. This generalizes [4], where the same result is proved for linear embeddings of Segre varieties. Also, it should be noted that homogeneous spaces include all varieties of finite representation type, with the only exception of the cubic scroll in $\mathbb{P}^4$. Anyway, all other rational normal scrolls (see [9]), as well as all other rational ACM surfaces in $\mathbb{P}^4$ (see [10]) are of wild representations type. All these examples lead to suspect that there are in fact no other smooth projective varieties of tame representation type, besides the example presented in this note and the elliptic curve.

This note is devoted to sketch the argument to prove our main result, full proofs will be given in [5]. Our method involves derived categories, more specifically monads and Beilinson-type theorems.

2. A family of ACM bundles parametrized by a projective line

We want to study ACM bundles on the variety X obtained as product of a projective line and a smooth projective conic. So $X = \mathbb{P}^1 \times \mathbb{P}^1$ is embedded in $\mathbb{P}^5$ by the linear system $\mathcal{O}_X(h)$ of bidegree $(1,2)$. In other words, we want to study vector bundles E on $X = \mathbb{P}^1 \times \mathbb{P}^1$ such that $H^1(E(t,2t)) = 0$, for any t.

As a preliminary step, let us list the ACM line bundles on X. Up to twist by $\mathcal{O}_X(t,2t)$, they are:

$$\mathcal{O}_X(0,-1), \mathcal{O}_X(-1,1), \mathcal{O}_X(-1,0), \mathcal{O}_X(-1,-1), \mathcal{O}_X(-1,-2).$$

The most important family of indecomposable ACM bundles on X is given by bundles fitting into (1), for integers $a, b \geq 0$ with $(a,b) \neq (0,0)$. These are classified by the following proposition, whose proof relies on Kronecker-Weierstrass' classification of matrix pencils.

Proposition 2. *Let E be a bundle fitting in* (1)*. Then E is a semistable Ulrich bundle and, if E is indecomposable, then $\mathrm{Ext}^2_X(E,E) = 0$ and $|a-b| \leq 1$. Further:*

(i) if $a = b \pm 1$ then E is indecomposable if and only if E is exceptional, and in this case E can be obtained by a general extension like (1)*;*

(ii) if $a = b$ and E is indecomposable then E varies a projective line. In fact, for all partitions $(\lambda_1, \dots, \lambda_r)$ of a, there are extensions of the above form such that E splits as direct sum of r bundles E_i, with $\operatorname{rank}(E_i) = 2\lambda_i$, and each E_i varying in a $\mathbb{P}^1$.

In the above proposition, by an Ulrich bundle we mean an ACM bundle E such that the module of global sections of E has the highest possible number of generators, namely $\operatorname{rank}(E)\deg(X)$.

3. Generators of the derived category adapted to ACM bundles

We consider the collection of line bundles over X: $E_3 = \mathcal{O}_X, E_2 = \mathcal{O}_X(0,-1), E_1 = \mathcal{O}_X(-1,-1), E_0 = \mathcal{O}_X(-1,-2)$. It is easy to see that these line bundles form a full strongly exceptional collection on X, so that the derived category $\mathcal{D}^b(X)$ of bounded complexes of coherent sheaves on X is given by:

$$\mathcal{D}^b(X) = \langle E_0, E_1, E_2, E_3 \rangle.$$

This choice of the generators of the derived category is adapted to the study of ACM bundles, as we shall see in a minute. We can compute the dual collection (F_0, F_1, F_2, F_3) of (E_0, E_1, E_2, E_3). We get:

$$F_3 = \mathcal{O}_X, \quad F_2 = \mathcal{O}_X(0,-1), \quad F_1 = \mathcal{O}_X(-1,1)[-1], \quad F_0 = \mathcal{O}_X(-1,0)[-1].$$

Given a vector bundle E over X, we construct a Beilinson complex quasi-isomorphic to E, by calculating $H^i(E \otimes E_j) \otimes F_j$, with $i, j \in \{0,1,2,3\}$. Now, if E is ACM, then we have $H^1(E \otimes E_0) = 0$ and $H^1(E \otimes E_3) = 0$. Set $a = h^1(E \otimes E_2)$ and $b = h^i(E \otimes E_1)$. We get the following table:

$*$	$*$	$*$	$*$
0	b	a	0
$*$	$*$	$*$	$*$
E_0	E_1	E_2	E_3

The central line of this table can be thought of as a distinguished triangle:

$$F_1^{\oplus b} \to F_2^{\oplus a} \to E' \to F_1^{\oplus b}[1],$$

or in other words as an exact sequence $0 \to \mathcal{O}_X(0,-1)^{\oplus a} \to E' \to \mathcal{O}_X(-1,1)^{\oplus b} \to 0$, which is the extension of the form (1). Looking more closely into the complex obtained by the above table, and using the fact that F_1 and F_3 are totally orthogonal to each other, as well as F_0 and F_2, one can show:

Lemma 3. *The bundle E' is a direct summand of E.*

By Proposition 2, E' moves in a family of dimension ≤ 1. By the lemma, we can split off the summand E' from E, to get another ACM bundle $E'' = E/E'$, with moreover $H^1(E''(0,-1)) = H^1(E''(-1,-1)) = 0$.

4. Splitting off line bundles from ACM bundles

In view of the above argument, to show that X is of tame representation type,it suffices to check that E'' splits as a direct sum of line bundles. To prove this fact we use the following notion, given in [3]:

Definition 4. *A coherent sheaf F on $Q_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ is (p,p')-Qregular if:*

$$H^1(F(p-1,p')) = H^1(F(p,p'-1)) = H^2(F(p-1,p'-1)) = 0.$$

We often say *regular* instead of $(0,0)$-regular, *p-regular* instead of (p,p)-regular, and *irregular* for not regular. This notion of regularity coincides with the definition of p-Qregularity on X if $p = p'$ (see [2]) and it coincides with the definition of (p,p')-regularity on $\mathbb{P}^1 \times \mathbb{P}^1$ by Hoffman and Wang (see [8]). Moreover if F is a regular coherent sheaf, then it is globally generated (gg for short) and $F(p,p')$ is regular for $p,p' \geq 0$ (see [3, Proposition 2.2 and Remark 2.3]). Any vector bundle F can be made into a regular one by twisting by $\mathcal{O}_X(t,2t)$. Summing up, our main result will be proved if we show the next lemma.

Lemma 5. *Let E be an irregular indecomposable ACM bundle on X, with $E(1,2)$ regular. If $H^1(E(-1,-1)) = H^1(E(0,-1)) = 0$ then E is isomorphic to $\mathcal{O}_X(-1,0)$, or $\mathcal{O}_X(-1,-1)$ or, $\mathcal{O}_X(-1,-2)$.*

To prove the lemma, we have to use the exact sequences:

$$0 \to E(s-1,t) \to E(s,t)^{\oplus 2} \to E(s+1,t) \to 0, \tag{2}$$

$$0 \to E(s,t-1) \to E(s,t)^{\oplus 2} \to E(s,t+1) \to 0. \tag{3}$$

Since E is irregular and $H^1(E(0,-1)) = 0$, we have $H^1(E(-1,0)) \neq 0$ or $H^2(E(-1,-1)) \neq 0$. Let us look at these two cases.

1. If $H^2(E(-1,-1)) \neq 0$, we study two sub-cases, according to whether $E(1,1)$ is regular or not.
   - 1.1 If $E(1,1)$ is regular, then it is gg. Since $H^2(E(-1,-1)) \cong H^0(E^\vee(-1,-1)) \neq 0$, a general global section of $E(1,1)$ thus splits off $\mathcal{O}_X$ from $E(1,1)$. But E is indecomposable so $E \cong \mathcal{O}_X(-1,-1)$.
   - 1.2 If the bundle $E(1,1)$ is irregular, we have $H^1(E(0,1)) \neq 0$, or $H^1(E(1,0)) \neq 0$ or $H^2(E) \neq 0$. The last condition cannot occur, since $H^2(E) \cong H^0(E(-2,-2)) \neq 0$, so since $E(2,2)$ is gg again $E \cong \mathcal{O}_X(-2,-2)$, which contradicts that $E(1,2)$ be regular. Moreover, E is ACM so $H^1(E) = 0$ and $E(1,2)$ is regular so $H^1(E(0,2)) = 0$, hence $H^1(E(0,1)) = 0$ from (3) with $s = 0$, $t = 1$.
     So we must have $H^1(E(1,0)) \neq 0$. Now, $E(1,2)$ is regular so $H^1(E(1,1)) = 0$ hence from (3) with $s = t = 1$ we get $H^0(E(1,2)) \neq 0$. On the other hand, $H^2(E) = H^1(E) = 0$ so from (2) with $s = t = 0$ we get $H^2(E(-1,0)) \neq 0$. Hence $H^0(E^\vee(-1,-2)) \neq 0$, and again $E \cong \mathcal{O}_X(-1,-2)$.

2. It remains to look at the case $H^1(E(-1,0)) \neq 0$, and we can now assume $H^2(E(-1,-1)) = 0$. Again from (2) with $s = t = 0$ we obtain $H^0(E(1,0)) \neq 0$. Also, using $H^1(E(-1,-1)) = H^2(E(-1,-1)) = 0$, from (3) with $s = t = -1$ we get $H^2(E(-1,-2)) \neq 0$. Then $H^0(E^\vee(-1,0)) \neq 0$, so $E \cong \mathcal{O}_X(-1,0)$.

Let us notice that all the indecomposable ACM bundles on X with rank ≥ 2 are Ulrich bundles.

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