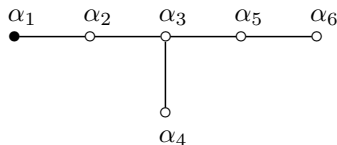# ON THE DERIVED CATEGORY OF THE CAYLEY PLANE II

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ABSTRACT. We find a full strongly exceptional collection for the Cayley plane $\mathbb{OP}^2$, the simplest rational homogeneous space of the exceptional group E_6. This collection, closely related to the one given by the second author in [Man11], consists of 27 vector bundles which are homogeneous for the group E_6, and is a minimal Lefschetz collection with respect to the minimal equivariant embedding of $\mathbb{OP}^2$.

INTRODUCTION

The Cayley plane. Let $X = \mathbb{OP}^2$ be the Cayley plane. This is the closed orbit in the projectivization of the minimal representation of the complex simply connected exceptional Lie group E_6. The Cayley plane can also be identified with the quotient E_6/P_1, where P_1 is the parabolic subgroup of E_6 corresponding to the root α_1 of E_6. This is sketched in the diagram below, where the simple roots $\alpha_1 \ldots, \alpha_6$ of E_6 are depicted. We will denote by $\omega_1, \ldots, \omega_6$ the corresponding fundamental weights.

The aim of this note is to provide a full strongly exceptional collection of the derived category of coherent sheaves on the Cayley plane. This completes [Man11], where a strongly exceptional collection closely related to ours was found. But it was not proved that this collection generates the whole derived category.

Our proof of this missing point has two important ingredients. On one hand, we use specific tensor relations between certain homogeneous bundles on the Cayley plane, to show that some of these bundles belong to the category generated by our strongly exceptional collection. This, after a brief account of the geometry of the Cayley plane (Section 1), will be the object of Section 2. This allows us, on the other hand, to use restriction to certain specific subvarieties of the Cayley plane, its maximal quadrics, for which we know a full exceptional collection of the derived category. We will do this in Section 3. Section 4 contains the proof of our result.

The main result. To state our main result, we need to define two vector bundles $\mathcal{S}$ and $\mathcal{S}_2$ on the Cayley plane. The first one is defined as $\mathcal{S} = \mathcal{N}(-1)$, where $\mathcal{N}$ is

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the normal bundle of the embedding of X in $\mathbb{P}(V_{\omega_1})$. The second one is defined by the direct sum decomposition (see [Man11]):

$$(1) \qquad\qquad S^2 \mathcal{S}^* = \mathcal{S}_2^* \oplus \mathcal{O}_X(-1).$$

Consider the collections:

$$\mathcal{A} = (\mathcal{S}_2^*, \mathcal{S}^*, \mathcal{O}_X),$$
$$\mathcal{C} = (\mathcal{S}^*, \mathcal{O}_X).$$

Theorem. *The Cayley plane admits the full strongly exceptional collection:*

$$\mathrm{D}^{\mathrm{b}}(\mathbb{OP}^2) = \langle \mathcal{A}, \mathcal{A}(1), \mathcal{A}(2), \mathcal{C}(3), \mathcal{C}(4), \dots, \mathcal{C}(11) \rangle.$$

This is a Lefschetz collection in the sense of [Kuz07], which is minimal in the sense of [Kuz08]. It was used in [IM11] to study the derived categories of seven-dimensional cubics.

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1. THE GEOMETRY OF THE CAYLEY PLANE

The Cayley plane $X = \mathbb{OP}^2$ is a smooth projective variety of dimension 16 and Picard number 1, homogeneous under the complex exceptional Lie group E_6. The ample generator $\mathcal{O}_X(1)$ of $\mathrm{Pic}(X) \simeq \mathbb{Z}$ embeds X equivariantly as the smallest orbit for E_6 in the 26-dimensional projective space, acted on linearly by the projectivized smallest non-trivial irreducible E_6-representation, and we have $\omega_X \simeq \mathcal{O}_X(-12)$. The Cayley plane is one of the two exceptional complex Hermitian symmetric varieties, and appears naturally as the "last Severi variety", classified by Zak, see [Zak93].

1.1. **Octonionic plane geometry.** The Cayley plane $X = \mathbb{OP}^2$ is the complexification of the projective plane over the Cayley octonions. In particular, it contains 8-dimensional quadrics called $\mathbb{O}$-lines, since they can be thought of as copies of the projective line $\mathbb{OP}^1$ over the octonions. These $\mathbb{O}$-lines define a (slightly degenerate) plane projective geometry over the octonions. Let us recall briefly the ingredients from this geometry that will be useful in this note, whereby explaining the terminology. For more details, see [IM11].

The Cayley plane $X = \mathbb{OP}^2 \subseteq \mathbb{P}(V_{\omega_1})$ has to be considered together with its twin $\mathbb{O\check{P}}^2 \subseteq \mathbb{P}(V_{\omega_6}) = \mathbb{P}(V_{\omega_1}^*)$. Both varieties are isomorphic as embedded projective varieties, but not equivariantly. If x is a point of X, the orthogonal to its tangent space in $\mathbb{P}(V_{\omega_1}^*)$ cuts the dual Cayley plane $\mathbb{O\check{P}}^2$ along a smooth 8-dimensional quadric $\check{Q}_x$. The linear span of $\check{Q}_x$ is the 10-dimensional vector space $\mathcal{S}_x^*$. Symmetrically, each point ℓ in $\mathbb{O\check{P}}^2$ defines a smooth 8-dimensional quadric Q_ℓ in the Cayley plane. Correspondingly we have a vector bundle $\check{\mathcal{S}}$ of rank 10 on $\mathbb{O\check{P}}^2$. The

incidence variety $\{(x, \ell) \in X \times \mathbb{O}\check{\mathbb{P}}^2 \mid x \in Q_\ell\}$ is the flag variety $E_6/P_{1,6}$, and we have:

$$E_6/P_{1,6}$$

$$X \qquad \mathbb{O}\check{\mathbb{P}}^2$$

where both maps are fibrations in smooth 8-dimensional quadrics.

We call these quadrics the $\mathbb{O}$-*lines*. They have (up to a grain of salt: the word *general* in the next statements) the characteristic properties of a plane projective geometry: two general $\mathbb{O}$-lines meet in a single point, and through two general points pass a unique $\mathbb{O}$-line. They can also be considered as *entry-loci* in the following sense. The secant variety of X is a cubic hypersurface in $\mathbb{P}(V_{\omega_1})$ which we call the Cartan cubic $\mathscr{C}$, see [IM11]. This cubic is also the dual variety of $\mathbb{O}\check{\mathbb{P}}^2$, so we have a birational map (in fact a resolution of singularities) $\nu : \mathbb{P}(\check{\mathcal{S}}^*) \to \mathscr{C} \subseteq \mathbb{P}(V_{\omega_1})$, which is given by the relatively ample line bundle on $\mathbb{P}(\check{\mathcal{S}}^*)$. The flag variety $E_6/P_{1,6}$ is the exceptional divisor of ν in $\mathbb{P}(\check{\mathcal{S}}^*)$.

The secant variety $\mathscr{C}$ is degenerate (and therefore equal to the tangent variety of X): this means that a point y of the $\mathscr{C} \setminus X$ belongs to infinitely many secant lines to the $\mathscr{C}$. The intersection of these secants with X, traditionally called the entry-locus of y, is a smooth 8-dimensional quadric, and in fact the $\mathbb{O}$-line Q_ℓ for some $\ell \in \mathbb{O}\check{\mathbb{P}}^2$. Summing up, we have a commutative diagram of E_6-equivariant morphisms:

$$\mathbb{P}(\check{\mathcal{S}}^*) \longleftarrow E_6/P_{1,6}$$

$$\mathscr{C} \longleftarrow X \qquad \mathbb{O}\check{\mathbb{P}}^2$$

1.2. Homogeneous bundles on the Cayley plane.

The category of E_6-homogeneous bundles on the Cayley plane $X = \mathbb{O}\mathbb{P}^2 = E_6/P_1$ is equivalent to the category of P_1-modules. Let $P_1 = LP^u$ be a Levi decomposition, where P^u is the unipotent radical and L is reductive. The center of the Levi factor L is one-dimensional. The quotient of L by its center is the semisimple part of P_1, and is isomorphic to Spin_{10}.

The unipotent radical P^u has to act trivially on an irreducible P_1-module, which is therefore completely determined by its L-module structure. An irreducible L-module is determined by its highest weight, an L-dominant weight of E_6. The set L-dominant of weights is given by linear combinations $\omega = a_1\omega_1 + \cdots + a_6\omega_6$, with $a_2, \ldots, a_6 \in \mathbb{Z}_{\geq 0}$ and $a_1 \in \mathbb{Z}$. We denote by $\mathcal{E}_\omega$ the E_6-homogeneous bundle on X corresponding to the irreducible L-module determined by the weight ω.

We have $E_{\omega_1} \simeq \mathcal{O}_X(1)$, the ample generator of $\mathrm{Pic}(X)$. Set $H = c_1(\mathcal{O}_X(1))$. Further recall that we defined $\mathcal{S} = \mathcal{N}(-1)$, so $\mathcal{S}$ is clearly E_6-homogeneous. Moreover, we can derive the Chern classes of $\mathcal{S}$ and hence of $\mathcal{S}_2$ from [IM05]. Some of the invariants of these bundles are:

$$\mathrm{rk}(\mathcal{S}) = 10, \qquad c_1(\mathcal{S}) = 5H, \qquad c_2(\mathcal{S}) = 12H^2, \qquad c_{10}(\mathcal{S}) = 0.$$
$$\mathrm{rk}(\mathcal{S}_2) = 54, \qquad c_1(\mathcal{S}_2(-1)) = 0, \qquad c_2(\mathcal{S}_2(-1)) = 9H^2.$$

Now it is clear that $\mathcal{S} \simeq \mathcal{E}_{\omega_6}$, since $\mathcal{E}_{\omega_6}$ is the only E_6-homogeneous bundles of rank 10 with the required Chern classes. We will see another way to identify

$\mathcal{E}_{\omega_6}$ and $\mathcal{N}(-1)$ at the beginning of the next section. Further, we have $\mathcal{S}_2 = \mathcal{E}_{2\omega_6}$ (see [Man11]).

By the Borel-Weil theorem $H^0(X, \mathcal{S}) \simeq V_{\omega_6}^* \simeq V_{\omega_1}$ and $\mathcal{S}$ is generated by its global sections. In particular for any point x in the Cayley plane the fibre $\mathcal{S}_x^*$ of the dual bundle can be considered as a subspace of $V_{\omega_6} = V_{\omega_1}^*$, and this subspace is nothing else than the linear span of the $\mathbb{O}$-line $\check{Q}_x$.

Recall from [Man11] the isomorphisms $\mathcal{S}^* \simeq \mathcal{S}(-1)$, $\mathcal{S}_2^* \simeq \mathcal{S}_2(-2)$.

We already mentioned that $\mathcal{S}$ is globally generated, but the fact that $c_{10}(\mathcal{S}) = 0$ indicates that a general global section s of $\mathcal{S}$ vanishes nowhere. We can thus define a rank-9 vector bundle $\mathcal{S}_s$ on X by the sequence:

$$(2) \qquad\qquad 0 \to \mathcal{O}_X \xrightarrow{s} \mathcal{S} \to \mathcal{S}_s \to 0.$$

Moreover $c_1(\mathcal{S}_s) = c_1(\mathcal{S}) = 5H$, and of course $\mathcal{S}_s$ is again globally generated. The next lemma will play a crucial role in our argument.

Lemma 1. *Consider a vector bundle $\mathcal{S}_s$ as above and a global section σ of $\mathcal{S}_s$.*

  i) *If the global section σ is general enough, then it vanishes on the disjoint union $\mathcal{Q}_\sigma$ of three smooth 7-dimensional quadrics.*
  ii) *Any smooth 7-dimensional quadric contained in X is a component of some $\mathcal{Q}_\sigma$.*

Proof. A section s of $\mathcal{S}$ is defined by a vector $v \in V_{\omega_1}$, and s vanishes at $x \in X$ if and only if v belongs to the affine tangent space $\hat{\mathcal{T}}_{X,x}$. This can happen only if v belongs to the union of the tangent spaces to the Cayley plane, which is nothing else than the cone over the Cartan cubic. In particular this cannot happen if v is general: this explains why $c_{10}(\mathcal{S}) = 0$.

A section σ of $\mathcal{S}_s$ is defined by a vector $\bar{u} \in V_{\omega_1}/\mathbb{C}v$. It vanishes at $x \in X$ if and only if u belongs to the span of v and the affine tangent space $\hat{\mathcal{T}}_{X,x}$. Generically, the projective line generated by u and v cuts the Cartan cubic in three points y_1, y_2, y_3, and the embedded tangent space $T_{X,x}$ must contain one of these three points. Suppose that this point is y_1. Then x must be on the entry-locus of y_1, a smooth 8-dimensional quadric Q_1. Moreover y_1 must belong to $T_{Q_1,x}$, which means that x is on the polar hyperplane H_1 to y_1 with respect to Q_1. Finally, x has to belong to the 7-dimensional quadric $Q_1 \cap H_1$. Switching to y_2 and y_3 we obtain this way three 7-dimensional quadrics that constitute the vanishing locus of σ. These quadrics have to be disjoint since $\mathcal{S}_s$ is globally generated so the vanishing locus of σ is smooth. The statement (i) follows.

To check (ii), let Q be a smooth 7-dimensional quadric. Observe that its linear span $\langle Q \rangle$ cannot be contained in X, whose maximal linear spaces are only five dimensional by [LM03, Corollary 7.19]. Since the Cayley plane is cut out by quadrics, this implies that $X \cap \langle Q \rangle = Q$. Note moreover that $\langle Q \rangle$, being the secant variety of Q, is contained in the secant variety of X. If we choose a point z in $\langle Q \rangle \setminus Q$, its entry locus $\tilde{Q}$ is an 8-dimensional quadric. Since the line joining z to any point of Q is contained in $\langle Q \rangle$, it meets Q along a scheme of length two, hence Q must be contained in the entry locus of z. Therefore Q must be a hyperplane section of $\tilde{Q}$, say $Q = \tilde{Q} \cap H$. Let y be the point of $\langle \tilde{Q} \rangle$ polar to H with respect to $\tilde{Q}$. Since Q is smooth, y does not belong to $\tilde{Q}$, hence not to the Cayley plane either. A general line through y will allow to define a section σ of some $\mathcal{S}_s$ such that Q is a component of $\mathcal{Q}_\sigma$. $\square$

2. GENERATING NEW BUNDLES FROM OUR EXCEPTIONAL COLLECTION

Given the exceptional collection appearing in our main result, we denote by $\mathscr{D}$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(X)$ generated by it:

$$\mathscr{D} = \langle \mathcal{A}, \mathcal{A}(1), \mathcal{A}(2), \mathcal{C}(3), \mathcal{C}(4), \ldots, \mathcal{C}(11) \rangle,$$

which means that $\mathscr{D}$ is the smalles triangulated full subcategory of $\mathrm{D}^{\mathrm{b}}(X)$ containing $(\mathcal{A}, \mathcal{A}(1), \mathcal{A}(2), \mathcal{C}(3), \mathcal{C}(4), \ldots, \mathcal{C}(11))$. One may ask, what are the sheaves on X that do lie in $\mathscr{D}$? Of course, the goal of this note is to answer: *all of them.* In this section, we will first prove that some particular sheaves lie in $\mathscr{D}$. We will use these sheaves further on, to prove that the orthogonal of $\mathscr{D}$ in $\mathrm{D}^{\mathrm{b}}(X)$ is zero, which amounts to our main result.

One of our tools for getting new bundles from the generators of $\mathscr{D}$ will be the following classical observation. Given a P_1-module E, we can consider it as an L-module, and then define a new P_1-module structure by extending trivially to P^u. We get the associated graded module $\mathrm{gr}(E)$, which is a direct sum of irreducible P_1-modules. Moreover, E can be reconstructed from the summands of $\mathrm{gr}(E)$ by a series of extensions. For instance, starting from the normal bundle sequence and the Euler sequence, we can identify $\mathcal{S} = \mathcal{E}_{\omega_6} \simeq \mathcal{N}(-1)$ by observing that $\mathrm{gr}(V_{\omega_1} \otimes \mathcal{O}_X) \simeq \mathcal{E}_{\omega_6} \oplus \mathcal{O}_S \oplus \mathcal{E}_{\omega_4}(-1)$, and comparing ranks.

If we consider an irreducible bundle $\mathcal{E}_\omega$ defined by an L-dominant weight ω, the corresponding irreducible Spin_{10}-representation has highest weight $\omega = a_2\omega_2 + \cdots + a_6\omega_6$. Relations between representations of Spin_{10} will therefore imply relations between the corresponding homogeneous bundles. For example, the following fact is general: if we consider a fundamental representation corresponding to an extremal node of some Dynkin diagram, its second skew-symmetric power contains the fundamental representation corresponding to the neighbouring node.
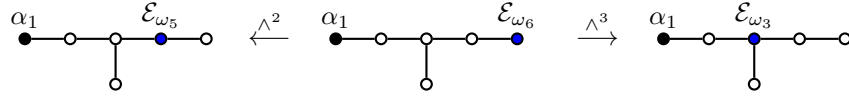
This works well till we attain a triple node or a multiple edge.
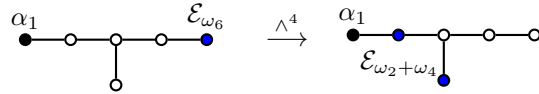
For Spin_{10} this is particularly neat since these inclusions are in fact always identities. In particular the fundamental representation corresponding to the triple node can be written as a wedge power in three different ways.

To get the correct relations between L-modules there remains to adjust the action of the one dimensional center, which can be done by computing the first Chern class. Once we cross a triple node, the wedge power contains (and will be equal to, in the cases we will be interested in) the irreducible representation whose highest weight is the sum of the two fundamental weights beyond the triple node (this follows from Theorem 2.1 in [LM04]). This way we get:

Lemma 2. *The E_6-homogeneous bundles defined by the fundamental representations of L can be described as:*

$$\mathcal{E}_{\omega_6} \simeq \mathcal{S}, \qquad \mathcal{E}_{\omega_5} \simeq \wedge^2 \mathcal{S}, \qquad \mathcal{E}_{\omega_3} \simeq \wedge^3 \mathcal{S},$$
$$\mathcal{E}_{\omega_4} \simeq \mathcal{T}_X, \qquad \mathcal{E}_{\omega_2} \simeq \Omega_X(2), \qquad \mathcal{E}_{\omega_3} \simeq \wedge^2 \mathcal{T}_X \simeq \Omega_X^2(3).$$

Moreover, the wedge powers that come next are:

$$\wedge^4 \mathcal{S} \simeq \mathcal{E}_{\omega_2+\omega_4}, \qquad \wedge^3 \mathcal{T}_X \simeq \mathcal{E}_{\omega_2+\omega_5}, \qquad \Omega_X^3(4) \simeq \mathcal{E}_{\omega_4+\omega_5}.$$

We already mentioned that we also have $\mathcal{E}_{\omega_1} \simeq \mathcal{O}_X(1)$, the ample generator of the Picard group $\mathrm{Pic}(X) \simeq \mathbb{Z}$. This very ample line bundle defines the natural embedding of X into $\mathbb{P}(V_{\omega_1})$. One has a natural duality $V_{\omega_1}^* \simeq V_{\omega_6}$, so $H^0(X, \mathcal{O}_X(1)) \simeq V_{\omega_6}$. Moreover, the bundle $\mathcal{S}(1)$ is the normal bundle of the embedding of X into $\mathbb{P}(V_{\omega_1})$. Let us denote $\Omega_\mathbb{P}^p = \Omega_{\mathbb{P}(V_{\omega_1})}^p|_X$ and $\mathcal{T}_\mathbb{P}^p = \wedge^p \mathcal{T}_{\mathbb{P}(V_{\omega_1})}|_X$. The Euler sequence reads:

$$(3) \qquad\qquad 0 \to \Omega_\mathbb{P} \to V_{\omega_6} \otimes \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0,$$

while the conormal bundle sequence is:

$$(4) \qquad\qquad 0 \to \mathcal{S}^*(-1) \to \Omega_\mathbb{P} \to \Omega_X \to 0.$$

We get a commutative exact diagram, where we omit zeroes all around the diagram for brevity:

$$(5) \qquad
\begin{array}{ccc}
\mathcal{S}^*(-1) & = & \mathcal{S}^*(-1) \\
\downarrow & & \downarrow \\
\Omega_\mathbb{P} \longrightarrow & V_{\omega_6} \otimes \mathcal{O}_X(-1) & \to \mathcal{O}_X \\
\downarrow & \downarrow & \| \\
\Omega_X \longrightarrow & \hat{\Omega}_X & \longrightarrow \mathcal{O}_X,
\end{array}$$

and its dual:

$$(6) \qquad
\begin{array}{ccc}
\mathcal{O}_X & = & \mathcal{O}_X \\
\downarrow & & \downarrow \\
\hat{\mathcal{T}}_X \longrightarrow & V_{\omega_1} \otimes \mathcal{O}_X(1) & \to \mathcal{S}^*(2) \\
\downarrow & \downarrow & \| \\
\mathcal{T}_X \longrightarrow & \mathcal{T}_\mathbb{P} & \longrightarrow \mathcal{S}^*(2),
\end{array}$$

where we denote by $\hat{\mathcal{T}}_X$ and $\hat{\Omega}_X$ the affine tangent and cotangent bundles of X.

Lemma 3. *Consider the bundles* $F_1 = \mathcal{T}_X(-1) \simeq \mathcal{E}_{\omega_4}(-1)$, $F_2 = \Omega_X^2(2) \simeq \mathcal{E}_{\omega_3}(-1)$, $F_3 = \mathcal{S} = \mathcal{E}_{\omega_6}$, *and* $F_4 = \Omega_X(2) \simeq \mathcal{E}_{\omega_2}$. *Then we have:*

$$\mathrm{Ext}_X^1(F_3, F_1)^{E_6} \simeq \mathrm{Ext}_X^1(F_2, F_1)^{E_6} \simeq \mathrm{Ext}_X^1(F_4, F_3)^{E_6} \simeq \mathrm{Ext}_X^1(F_4, F_2)^{E_6} \simeq \mathbb{C},$$

$$\mathrm{Ext}_X^2(F_4, F_1)^{E_6} \simeq \mathbb{C},$$

and all other E_6-invariant components of $\mathrm{Ext}^i(F_j, F_k)$ are zero, for $i = 1, 2$ and $j, k = 1, \dots, 4$. Moreover, the Yoneda pairing induces isomorphisms:

$$\Upsilon : \mathrm{Ext}_X^1(F_4, F_3)^{E_6} \otimes \mathrm{Ext}_X^1(F_3, F_1)^{E_6} \to \mathrm{Ext}_X^2(F_4, F_1)^{E_6},$$

$$\Upsilon : \mathrm{Ext}_X^1(F_4, F_2)^{E_6} \otimes \mathrm{Ext}_X^1(F_2, F_1)^{E_6} \to \mathrm{Ext}_X^2(F_4, F_1)^{E_6}.$$

Proof. Using representations of Spin_{10} and adjusting the first Chern class, we get:

$$\mathrm{gr}(F_1^* \otimes F_2) \simeq \mathcal{E}_{\omega_4} \oplus \mathcal{E}_{\omega_4+\omega_5}(-1) \oplus \mathcal{E}_{\omega_2+\omega_6}(-1) \oplus \mathcal{E}_{\omega_2+\omega_3}(-2),$$

$$(7) \qquad \mathrm{gr}(F_1^* \otimes F_3) \simeq \mathcal{E}_{\omega_4} \oplus \mathcal{E}_{\omega_2+\omega_6}(-1),$$

$$\mathrm{gr}(F_1^* \otimes F_4) \simeq \mathcal{E}_{2\omega_2}(-1) \oplus \mathcal{E}_{\omega_3} \oplus \mathcal{E}_{\omega_6}(1).$$

For the bundle F_2 we get:

$$\mathrm{gr}(F_2^* \otimes F_1) \simeq \mathcal{E}_{\omega_2}(-2) \oplus \mathcal{E}_{\omega_4+\omega_6}(-2) \oplus \mathcal{E}_{\omega_2+\omega_5}(-3) \oplus \mathcal{E}_{\omega_3+\omega_4}(-3),$$
$$\mathrm{gr}(F_2^* \otimes F_3) \simeq \mathcal{E}_{\omega_5}(-1) \oplus \mathcal{E}_{\omega_2+\omega_4}(-2) \oplus \mathcal{E}_{\omega_3+\omega_6}(-2),$$
$$F_2^* \otimes F_4 \simeq F_1^* \otimes F_2.$$

For F_3, we get:

$$\text{(8)} \qquad \mathrm{gr}(F_3^* \otimes F_1) \simeq \mathcal{E}_{\omega_2}(-2) \oplus \mathcal{E}_{\omega_4+\omega_6}(-2),$$
$$F_3^* \otimes F_2 \simeq F_2^* \otimes F_3,$$
$$F_3^* \otimes F_4 \simeq F_1^* \otimes F_3.$$

Finally for F_4:

$$\mathrm{gr}(F_4^* \otimes F_1) \simeq \mathcal{E}_{\omega_6}(-2) \oplus \mathcal{E}_{\omega_3}(-3) \oplus \mathcal{E}_{2\omega_4}(-3),$$
$$F_4^* \otimes F_2 \simeq F_2^* \otimes F_1,$$
$$F_4^* \otimes F_3 \simeq F_3^* \otimes F_1.$$

Now note that the leftmost column of (5) and the bottom row of (6) give non-zero elements in $\mathrm{Ext}_X^1(F_4, F_3)^{E_6}$ and $\mathrm{Ext}_X^1(F_3, F_1)^{E_6}$. Taking the exterior square of the central column of (5) we get:

$$\text{(9)} \qquad 0 \to \Omega_X^2 \to \hat{\Omega}_X^2 \to \Omega_X \to 0,$$

which gives a non-zero element $\eta \in \mathrm{Ext}_X^1(F_4, F_2)^{E_6}$. Dualizing and using $\mathcal{T}_X^2 \simeq \Omega_X^2(3)$ (see the previous lemma) we get a non-zero element $\zeta \in \mathrm{Ext}_X^1(F_2, F_1)^{E_6}$. Using Bott's theorem and the above decompositions of $\mathrm{gr}(F_j^* \otimes F_k)$, we get that $\mathrm{Ext}_X^2(F_4, F_1)$ is at most 1-dimensional, and we check that there are no other E_6-invariant elements in $\mathrm{Ext}^i(F_j, F_k)$, for $i = 1, 2$. So it only remains to prove the statement on the non-degeneracy of the Yoneda maps.

For the first map, we apply $\mathrm{Hom}_X(-, F_3(-2))$ to the leftmost column of (5). We get an exact sequence:

$$\mathrm{Ext}_X^1(\Omega_{\mathbb{P}}(2), F_3) \to \mathrm{Ext}_X^1(F_4, F_3) \to \mathrm{Ext}_X^1(F_1, F_3),$$

and the rightmost map is $\Upsilon(\zeta \otimes -)$. So, it suffices to prove that $\mathrm{Ext}_X^1(\Omega_{\mathbb{P}}(2), F_3) = 0$. But this is clear by applying $\mathrm{Hom}_X(-, F_3(-2))$ to the central row of (5), since $H^i(X, F_3(-2)) = H^i(X, F_3(-1)) = 0$ for all i.

For the second map, we apply $\mathrm{Hom}_X(-, F_1(-2))$ to (9). We get the exact sequence:

$$\mathrm{Ext}_X^1(\hat{\Omega}_X^2(2), F_1) \to \mathrm{Ext}_X^1(F_2, F_1) \to \mathrm{Ext}_X^2(F_4, F_1),$$

and the rightmost map is $\Upsilon(\eta \otimes -)$. So it suffices to check that the leftmost space is zero. Taking exterior square of the central column of (5) and using (1) we get:

$$0 \to \mathcal{S}_2(-4) \oplus \mathcal{O}_X(-3) \to V_{\omega_6} \otimes \mathcal{S}(-3) \to V_{\omega_5} \otimes \mathcal{O}_X(-2) \to \hat{\Omega}_X^2 \to 0.$$

Note that $H^1(X, F_1) = 0$ and $\mathrm{Hom}(\mathcal{S}(-1), F_1) = 0$ by (8), since $\mathrm{gr}(\mathcal{S} \otimes F_1) = \mathrm{gr}(F_3^* \otimes F_1(1))$. Then, applying $\mathrm{Hom}_X(-, F_1(-2))$ to the above sequence we easily get the desired vanishing $\mathrm{Ext}_X^1(\hat{\Omega}_X^2(2), F_1) = 0$. $\qquad\square$

We will simply denote $\Upsilon(a \otimes b) = ab$.

Lemma 4. *We have an exact sequence:*

$$0 \to \mathcal{S}_2^* \to V_{\omega_6} \otimes \mathcal{S}^* \to (V_{\omega_1} \oplus V_{\omega_5}) \otimes \mathcal{O}_X \to$$
$$\to (V_{\omega_6} \oplus V_{\omega_2}) \otimes \mathcal{O}_X(1) \to V_{\omega_1} \otimes \mathcal{S}^*(2) \to \mathcal{S}_2^*(3) \to 0.$$

In particular, $\mathcal{S}_2^*(t)$ lies in $\mathscr{D}$ for $3 \le t \le 12$.

Proof. We start with the exact sequence appearing as middle row of (6). Taking its symmetric square, using (1), and twisting by $\mathcal{O}_X(-1)$ we get:

$$(10) \qquad 0 \to \hat{\mathcal{T}}_X^2(-1) \to \wedge^2 V_{\omega_1} \otimes \mathcal{O}_X(1) \to V_{\omega_1} \otimes \mathcal{S}^*(2) \to \mathcal{S}_2^*(3) \oplus \mathcal{O}_X(2) \to 0,$$

Let K be the image of the middle map in the above sequence, and note that $\wedge^2 V_{\omega_1} \simeq V_{\omega_2}$. Using the surjection $\mathcal{S}_2^*(3) \oplus \mathcal{O}_X(2) \to \mathcal{S}_2^*(3)$, we get an epimorphism $V_{\omega_1} \otimes \mathcal{S}^*(2) \to \mathcal{S}_2^*(3)$, whose kernel $\tilde{K}$ fits into:

$$0 \to K \to \tilde{K} \to \mathcal{O}_X(2) \to 0.$$

By the above sequence, using the middle row of (5) and (10), we get an exact sequence:

$$(11) \qquad 0 \to E \to (V_{\omega_6} \oplus V_{\omega_2}) \otimes \mathcal{O}_X(1) \to V_{\omega_1} \otimes \mathcal{S}^*(2) \to \mathcal{S}_2^*(3) \to 0,$$

where E fits into:

$$0 \to \hat{\mathcal{T}}_X^2(-1) \to E \to \Omega_{\mathbb{P}}(2) \to 0.$$

Now, we wish to prove that $E \simeq E^*(1)$. Note that this will finish the proof, for in this case we can glue (11) with its dual, twisted by $\mathcal{O}_X(1)$, to get the desired resolution.

In order to check $E \simeq E^*(1)$, first notice that E is exceptional, indeed (11) is nothing but the left mutation of $\mathcal{S}_2^*(3)$ through $\mathcal{S}^*(2)$ and $\mathcal{O}_X(1)$, and $(\mathcal{O}_X(1), \mathcal{S}^*(2), \mathcal{S}_2^*(3))$ is an exceptional collection. In particular, E is indecomposable, and homogeneous for E_6. Further, note that E fits in the commutative E_6-invariant exact diagram (where again we omit zeroes all around):

$$(12) \qquad \begin{array}{ccc} \mathcal{T}_X(-1) \longrightarrow \hat{\mathcal{T}}_X^2(-1) \longrightarrow \mathcal{T}_X^2(-1) \\ \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \\ \mathcal{T}_{\mathbb{P}}(-1) \longrightarrow E \longrightarrow \hat{\Omega}_X^2(2) \\ \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \\ \mathcal{S}^* \longrightarrow \Omega_{\mathbb{P}}(2) \longrightarrow \Omega_X(2) \end{array}$$

In other words, $\mathrm{gr}(E) \simeq \bigoplus_{i=1}^4 F_i$ in the notation of the previous lemma. Therefore, we are in position to use [OR06, Theorem 5.6], and the bundle E is determined by 4 elements $(a_1, \ldots, a_4)$ in $\mathrm{Ext}_X^1(\mathrm{gr}(E), \mathrm{gr}(E))^{E_6}$, where we order the spaces $\mathrm{Ext}_X^1(F_j, F_k)^{E_6}$ according to the display of Lemma 3. Moreover, again by Lemma 3 and [OR06, Theorem 5.6], we must have $a_3 a_1 = a_4 a_2$. This means that E amounts to a representation of the following quiver, with the relation $a_3 a_1 = a_4 a_2$.

$$(13) \qquad \begin{array}{ccc} \overset{F_1}{\bullet} \overset{a_1}{\longleftarrow} \overset{F_2}{\bullet} \\ \uparrow a_2 \qquad \uparrow a_3 \\ \underset{F_3}{\bullet} \underset{a_4}{\longleftarrow} \underset{F_4}{\bullet} \end{array}$$

Now, observe that none of the a_i is zero. Indeed if for instance $a_1 = 0$, then $a_3 a_1 = a_4 a_2$ forces $a_2 = 0$ or $a_4 = 0$, and in both cases E is decomposable. Moreover, note that dualizing (12) and twisting by $\mathcal{O}_X(1)$, we get $\mathrm{gr}(E^*(1)) \simeq \mathrm{gr}(E)$, and clearly $E^*(1)$ is exceptional too. Then, $E^*(1)$ is also given as a representation $(b_1, \ldots, b_4)$ of the quiver appearing in (13), where a_i is replaced by b_i so again $b_3 b_1 = b_4 b_2$ and $b_i \neq 0$ for all i. Therefore, we can find a 4-tuple $(k_1, \ldots, k_4) \in (\mathbb{C}^*)^4$, operating on the vertexes of our quiver, that takes $(b_1, \ldots, b_4)$ to $(a_1, \ldots, a_4)$, namely $k_1 = a_4 a_2 / b_2 b_4$, $k_2 = a_4/b_4$, $k_3 = a_3/b_3$ and $k_4 = 1$. This says that E and $E^*(1)$ correspond to the same representation of our quiver, i.e., $E \simeq E^*(1)$. $\qquad \square$

Lemma 5. *The bundle $\wedge^2 \mathcal{S} \simeq \wedge^2 \mathcal{S}^*(2)$ belongs to the full subcategory of $\mathrm{D}^{\mathrm{b}}(X)$:*

$$\mathscr{D}_2 = \langle \mathcal{S}^*, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2), \mathcal{S}^*(3) \rangle.$$

In particular,

$$\wedge^2 \mathcal{S}^*(t) \in \mathscr{D}, \qquad for \ 2 \leq t \leq 10.$$

Proof. Consider the adjoint representation V_{ω_4}. We can compute its restriction to Spin_{10} by the following simple argument. We observe that highest weights for the action of Spin_{10} are given by the highest roots with a fixed coefficient k on the simple root α_1. This coefficient k can be equal to 0 or ± 1; for $k = 0$ we get the adjoint representation of Spin_{10} and for $k = \pm 1$ we get the two spin representations. Since $45 + 16 + 16 = 77 = 78 - 1$ there is only one trivial factor missing. Taking into account the action of the center of the Levi factor L of P_1, that is, adjusting the first Chern class, we obtain:

$$(14) \qquad \mathrm{gr}(V_{\omega_4} \otimes \mathcal{O}_X) \simeq \mathcal{T}_X \oplus \Omega_X \oplus \wedge^2 \mathcal{S}(-1) \oplus \mathcal{O}_X.$$

Now, using (5) and (6) we get that $\mathcal{T}_X$ belongs to $\langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{S}^*(2) \rangle$ and Ω_X to $\langle \mathcal{S}^*(-1), \mathcal{O}_X(-1), \mathcal{O}_X \rangle$, so that (14) ensures that $\wedge^2 \mathcal{S}(-1) \simeq \wedge^2 \mathcal{S}^*(1)$ lies in $\langle \mathcal{S}^*(-1), \mathcal{O}_X(-1), \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{S}^*(2) \rangle$. This clearly implies our statement. $\qquad \square$

Lemma 6. *The bundle $\wedge^3 \mathcal{S} \simeq \wedge^3 \mathcal{S}^*(3)$ lies in:*

$$\mathscr{D}_3 = \langle \mathcal{O}_X, \mathcal{S}_2^*(1), \mathcal{S}^*(1), \mathcal{O}_X(1), \mathcal{S}^*(2), \mathcal{O}_X(2), \mathcal{O}_X(3) \rangle.$$

In particular,

$$\wedge^3 \mathcal{S}^*(t) \in \mathscr{D}, \qquad for \ 3 \leq t \leq 11.$$

Proof. We have to check that $\wedge^3 \mathcal{S} \simeq \Omega_X^2(3)$ (by Lemma 2) lies in $\mathscr{D}_3$. To do this, we take the exterior square of the conormal sequence (4), and we twist by $\mathcal{O}_X(3)$, thus obtaining an exact sequence:

$$0 \to S^2 \mathcal{S}^*(1) \to \mathcal{S}^* \otimes \Omega_{\mathbb{P}}(2) \to \Omega_{\mathbb{P}}^2(3) \to \wedge^3 \mathcal{S}(3) \to 0.$$

We are now lead to ask if $S^2 \mathcal{S}^*(1)$, $\mathcal{S}^* \otimes \Omega_{\mathbb{P}}(2)$ and $\Omega_{\mathbb{P}}^2(3)$ belong to $\mathscr{D}_3$.

For the first bundle, recall from (1) the isomorphism $S^2 \mathcal{S}^*(1) \simeq \mathcal{S}_2^*(1) \oplus \mathcal{O}_X$. For the second one, twisting by $\mathcal{S}^*$ the Euler sequence (3) we see that $\mathcal{S}^* \otimes \Omega_{\mathbb{P}}(2)$ belongs to $\langle \mathcal{S}^*(1), \mathcal{S}^*(2) \rangle$. For the third bundle, we take the exterior square of the Euler sequence:

$$0 \to \Omega_{\mathbb{P}}^2 \to \wedge^2 V_{\omega_6} \otimes \mathcal{O}_X(-2) \to V_{\omega_6} \otimes \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0.$$

This readily says that $\Omega_{\mathbb{P}}^2(3)$ lies in $\langle \mathcal{O}_X(1), \mathcal{O}_X(2), \mathcal{O}_X(3) \rangle$. We get $\wedge^3 \mathcal{S} \in \mathscr{D}_3$.

Using the isomorphism $\wedge^3 \mathcal{S}^* \simeq \wedge^3 \mathcal{S}(-3)$, and Lemma 4, we deduce that $\wedge^3 \mathcal{S}^*(t)$ lies in $\mathscr{D}$, for $3 \leq t \leq 11$. $\qquad \square$

Lemma 7. *The bundle $\wedge^4 \mathcal{S} \simeq \wedge^4 \mathcal{S}^*(4)$ lies in:*

$$\mathscr{D}_4 = \langle \mathcal{S}^*(1), \mathcal{O}_X(1), \mathcal{S}^*(2), \mathcal{O}_X(2), \mathcal{S}_2^*(3), \mathcal{S}^*(3), \mathcal{O}_X(3), \mathcal{S}^*(4) \rangle.$$

In particular,

$$\wedge^4 \mathcal{S}^*(t) \in \mathscr{D}, \qquad for\ 3 \le t \le 11.$$

Proof. By Lemma 2, we have $\wedge^4 \mathcal{S} \simeq \mathcal{E}_{\omega_2 + \omega_4}$. Using the same method as before we observe that the completely irreducible bundle associated to $\mathcal{E}_{\omega_2} \otimes \mathcal{E}_{\omega_4} \simeq \Omega_X(2) \otimes \mathcal{T}_X$ is the following:

$$\mathrm{gr}(\mathcal{E}_{\omega_2} \otimes \mathcal{E}_{\omega_4}) = \mathcal{E}_{\omega_2 + \omega_4} \oplus \wedge^2 \mathcal{S}(1) \oplus \mathcal{O}_X(2).$$

Therefore, it we will be enough to show that the three bundles $\Omega_X(2) \otimes \mathcal{T}_X$, $\wedge^2 \mathcal{S}(1)$ and $\mathcal{O}_X(2)$ lie in the subcategory $\mathscr{D}_4$. This is obviously the case for $\mathcal{O}_X(2)$, and also for $\wedge^2 \mathcal{S}(1)$ in view of Lemma 5. To show that $\Omega_X(2) \otimes \mathcal{T}_X$ lies in $\mathscr{D}_4$, we first consider the complexes lying on the diagonal of (5), twisted by $\mathcal{O}_X(2)$, together with its dual, lying on the diagonal of (6). Tensoring these two complexes, we get a third complex of the form:

$$
\mathcal{S}^*(1) \to
\begin{matrix} V_{\omega_1} \otimes \mathcal{S}^*(2) \\ \oplus \\ V_{\omega_6} \otimes \mathcal{O}_X(1) \end{matrix}
\to
\begin{matrix} \mathcal{S}^* \otimes \mathcal{S}^*(3) \\ \oplus \\ V_{\omega_1} \otimes V_{\omega_6} \otimes \mathcal{O}_X(2) \\ \oplus \\ \mathcal{O}_X(2) \end{matrix}
\to
\begin{matrix} V_{\omega_6} \otimes \mathcal{S}^*(3) \\ \oplus \\ V_{\omega_1} \otimes \mathcal{O}_X(3) \end{matrix}
\to \mathcal{S}^*(4).
$$

The proof will be finished once we show that all the terms appearing in this complex lie in $\mathscr{D}_4$. This is clear for the twists of $\mathcal{O}_X$ and of $\mathcal{S}^*$. Note that the only term which is not of this form is:

$$\mathcal{S}^* \otimes \mathcal{S}^*(3) \simeq \wedge^2 \mathcal{S}^*(3) \oplus \mathcal{S}_2^*(3) \oplus \mathcal{O}_X(2),$$

see (1). But we have already seen that $\wedge^2 \mathcal{S}^*(3) \simeq \wedge^2 \mathcal{S}(1)$ lies in $\mathscr{D}_4$, as well as $\mathcal{S}_2^*(3)$ and $\mathcal{O}_X(2)$, so the lemma is proved. $\qquad \square$

Set $\mathcal{S}_3 = \mathcal{E}_{3\omega_1}$ and recall from [Man11] that $S^3 \mathcal{S} \simeq \mathcal{S}_3 \oplus \mathcal{S}(1)$ and $\mathcal{S}_3^* \simeq \mathcal{S}_3(-3)$.

Lemma 8. *The bundle $\mathcal{S}_3^*(t)$ lies in $\mathscr{D}$ for $1 \le t \le 6$.*

Proof. Taking the symmetric cube of (4) twisted by $\mathcal{O}_X(3)$, we get:

$$0 \to S^3 \mathcal{S}^* \to S^2 \mathcal{S}^* \otimes \Omega_{\mathbb{P}}(1) \to \mathcal{S}^* \otimes \Omega_{\mathbb{P}}^2(2) \to \Omega_{\mathbb{P}}^3(3) \to \Omega_X^3(3) \to 0.$$

We look at the various terms of this sequence. The first one decomposes as $\mathcal{S}^*(-1) \oplus \mathcal{S}_3^*$, and our aim is to understand which of its twists lie in $\mathscr{D}$, so let us study the other terms. The second term lies in $\langle \mathcal{O}_X(-1), \mathcal{S}_2^*, \mathcal{O}_X, \mathcal{S}_2^*(1) \rangle$, the third one in $\langle \mathcal{S}^*, \mathcal{S}^*(1), \mathcal{S}^*(2) \rangle$, and the fourth one in $\langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2), \mathcal{O}_X(3) \rangle$. So it remains to study the last term $\Omega_X^3(3)$.

By Lemma 2 we know that $\Omega_X^3(3) \simeq \mathcal{E}_{\omega_4 + \omega_5}(-1)$. Moreover we check that:

$$\mathrm{gr}(\mathcal{E}_{\omega_4} \otimes \mathcal{E}_{\omega_5}(-1)) \simeq \mathcal{E}_{\omega_4 + \omega_5}(-1) \oplus \mathcal{E}_{\omega_2 + \omega_6}(-1) \oplus \mathcal{E}_{\omega_4},$$
$$\mathrm{gr}(\mathcal{E}_{\omega_2} \otimes \mathcal{E}_{\omega_6}(-1)) \simeq \mathcal{E}_{\omega_2 + \omega_6}(-1) \oplus \mathcal{E}_{\omega_4},$$

where by the way the second formula is just (7). We deduce that $\Omega_X^3(3)$ lies in the full subcategory of $\mathrm{D}^{\mathrm{b}}(X)$ generated by $\mathcal{E}_{\omega_4} \otimes \mathcal{E}_{\omega_5}(-1)$, $\mathcal{E}_{\omega_2} \otimes \mathcal{E}_{\omega_6}(-1)$ and $\mathcal{E}_{\omega_4}$. Some of these terms can be settled rather easily, for instance

$$\mathcal{E}_{\omega_4} \simeq \mathcal{T}_X \in \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{S}^*(2) \rangle,$$

by (6). Next, $\mathcal{E}_{\omega_6}(-1) \simeq \mathcal{S}^*$ and

$$\mathcal{E}_{\omega_2} \simeq \Omega_X(2) \in \langle \mathcal{S}^*(1), \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle,$$

by (5), so $\mathcal{E}_{\omega_2} \otimes \mathcal{E}_{\omega_6}(-1)$ lies in $\langle \wedge^2 \mathcal{S}^*(1), \mathcal{O}_X, \mathcal{S}_2^*(1), \mathcal{S}^*(1), \mathcal{S}^*(2) \rangle$, so Lemma 5 will take care of this term.

It remains to analyze the summand $\mathcal{E}_{\omega_4} \otimes \mathcal{E}_{\omega_5}(-1)$. First recall by Lemma 2 that this bundle is isomorphic to $\mathcal{T}_X \otimes \wedge^2 \mathcal{S}^*(1)$. Moreover, $\mathcal{T}_X \in \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{S}^*(2) \rangle$, and by Lemma 5, $\wedge^2 \mathcal{S}^*(1)$ belongs to $\langle \mathcal{S}^*(-1), \mathcal{O}_X(-1), \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{S}^*(2) \rangle$. Recalling that $\mathcal{S}^* \otimes \mathcal{S}^* \simeq \mathcal{S}_2^* \oplus \wedge^2 \mathcal{S}^* \oplus \mathcal{O}_X(-1)$ we deduce that $\mathcal{T}_X \otimes \wedge^2 \mathcal{S}^*(1)$ belongs to:

$$\left\langle \wedge^2 \mathcal{S}^*(1), \mathcal{S}_2^*(1), \{\mathcal{S}^*(r), \mathcal{O}_X(s), \ \text{for } -1 \leq r, s \leq 3\}, \mathcal{S}_2^*(4), \wedge^2 \mathcal{S}^*(4) \right\rangle.$$

Putting all these elements together, we get that $\mathcal{S}_3^*$ lies in:

$$\left\langle \{\mathcal{S}_2^*(p), \mathcal{S}^*(r), \mathcal{O}_X(s) \mid 0 \leq p \leq 4, -1 \leq r \leq 3, -1 \leq s \leq 3\}, \wedge^2 \mathcal{S}^*(1), \wedge^2 \mathcal{S}^*(4) \right\rangle.$$

Using Lemma 5 and Lemma 4, we deduce that $\mathcal{S}_3^*(t)$ lies in $\mathscr{D}$ for $1 \leq t \leq 6$. $\quad\square$

One can deduce from these lemmas that many more bundles lie in $\mathscr{D}$. One example of this is the next result, that we will need further on.

Corollary 9. *We have:*

(15) $$\wedge^2 \mathcal{S}^* \otimes \mathcal{S}^*(t) \in \mathscr{D}, \qquad\qquad \text{for } 2 \leq t \leq 9,$$

(16) $$\wedge^3 \mathcal{S}^* \otimes \mathcal{S}^*(t) \in \mathscr{D}, \qquad\qquad \text{for } 5 \leq t \leq 8,$$

(17) $$\wedge^4 \mathcal{S}^* \otimes \mathcal{S}^*(t) \in \mathscr{D}, \qquad\qquad \text{for } 5 \leq t \leq 7.$$

Proof. In order to prove (15), first recall that by Lemma 5, we know that $\wedge^2 \mathcal{S}^*(2) \in \langle \mathcal{S}^*, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2), \mathcal{S}^*(3) \rangle$. We deduce that $\wedge^2 \mathcal{S}^* \otimes \mathcal{S}^*(4)$ lies in:

$$\langle \wedge^2 \mathcal{S}^*(2), \mathcal{O}_X(1), \mathcal{S}_2^*(2), \mathcal{S}^*(2), \mathcal{S}^*(3), \mathcal{S}^*(4), \mathcal{O}_X(4), \mathcal{S}_2^*(5), \wedge^2 \mathcal{S}^*(5) \rangle.$$

Therefore $\wedge^2 \mathcal{S}^* \otimes \mathcal{S}^*(t) \in \mathscr{D}$ for $4 \leq t \leq 9$.

Now we use the relation:

(18) $$\wedge^2 \mathcal{S}^* \otimes \mathcal{S}^* \oplus \mathcal{S}_3^* \simeq \mathcal{S}_2^* \otimes \mathcal{S}^* \oplus \wedge^3 \mathcal{S}^*,$$

which is easy to establish using Spin_{10}-representations. With Lemmas 8 and 6 this implies that $\mathcal{S}_2^* \otimes \mathcal{S}^*(t) \in \mathscr{D}$ for $4 \leq t \leq 6$. Then we can use the complex from Lemma 4, twisted by $\mathcal{S}^*$, and deduce that this is still true for $t = 2, 3$. Then $\wedge^2 \mathcal{S}^* \otimes \mathcal{S}^*(t)$ also belongs to $\mathscr{D}$ for $t = 2, 3$.

To prove (16), we use Lemma 6, and we tensor by $\mathcal{S}^*(-3)$ the generators of the subcategory $\mathscr{D}_3$ appearing in this Lemma. We find that $\wedge^3 \mathcal{S}^* \otimes \mathcal{S}^*$ lies in the subcategory generated by $\mathcal{S}^*(r)$, with $-3 \leq r \leq 0$ and by $\mathcal{S}^* \otimes \mathcal{S}^*(-2)$, $\mathcal{S}^* \otimes \mathcal{S}^*(-1)$, $\mathcal{S}_2^* \otimes \mathcal{S}^*(-2)$. All these terms have been previously encountered. Using the relation (18), Lemma 8, Lemma 5 and Lemma 4 we get (16).

It remains to prove (17). We use Lemma 7 to see that:

$$\wedge^4 \mathcal{S}^* \otimes \mathcal{S}^* \in \left\langle \wedge^2 \mathcal{S}^*(q), \mathcal{S}_2^*(p), \mathcal{S}^*(r), \mathcal{O}_X(s) \right\rangle,$$

with $-3 \leq q, p \leq 0$, $-3 \leq r \leq -1$, and $-4 \leq s \leq -1$. Now the relation (17) can be deduced from Lemmas 4, 5 and 8. $\quad\square$

3. Restrictions to quadrics of dimension seven and eight

In order to prove our main result, our strategy will consist in restricting to quadrics in the Cayley plane: either $\mathbb{O}$-lines, which are eight-dimensional, or the seven-dimensional quadrics that appear in Lemma 1. In this section we prove that being orthogonal to $\mathscr{D}$ imposes drastic conditions on the restrictions to such quadrics. The final conclusion that $^{\perp}\mathscr{D} = 0$ will be obtained in the last section.

The following lemma, certainly known to experts, closely resembles [OSS80, Theorem 2.3.2]. We provide the proof for the reader's convenience.

Let us denote by Q_n a smooth quadric hypersurface in $\mathbb{P}^{n+1}$. If n is odd, we denote by Σ the spinor bundle on Q_n. If n is even, we denote by Σ_+ and Σ_- the two spinor bundle on Q_n (see [Ott88]).

Lemma 10. *Let $n \geq 3$ and let E be a vector bundle on Q_n. Then E splits as $\oplus_i \mathcal{O}_{Q_n}(a_i)$ if and only if the restriction of E to some quadric surface $Q_2 \subseteq Q_n$ splits as $\oplus_i \mathcal{O}_{Q_2}(a_i)$.*

Proof. One direction is obvious, and it suffices to prove that E splits as $\oplus_i \mathcal{O}_{Q_n}(a_i)$ if the restriction $E|_{Q_{n-1}}$ to a codimension-one quadric splits as $\oplus_i \mathcal{O}_{Q_{n-1}}(a_i)$, for $n \geq 3$. Consider the exact sequence:

$$(19) \qquad\qquad 0 \to E(t-1) \to E(t) \to E|_{Q_{n-1}}(t) \to 0.$$

Fix $t \in \mathbb{Z}$. Since the restricted bundle $E|_{Q_{n-1}}$ splits as $\oplus_i \mathcal{O}_{Q_{n-1}}(a_i)$, and $n \geq 3$, we have $H^1(Q_{n-1}, E|_{Q_{n-1}}(t)) = 0$ so $H^1(Q_n, E(t-1))$ maps onto $H^1(Q_n, E(t))$. Moreover, since E is locally free we must have $H^1(Q_n, E(t)) = 0$ for $t \ll 0$. This gives $H^1(Q_n, E(t)) = 0$ for all $t \in \mathbb{Z}$.

Consider now the bundle $F = \oplus_i \mathcal{O}_{Q_n}(a_i)$. Clearly, we have an epimorphism $F \to E|_{Q_{n-1}}$. Applying $\mathrm{Hom}_{Q_n}(F, -)$ to (19) for $t = 0$, we see that this epimorphism lifts to a morphism $\phi : F \to E$ if $\mathrm{Ext}^1_{Q_n}(F, E(-1)) = 0$. This is indeed the case since we proved that $H^1(Q_n, E(t)) = 0$ for all $t \in \mathbb{Z}$.

We have thus a morphism $\phi : F \to E$. The determinant of ϕ is a constant, since $c_1(F) = c_1(E)$, and this constant is non-zero, since the restriction of ϕ to Q_{n-1} is an isomorphism. Then ϕ is an isomorphism, and the lemma is proved. $\qquad\square$

The next proposition is the key ingredient of the proof of our main result. Roughly speaking, it uses the lemmas of the previous section to describe the restriction to 7-dimensional quadrics of a complex orthogonal to our collection. We will write $\mathbf{R}\Phi$ for the right-derived functors of a left-exact functor Φ, and similarly for the left-derived $\mathbf{L}\Phi$ of a right-exact functor. Given a subvariety Z of X, we will denote by i_Z the embedding of Z in X.

Proposition 11. *Let E be an object of the subcategory $^{\perp}\mathscr{D}$ of $\mathrm{D}^{\mathrm{b}}(X)$. Let $Q \subseteq X$ be a smooth 7-dimensional quadric. Then $\mathbf{L}i_Q^*(E)$ is a direct sum of shifts of $\mathcal{O}_Q(11)$.*

Proof. By Lemma 1, there exist $s \in H^0(X, \mathcal{S})$ and $\sigma \in H^0(X, \mathcal{S}_s)$ such that Q is a component of $\mathscr{Q} = \mathscr{Q}_\sigma$, so that $\mathscr{Q}$ is the disjoint union of Q and two additional 7-dimensional quadrics Q' and Q''. Set $j = i_{\mathscr{Q}}$, and let ι, ι', ι'' be the embedding respectively of Q, Q' and Q'' into $\mathscr{Q}$, so that $i_Q = j \circ \iota$, $i_{Q'} = j \circ \iota'$, $i_{Q''} = j \circ \iota''$. Clearly, since $\mathscr{Q} = Q \sqcup Q' \sqcup Q''$, for all $t \in \mathbb{Z}$, we have:

$$(20) \qquad\qquad \mathcal{O}_{\mathscr{Q}}(t) \simeq \iota_*(\mathcal{O}_Q(t)) \oplus \iota'_*(\mathcal{O}_{Q'}(t)) \oplus \iota''_*(\mathcal{O}_{Q''}(t)).$$

Denote by Σ' and Σ'' the spinor bundles, respectively on Q' and Q'', and write:

$$(21) \qquad \Sigma_{\mathscr{Q}} = \iota_*(\Sigma) \oplus \iota'_*(\Sigma') \oplus \iota''_*(\Sigma'').$$

This is a vector bundle of rank 8 on $\mathscr{Q}$ that restricts to the spinor bundle on each of the quadrics Q, Q' and Q''.

Now comes the main goal of this proof, which is to show that:

$$(22) \qquad j_*(\mathcal{O}_{\mathscr{Q}}(t)) \in \mathscr{D}, \qquad\qquad \text{for } 5 \leq t \leq 10,$$

$$(23) \qquad j_*(\Sigma_{\mathscr{Q}}(6)) \in \mathscr{D}.$$

To accomplish this task, we first note that $\mathcal{S}|_{\mathscr{Q}} \simeq \mathcal{O}_{\mathscr{Q}} \oplus \mathcal{O}_{\mathscr{Q}}(1) \oplus \Sigma_{\mathscr{Q}}$. Indeed, this is essentially just the statement that the vector representation of Spin_{10} restricted to Spin_8 is the sum of the vector representation and two copies of the trivial representation – there remains to take care of the actions of the center of the Levi subgroups of the parabolics (i.e., to compute c_1) to obtain the precise identity. Then, we consider the Koszul complex of the global section σ of $\mathcal{S}_s$:

$$0 \to \wedge^9 \mathcal{S}_s^* \simeq \mathcal{O}_X(-5) \to \wedge^8 \mathcal{S}_s^* \to \cdots \to \wedge^2 \mathcal{S}_s^* \to \mathcal{S}_s^* \to \mathcal{O}_X \to j_*(\mathcal{O}_{\mathscr{Q}}) \to 0.$$

It is now clear that, to prove (22) and (23) it suffices to prove, respectively, that:

$$(24) \qquad \wedge^p \mathcal{S}_s^*(t) \in \mathscr{D} \qquad\qquad \text{for } 0 \leq p \leq 9, \qquad \text{and for } 5 \leq t \leq 10,$$

$$(25) \qquad \wedge^p \mathcal{S}_s^* \otimes \mathcal{S}^*(7) \in \mathscr{D}, \qquad \text{for } 0 \leq p \leq 9,$$

Where the degree shift between (23) and (25) is explained by $\mathcal{S}^*(7) \simeq \mathcal{S}(6)$.

Let us first focus on (24). Taking exterior powers of (2), and of its dual, we get:

$$\wedge^p \mathcal{S}_s^* \in \langle \wedge^p \mathcal{S}^*, \wedge^{p-1} \mathcal{S}^*, \dots, \mathcal{O}_X \rangle.$$

Note also that $\wedge^p \mathcal{S}_s^* \simeq \wedge^{9-p} \mathcal{S}_s(-5)$. We deduce that (24) holds as soon as:

$$\wedge^p \mathcal{S}^*(t) \in \mathscr{D}, \quad \text{and} \quad \wedge^p \mathcal{S}^*(p+t-5) \in \mathscr{D}, \quad \text{for } 0 \leq p \leq 4 \text{ and } 5 \leq t \leq 10.$$

This last fact is clear for $p = 0, 1$. Moreover, it follows from Lemma 5 for $p = 2$, Lemma 6 for $p = 3$ and Lemma 7 for $p = 4$.

We now turn to the proof of (25). By the same argument as above, we reduce the statement to:

$$\wedge^p \mathcal{S}^*(7) \otimes \mathcal{S}^* \in \mathscr{D}, \quad \text{and} \quad \wedge^p \mathcal{S}^*(p+2) \otimes \mathcal{S}^* \in \mathscr{D}, \quad \text{for } 0 \leq p \leq 4.$$

This is clear for $p = 0$. For $p = 1$, it amounts to:

$$\wedge^2 \mathcal{S}^*(7), \mathcal{S}_2^*(7), \mathcal{O}_X(7), \wedge^2 \mathcal{S}^*(3), \mathcal{S}_2^*(3), \mathcal{O}_X(2) \in \mathscr{D},$$

which follows from Lemma 5 and Lemma 4. For $p = 2, 3, 4$, we can apply Corollary 9, and we are done. So, (22) and (23) are proved.

Let us now go back to the proof of the proposition. Consider the following subcategory of $\mathrm{D}^b(Q)$:

$$\mathscr{U} = \langle \mathcal{O}_Q(5), \mathcal{O}_Q(6), \Sigma(6), \mathcal{O}_Q(7), \mathcal{O}_Q(8), \mathcal{O}_Q(9), \mathcal{O}_Q(10) \rangle.$$

What we have to prove is that $\mathbf{L}i_Q^*(E)$ lies in $^\perp \mathscr{U}$. Indeed, by [Kap88], we have $\mathrm{D}^b(Q) = \langle \mathscr{U}, \mathcal{O}_Q(11) \rangle$, so that $^\perp \mathscr{U} = \langle \mathcal{O}_Q(11) \rangle$. Hence any object of $^\perp \mathscr{U}$ is a direct sum of shifts of $\mathcal{O}_Q(11)$.

Now note that, by (20), for any $t, q \in \mathbb{Z}$, we have:

$$\mathrm{Hom}_{\mathrm{D^b}(Q)}(\mathbf{L}i_Q^*(E), \mathcal{O}_Q(t)[q]) \simeq \mathrm{Hom}_{\mathrm{D^b}(\mathscr{Q})}(\mathbf{L}j^*(E), \iota_*(\mathcal{O}_Q(t))[q]) \subseteq$$
$$\subseteq \mathrm{Hom}_{\mathrm{D^b}(\mathscr{Q})}(\mathbf{L}j^*(E), \mathcal{O}_{\mathscr{Q}}(t)[q]) \simeq$$
$$\simeq \mathrm{Hom}_{\mathrm{D^b}(X)}(E, j_*(\mathcal{O}_{\mathscr{Q}}(t))[q]),$$

where we have used that $i_Q = j \circ \iota$ hence $\mathbf{L}\iota^* \circ \mathbf{L}j^* \simeq \mathbf{L}i_Q^*$, and that $\mathbf{L}\iota^*$ and $\mathbf{L}j^*$ are left adjoint to ι_* and j_* respectively. By the same reason, using (21) we get:

$$\mathrm{Hom}_{\mathrm{D^b}(Q)}(\mathbf{L}i_Q^*(E), \Sigma(t)[q]) \subseteq \mathrm{Hom}_{\mathrm{D^b}(X)}(E, j_*(\Sigma_{\mathscr{Q}}(t))[q]).$$

So, by (22) and (23), the assumption that E lies in $^\perp\mathscr{D}$ implies that $\mathbf{L}i_Q^*(E)$ lies in $^\perp\mathscr{U}$. This finishes the proof. $\qquad\square$

Next comes the last lemma that we need. We give here both our original argument that mimics the analogous result for bundle on the projective plane, cf. [OSS80, Theorem 3.2.1], and the simpler, more general argument, suggested by the referee.

Lemma 12. *Fix a point $x \in X$, and let E be an object of $\mathrm{D^b}(X)$ such that, for any 8-dimensional quadric $Q_8 \subseteq X$ through x, we have:*

$$\mathbf{L}i_{Q_8}^*(E) \simeq \bigoplus_k \mathcal{O}_{Q_8}^{r_k}[-k],$$

for some integers r_k. Then:

$$E \simeq \bigoplus_k \mathcal{O}_X^{r_k}[-k].$$

Proof. The argument follows the proof given in [OSS80] that any bundle on the projective plane, which is trivial on any line through a given point, must be trivial. This proof applies almost verbatim since the Cayley plane and its dual define a kind of plane projective geometry, and since the argument carries over to derived functors. Let:

$$\mathcal{B}_x = \{(y, \ell) \in X \times \mathbb{O}\check{\mathbb{P}}^2, \quad x, y \in Q_\ell\}.$$

The second projection $q : \mathcal{B}_x \to \mathbb{O}\check{\mathbb{P}}^2$ maps $\mathcal{B}_x$ to $\check{Q}_x$, and makes of $\mathcal{B}_x$ a locally trivial fibration in smooth 8-dimensional quadrics over $\check{Q}_x$. In particular, $\mathcal{B}_x$ is smooth of dimension 16. The first projection $p : \mathcal{B}_x \to X$ is birational since a general point of the Cayley plane belongs to a unique $\mathbb{O}$-line Q_ℓ passing through x. Recall also that q has a section ξ defined by $\xi(\ell) = (x, \ell)$ so $p \circ \xi$ is the constant map x, while of course $q \circ \xi$ is the identity.

$$\begin{array}{ccc} & \mathcal{B}_x & \\ {}^{p}\swarrow & {\Big\downarrow}{\scriptstyle q} & \nwarrow{}^{\xi} \\ X & & \check{Q}_x \end{array}$$

Just as in [OSS80], we would like to find a complex F in $\mathrm{D^b}(\check{Q}_x)$ such that:

$$(26) \qquad\qquad\qquad \mathbf{L}q^*(F) \simeq \mathbf{L}p^*(E).$$

Let us first check that this implies our statement. Note that, by (26), we have:

$$F \simeq \mathbf{L}(q \circ \xi)^*(F) \simeq \mathbf{L}\xi^*(\mathbf{L}q^*(F)) \simeq \mathbf{L}\xi^*(\mathbf{L}p^*(E)) \simeq \mathbf{L}(p \circ \xi)^*(E) \simeq \bigoplus_k \mathcal{O}_{\check{Q}_x}^{r_k}[-k],$$

where the last isomorphism holds since $p \circ \xi$ has constant value x, so $\mathbf{L}(p \circ \xi)^*(E)$ is obtained as $(p \circ \xi)^*(\mathbf{L}i_x^*(E))$, and clearly $\mathbf{L}i_x^*(E) \simeq \bigoplus_k \mathbb{C}_x^{r_k}[-k]$, by restriction from Q_8, since all the $\mathbf{L}^k i_{Q_8}^*(E)$ are free. Hence, we get:

$$\mathbf{L}p^*(E) \simeq \mathbf{L}q^*(F) \simeq \bigoplus_k \mathcal{O}_{\mathcal{B}_x}^{r_k}[-k],$$

and, in turn:

$$E \simeq \mathbf{R}p_*(\mathbf{L}p^*(E)) \simeq \bigoplus_k \mathbf{R}p_*(\mathcal{O}_{\mathcal{B}_x})^{r_k}[-k] \simeq \bigoplus_k \mathcal{O}_X^{r_k}[-k],$$

where the last isomorphism is clear, since p satisfies $\mathbf{R}p_*(\mathcal{O}_{\mathcal{B}_x}) \simeq \mathcal{O}_X$.

It remains to prove (26). We set:

$$F = \mathbf{R}q_*(\mathbf{L}p^*(E)),$$

and we have a natural morphism in $\mathrm{D}^{\mathrm{b}}(\mathcal{B}_x)$:

$$\varphi : \mathbf{L}q^*(F) = \mathbf{L}q^*(\mathbf{R}q_*(\mathbf{L}p^*(E))) \to \mathbf{L}p^*(E).$$

We would like to prove that, for any $\ell \in \check{Q}_x$, this morphism restricts to an isomorphism in the derived category $\mathrm{D}^{\mathrm{b}}(Q_\ell)$ of the fibre Q_ℓ of q over ℓ, and this will finish the proof since these fibres cover $\mathcal{B}_x$. Denote by α the embedding of $\{\ell\}$ into $\check{Q}_x$, by β the restriction of q to $Q_\ell \to \{\ell\}$ and by γ the embedding $Q_\ell \to \mathcal{B}_x$ so that $p \circ \gamma = i_{Q_\ell}$. We have to prove that $\mathbf{L}\gamma^*(\varphi)$ is an isomorphism. We write $\mathbf{L}\gamma^*(\varphi)$ as:

$$(27) \qquad \mathbf{L}\gamma^* \mathbf{L}q^* \mathbf{R}q_* \mathbf{L}p^* E \to \mathbf{L}\gamma^* \mathbf{L}p^* E.$$

On the right-hand-side, we have natural isomorphisms:

$$\mathbf{L}\gamma^* \mathbf{L}p^* E \simeq \mathbf{L}(p \circ \gamma)^* E \simeq \mathbf{L}i_{Q_\ell}^* E \simeq \bigoplus_k \mathcal{O}_{Q_\ell}^{r_k}[-k].$$

On the left-hand-side of (27), we have natural isomorphisms:

$$\mathbf{L}\gamma^* \mathbf{L}q^* \mathbf{R}q_* \mathbf{L}p^* E \simeq \mathbf{L}\beta^* \mathbf{L}\alpha^* \mathbf{R}q_* \mathbf{L}p^* E \simeq \mathbf{L}\beta^* \mathbf{R}\beta_* \mathbf{L}\gamma^* \mathbf{L}p^* E \simeq \mathbf{L}\beta^* \mathbf{R}\beta_* \mathbf{L}i_{Q_\ell}^* E,$$

where the middle one is given by smooth base-change. So $\mathbf{L}\gamma^*(\varphi)$ is the natural map:

$$\mathbf{L}\beta^* \mathbf{R}\beta_* \mathbf{L}i_{Q_\ell}^* E \to \mathbf{L}i_{Q_\ell}^* E,$$

which is clearly an isomorphism since $\mathbf{L}i_{Q_\ell}^* E \simeq \bigoplus_k \mathcal{O}_{Q_\ell}^{r_k}[-k]$. $\qquad\qquad \square$

Second proof of Lemma 12. The proof of (26) can be deduced from the following more general fact. Let Y and S be smooth projective varieties, and $q : Y \to S$ be a flat morphism satisfying $\mathbf{R}q_*(\mathcal{O}_Y) \simeq \mathcal{O}_S$. For $s \in S$, write $Y_s = q^{-1}(\{s\})$ and $i_s : Y_s \hookrightarrow Y$ the embedding of the fibre Y_s. Then, an object F of $\mathrm{D}^{\mathrm{b}}(Y)$ such that, for all $s \in S$, $\mathbf{L}i_s^*(F)$ lies in $\langle \mathcal{O}_{Y_s} \rangle$ is isomorphic to $\mathbf{L}q^*(G)$, for some $G \in \mathrm{D}^{\mathrm{b}}(S)$.

To prove this fact, consider the subcategory $\mathbf{L}q^*(\mathrm{D}^{\mathrm{b}}(S))^\perp$ of $\mathrm{D}^{\mathrm{b}}(Y)$. Since $\mathbf{R}q_*$ is right-adjoint to $\mathbf{L}q^*$, this category consists of objects $F \in \mathrm{D}^{\mathrm{b}}(Y)$ such that $\mathbf{R}q_*(F) = 0$, hence we write it as $\ker(\mathbf{R}q_*)$. We have thus a semiorthogonal decomposition:

$$\mathrm{D}^{\mathrm{b}}(Y) = \langle \ker(\mathbf{R}q_*), \mathbf{L}q^*(\mathrm{D}^{\mathrm{b}}(S)) \rangle.$$

This induces a distinguished triangle:

$$(28) \qquad\qquad\qquad F'' \to F \to F' \to F''[1],$$

with $\mathbf{R}q_*(F') = 0$, and $F'' \simeq \mathbf{L}q^*(G)$ for some $G \in \mathrm{D}^\mathrm{b}(S)$. We want to prove that $F' = 0$ under our assumptions. Assume $F' \neq 0$, the object F' must restrict non-trivially to some fibre of q, say $\mathbf{L}i_s^*(F') \neq 0$. Write $q_s = q|_{Y_s}$, set j_s for the embedding $\{s\} \hookrightarrow S$, and define $F_s = \mathbf{L}i_s^*(F')$. By flat base-change:

$$\mathbf{R}q_{s*}(F_s') \simeq \mathbf{L}j_s^*(\mathbf{R}q_*(F')) = 0,$$

so F_s' sits in $\ker(\mathbf{R}q_{s*})$. In other words, F_s' lies in the subcategory $\langle \mathcal{O}_{Y_s} \rangle^\perp$ of $\mathrm{D}^\mathrm{b}(Y_s)$, indeed $\mathbf{R}q_{s*}$ is just the derived global section functor $\mathbf{R}\operatorname{Hom}_{Y_s}(\mathcal{O}_{Y_s}, -)$.

Now, apply $\mathbf{L}i_s^*$ to (28) to obtain an exact triangle:

$$\mathbf{L}i_s^*(F'') \to \mathbf{L}i_s^*(F) \to F_s' \to \mathbf{L}i_s^*(F'')[1].$$

In turn, this is the distinguished triangle associated with $F_s = \mathbf{L}i_s^*(F)$ according to the semiorthogonal decomposition:

$$\mathrm{D}^\mathrm{b}(Y_s) = \langle \langle \mathcal{O}_{Y_s} \rangle^\perp, \mathcal{O}_{Y_s} \rangle.$$

So, F_s lies in $\langle \mathcal{O}_{Y_s} \rangle$ if and only if $F_s' = 0$, so our assumption contradicts $F_s' \neq 0$, and the lemma is proved. $\square$

4. Proof of the main result

Here we prove that our subcategory $\mathscr{D}$ generates the whole $\mathrm{D}^\mathrm{b}(X)$. We have to show that $^\perp\mathscr{D} = 0$. The idea, inspired on an argument appearing in [BO95], is to restrict E to the family of 8-dimensional quadrics in X through a given point. So let E be a complex of coherent sheaves on X, lying in $^\perp\mathscr{D} = 0$, and let us prove that $E = 0$.

Let x be a point of X such that, for all k, $\mathcal{H}^k(E)$ is locally free around x (taking x general in X will do). We have:

$$\mathbf{L}i_x^*(E) \simeq \bigoplus_k \mathbf{L}^k i_x^*(E)[-k] \simeq \bigoplus_k \mathcal{H}^k(E)|_x[-k] \simeq \bigoplus_k \mathbb{C}_x^{r_k}[-k],$$

for some integers r_k, with $r_k \neq 0$ for finitely many k's.

Let us first use 7-dimensional quadrics. So let Q_7 be a smooth 7-dimensional quadric contained in X and passing through x. In view of Proposition 11, $\mathbf{L}i_{Q_7}^*(E)$ is the direct sum of shifts of $\mathcal{O}_{Q_7}(11)$, so that:

$$\mathbf{L}i_{Q_7}^*(E(-11)) \simeq \bigoplus_k \mathbf{L}^k i_{Q_7}^*(E(-11))[-k] \simeq \bigoplus_k \mathcal{O}_{Q_7}^{t_k}[-k],$$

for some (finite) sequence of integers t_k. All of these cohomology sheaves are free, so restricting from Q_7 to x each of the $\mathbf{L}^k i_{Q_7}^*(E(-11))$ we must get $\mathbf{L}^k i_x^*(E(-11)) \simeq \mathbb{C}_x^{r_k}$. Hence $t_k = r_k$ for each k.

Further, note that, given another smooth 7-dimensional quadric $Q_7' \subseteq X$, meeting Q_7, we will have:

$$\mathbf{L}i_{Q_7'}^*(E(-11)) \simeq \bigoplus_k \mathbf{L}^k i_{Q_7'}^*(E(-11))[-k] \simeq \bigoplus_k \mathcal{O}_{Q_7'}^{r_k}[-k],$$

for the same sequence of integers r_k. Indeed, the rank of the restricted free sheaves to any point of $Q_7 \cap Q_7'$ must agree.

Now let us move to 8-dimensional quadrics. For any smooth such quadric $Q_8 \subseteq X$ containing x, we consider the (finitely many) non-zero sheaves $\mathbf{L}^k i_{Q_8}^*(E)$. We get finitely many proper subschemes of Q_8 as torsion loci of these sheaves, and we

denote by $\tau(\mathbf{L}^k i_{Q_8}^*(E))$ the torsion locus of each $\mathbf{L}^k i_{Q_8}^*(E)$. So, we may find a family of smooth hyperplane sections Q_7 of Q_8, covering the whole Q_8, such that, for any given k, $\tau(\mathbf{L}^k i_{Q_8}^*(E))$ does not contain Q_7. Let f be the linear form cutting Q_7 in Q_8. We have a long exact sequence:

$$\cdots \to \mathbf{L}^{k-1} i_{Q_7}^*(E) \to \mathbf{L}^k i_{Q_8}^*(E(-1)) \xrightarrow{f} \mathbf{L}^k i_{Q_8}^*(E) \to$$
$$\to \mathbf{L}^k i_{Q_7}^*(E) \to \mathbf{L}^{k+1} i_{Q_8}^*(E(-1)) \to \cdots$$

Since $\tau(\mathbf{L}^k i_{Q_8}^*(E)) \not\supset Q_7$ for all k, all the maps $f = \mathbf{L}^k i_{Q_8}^*(f)$ in the above sequence are injective. So $\mathbf{L}^k i_{Q_7}^*(E) \simeq (\mathbf{L}^k i_{Q_8}^*(E))|_{Q_7}$, and we may split the above long sequence into short exact sequences:

$$(29) \qquad 0 \to \mathbf{L}^k i_{Q_8}^*(E(-1)) \xrightarrow{f} \mathbf{L}^k i_{Q_8}^*(E) \to \mathbf{L}^k i_{Q_7}^*(E) \to 0.$$

Using Proposition 11, this means that, for any k, the sheaf $\mathbf{L}^k i_{Q_8}^*(E(-11))$ restricts to Q_7 as $\mathcal{O}_{Q_7}^{r_k}$.

We have a similar exact sequence for any other Q_7' in our covering family of hyperplane sections of Q_8. Since all these sections Q_7' obviously meet Q_7, using again (29) we get that, for any k, the sheaf $\mathbf{L}^k i_{Q_8}^*(E(-11))$ restricts to Q_7' as $\mathcal{O}_{Q_7'}^{r_k}$ (for the same integers r_k). In particular $\mathbf{L}^k i_{Q_8}^*(E(-11))$ is locally free of rank r_k. Lemma 10 now gives $\mathbf{L}^k i_{Q_8}^*(E(-11)) \simeq \mathcal{O}_{Q_8}^{r_k}$.

We have thus, for all k, an isomorphism $\mathbf{L}^k i_{Q_8}^*(E(-11)) \simeq \mathcal{O}_{Q_8}^{r_k}$. We would like to conclude that $E(-11)$ is itself isomorphic to $\bigoplus_k \mathcal{O}_X^{r_k}[-k]$. To achieve this, consider any other smooth 8-dimensional quadric Q_8' contained in X. By the same argument as above, we get that, given any k, the sheaf $\mathbf{L}^k i_{Q_8'}^*(E(-11))$ is free. The rank of this free sheaf must be r_k, since Q_8 and Q_8' meet at x, and restricting $\mathbf{L}^k i_{Q_8'}^*(E(-11))$ to x must give $\mathbf{L}^k i_x^*(E(-11)) \simeq \mathbb{C}^{r_k}$.

We are now in position to apply Lemma 12 to obtain an isomorphism $E(-11) \simeq \bigoplus_k \mathcal{O}_X^{r_k}[-k]$, so that $E \simeq \bigoplus_k \mathcal{O}_X^{r_k}(11)[-k]$. But E belongs to $^\perp \mathscr{D}$, in particular $\mathrm{Ext}_X^i(E, \mathcal{O}_X(11)) = 0$ for all i. Therefore $r_k = 0$ for all k, so $E = 0$. The proof is now complete.

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