Moduli space of homogeneous bundles on $\mathbb{P}^2$ with self–dual gradation

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October 31, 2013

1 Introduction

Let G be a semisimple, simply connected complex Lie group and P its parabolic subgroup. Let $X = \mathsf{G}/\mathsf{P}$ be a homogeneous variety. It is known that the category of $\mathsf{G}-$homogeneous bundles is equivalent to the category of representations of P (see [BK90]). Since group P is not reductive, to deal with its representations we consider the Levi decomposition of P, that is

$$\mathsf{P} = \mathsf{R} \cdot \mathsf{N},$$

where R is reductive and N is nilpotent. Let $\mathfrak{r} = \operatorname{Lie} \mathsf{R}, \mathfrak{n} = \operatorname{Lie} \mathsf{N}$ be corresponding Lie algebras. We say that X is a Hermitian symmetric variety if the representation of P, corresponding to tangent bundle of X, is trivial on N. This is equivalent to the condition $[\mathfrak{n}, \mathfrak{n}] = 0$ (see, for example [OR06, Chapter 2]). By Ottaviani and Rubei [OR06, Theorem 5.9], it turns out that for a Hermitian symmetric variety X the representations of P can be described in terms of representations of some (infinite) quiver $\mathcal{Q}_X$ with relations. Vertices of quiver $\mathcal{Q}_X$ correspond to irreducible representations of R, denoted v_λ. To any vertex v_λ associate an unique irreducible $\mathsf{G}-$homogeneous vector bundle E_λ (obtained by trivial extension of $\mathsf{R}-$module to $\mathsf{P}-$module). There is an arrow from $v_\lambda \to v_\mu$ in $\mathcal{Q}_X$ if the G-equivariant extension $\operatorname{Ext}^1_X(E_\lambda, E_\mu)^{\mathsf{G}}$ does not vanish. Relations between arrows will be made explicit later, for a particular case of $X = \mathbb{P}^2$ (see Definition (2.1) and [OR06, Definition 5.7] for general description). For any $\mathsf{G}-$homogeneous bundle E, define $\operatorname{gr} E$ to be an $\mathsf{R}-$module, obtained by restriction of $\mathsf{P}-$module given by E to R. It turns out that Mumford–Takemoto semistability of a G-homogeneous bundle can be reduced to $\theta-$semistability of corresponding quiver representation (cf. [OR06, Theorem 7.1]). Hence, by King construction [Kin94], it is possible to construct moduli spaces of semistable $\mathsf{G}-$homogeneous bundles E on X with fixed $\operatorname{gr} E$. In this chapter we will compute explicitly those moduli spaces for some homogeneous bundles on $\mathbb{P}^2$ with fixed self–dual gradation.

2 Basic definitions

From now on, let $\mathsf{G} = \mathsf{SL}(3, \mathbb{C})$ and let

$$\mathsf{P} = \{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathsf{SL}(3) \},$$

be a parabolic subgroup where $*$ denotes any complex number. Then

$$\mathsf{R} = \{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathsf{SL}(3) \},$$

$$\mathsf{N} = \{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathsf{SL}(3) \}.$$

Let $\mathcal{Q}_{\mathbb{P}^2}$ be the quiver with relations associated to $\mathbb{P}^2 = \mathsf{SL}(3)/\mathsf{P}$, as defined in [OR06, Definition 5.2]. The quiver $\mathcal{Q}_{\mathbb{P}^2}$ has 3 connected components (see [OR06, Theorem 5.12]), slopes of bundles attached to vertices in each component are congruent modulo $\frac{2}{3}$). Denote by $\mathcal{Q}$ one of the connected components of $\mathcal{Q}_{\mathbb{P}^2}$ containing vertex corresponding to bundle $\Omega_{\mathbb{P}^2}(2)$. Let $\mathcal{Q}^0$ (resp. $\mathcal{Q}^1$) be the set of vertices (resp. arrows) of $\mathcal{Q}$. The set $\mathcal{Q}^0$ is in one–to–one correspondence with the subset of the lattice $\mathbb{Z}^2$ given by points (i, j) with $i \geq j$. The point (i, j) represents a homogeneous vector bundle $E_{i,j}$, associated to a completely reducible representation of P. Setting $Q = \Omega_{\mathbb{P}^2}(2)$, we have $E_{i,j} \simeq \mathrm{S}^{i-j}\, Q(i + 2\,j) = \mathrm{S}^{i-j}\, Q \otimes \mathcal{O}_{\mathbb{P}^2}(i + 2\,j)$, where S denotes symmetric power. Thus we have

$$c_1(E_{i,j}) = \frac{3}{2}\,(i + j)\,(i - j + 1). \tag{2.1}$$

There is an arrow $E_{i,j} \to E_{h,k}$ if $\mathrm{Ext}^1(E_{i,j}, E_{h,k})^{\mathsf{SL}(3)}$ does not vanish. This implies that $\mathcal{Q}^1$ consists of arrows $(i, j) \to (i-1, j)$, denoted $v_{(i-1,j),(i,j)}$, for $i > j$ and $(i, j) \to (i, j-1)$, denoted $v_{(i,j-1),(i,j)}$, for $i \geq j$.

Let $\alpha \in \mathbb{N}^{\mathcal{Q}^0}$ be nonzero for a finite number of elements (i, j) in $\mathcal{Q}^0$. We call α a dimension vector. Define a homogeneous bundle $F(\alpha) = \oplus_{i,j} V_{i,j} \otimes E_{i,j}$, where $V_{i,j}$ is a complex vector space of dimension $\alpha_{i,j}$ and $E_{i,j}$ is defined above. Accordingly define $\mathsf{G}(\alpha) = \prod_{i,j} \mathsf{GL}(V_{i,j})$. A character $\chi_\theta : \mathsf{G}(\alpha) \to \mathbb{C}^*$ is defined for any $g = (g_{i,j}) \in \mathsf{G}$ by $\chi_\theta(g_{i,j}) = \prod_{i,j} \det(g_{i,j})^{\theta_{i,j}}$, with $\theta_{i,j} \in \mathbb{Z}$.

Following [Kin94] we choose the function $\theta \in \mathbb{Z}^{\mathcal{Q}^0}$ as follows:

$$\theta_{i,j} = \mathrm{rk}(F(\alpha))\, c_1(E_{i,j}) - c_1(F(\alpha))\, \mathrm{rk}(E_{i,j}). \tag{2.2}$$

With this choice, $\theta-$semistability of representations according to King is equivalent to slope semistability according to Mumford and Takemoto (see [OR06, Theorem 7.1 and 7.2]). We will always assume θ to be of this form.

Definition 2.1. *[Moduli space of homogeneous bundles] Define the space* $K(\mathcal{Q}, V(\alpha)) = \bigoplus_{a\in\mathcal{Q}^1} \mathrm{Hom}(V_{\mathsf{t}\,a}, V_{\mathsf{h}\,a})$ *where* $\mathsf{t}\,a \in \mathcal{Q}^0$ *(resp.* $\mathsf{h}\,a \in \mathcal{Q}^0$*) denotes the tail (resp. head) of* $a \in \mathcal{Q}^1$*, and consider the closed subset* $V(\alpha) \subset K(\mathcal{Q}, V(\alpha))$ *given by the following relations in* $\mathcal{Q}$

$$v_{(i-1,j-1),(i,j-1)}v_{(i,j-1),(i,j)} - v_{(i-1,j-1),(i-1,j)}v_{(i-1,j),(i,j)}, \quad \textit{for } j < i,$$

$$v_{(i-1,i-1),(i,i-1)}v_{(i,i-1),(i,i)}, \quad \textit{for any } i \in \mathbb{N}.$$

(see [OR06, Section 7]). The group $\mathsf{G}(\alpha)$ *acts on* $V(\alpha)$ *and we define the space of semi–invariants of degree* m *as*

$$\begin{aligned} A_m(\alpha) =&\mathbb{C}[V(\alpha)]^{\mathsf{G}(\alpha),\chi_{m\theta}} = \\ =&\{f \in \mathbb{C}[V(\alpha)] \mid f(g\,x) = \chi_\theta(g)^m f(x), \forall x \in V(\alpha)\} \end{aligned}$$

Given a dimension vector α*, define the associated moduli space*

$$M_\theta(\mathcal{Q}, \alpha) := \mathrm{Proj}(\bigoplus_m A_m(\alpha))$$

Remark 2.2. *There are two, equivalent choices of* $\mathsf{G}(\alpha)-$*action on* $V(\alpha)$*. Note that Ottaviani and Rubei in [OR06] use (implicitly) the following convention: if* $a \in \mathcal{Q}^1, f \in Hom(V_{\mathsf{t}\,a}, V_{\mathsf{h}\,a})$ *and* $g = (g_{i,j}) \in \mathsf{G}(\alpha)$*, then*

$$g \cdot f := g_{\mathsf{h}a}^{-1} f g_{\mathsf{t}a}.$$

With this choice, the direction of inequality in definition of King's $\theta-$*(semi)stability is as follows: a representations* V *is* $\theta-$*(semi)stable if, and only if, for every proper subrepresentation* W *of* V

$$\theta(W) < (\textit{resp. } \leq)\ 0 = \theta(V).$$

Following theorem is a main result of [OR06]:

Theorem 2.3 (Ottaviani, Rubei, King)**.** *For* θ *chosen as above the projective variety* $M_\theta(\mathcal{Q}, \alpha)$ *is a coarse moduli space parameterizing* $S-$*equivalence classes of semistable homogeneous bundles* F *over* $\mathbb{P}^2$ *such that the completely reducible bundle associated to* F *(i.e.* $\mathrm{gr}\, F$*) is isomorphic to* $F(\alpha)$*.*

Proof. See [OR06, Section 7] and [Kin94, Proposition 5.2].

We say that two semistable bundles are $S-$equivalent if they have the same composition factors in Jordan-Hölder filtration (see [Kin94, §3] and [LP97, p. 76]). Note that Mumford–Takemoto stability is a stronger condition than $\theta-$stability (see [OR06, Theorem 7.2]).

Definition 2.4. *A vector bundle* F *on* $\mathbb{P}^2$ *is called self–dual if* $F^\vee \simeq F$*.*

Notice that the support of the representation of $\mathcal{Q}$ (i.e. those pairs (i,j), such that $\alpha_{i,j}$ does not vanish) corresponding to a homogeneous self–dual bundle is a symmetric set along the direction $i+j=0$ (that is $\alpha_{i,j} = \alpha_{-j,-i}$). Since a self–dual bundle F satisfies $c_1(F) = 0$, by formulas (2.1) and (2.2) we deduce the following straightforward lemma.

Lemma 2.5. *For a self–dual homogeneous vector bundle F with dimension vector α, the character coefficients of $\theta \in \mathbb{Z}^{\mathcal{Q}^0}$ corresponding to slope semistability are (up to a common positive, rational scalar) given by*

$$\theta_{i,j} = (i-j+1)(i+j). \tag{2.3}$$

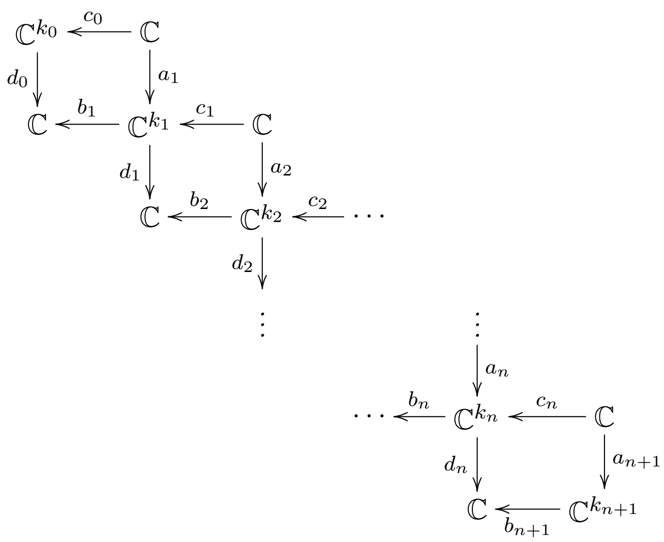
Proof. See [OR05, Remark 27.III].

The formula (2.3) says that the coefficient $\theta_{i,j}$ is $\mathrm{rk}(E_{i,j}) = i - j + 1$, times the distance from the diagonal $i + j = 0$. If c is a positive, rational scalar, then stability parameters θ and $c\,\theta$ give isomorphic moduli spaces (see [Har77, Ex. II.5.13]).

Definition 2.6. *For $n \geq 1$ and for any sequence of natural numbers $k_0, \dots, k_{n+1}$ with $k_0 \geq 1$, $k_{n+1} \geq 1$, and $k_i \geq 2$ for $2 \leq i \leq n$, define the dimension vector α_n as*

$$(\alpha_n)_{i,j} = \begin{cases} k_i & i = -j \text{ for } 0 \leq i \leq n+1, \\ 1 & i = -j+1, 1 \leq i \leq n+1, \\ 1 & i = -j-1, 0 \leq i \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

according to the diagram, drawn as a plane with coordinates (i,j)



(2.4)

where the main diagonal sits on the line $i = -j$. We call the vertex $(0,0)$, corresponding to k_0 (i.e. the upper left corner) the origin *of α_n. Matrix representing arrow $v_{(i,-i)(i,-i+1)}$ (resp. $v_{((i-1,-i)(i,-i)}, v_{(i,-i)(i+1,-i)}, v_{(i,-i-1)(i,-i)}$) is denoted by a_i (resp. b_i, c_i, d_i).*

We choose an interesting, however special, dimension vector α_n of a particular self–dual bundle. Note that a general element in $M_\theta(\mathcal{Q}, \alpha_n)$ will not correspond to a self–dual bundle as we do not impose additional relation on matrices representing arrows in $\mathcal{Q}$. It will turn out that the moduli space depends on n but not on $k_0, \dots, k_{n+1}$.

Notation 2.7. *By*

$$\alpha_n = (k_0, 1, 1, k_1, 1, 1, k_2, 1, \dots, 1, k_n, 1, 1, k_{n+1}),$$

we mean dimension vector α_n as defined above. That is, we order coefficients first in rows, then in columns, i.e.

$$\alpha_n = (\alpha_{0,0}, \alpha_{1,0}, \alpha_{0,-1}, \alpha_{1,-1}, \alpha_{2,-1}, \alpha_{1,-2}, \alpha_{2,-2}, \dots,$$
$$\dots, \alpha_{n,-n}, \alpha_{n+1,-n}, \alpha_{n,-n-1}, \alpha_{n+1,-n-1}).$$

Accordingly, for fixed n, we will write in analogous way the corresponding stability parameter θ, that is

$$\theta = (\theta_{0,0}, \theta_{1,0}, \theta_{0,-1}, \theta_{1,-1}, \theta_{2,-1}, \theta_{1,-2}, \theta_{2,-2}, \dots,$$
$$\dots, \theta_{n,-n}, \theta_{n+1,-n}, \theta_{n,-n-1}, \theta_{n+1,-n-1}).$$

Theorem 2.8. *The moduli space $M_\theta(\mathcal{Q}, \alpha_1)$ is isomorphic to $\mathbb{P}^1$ and $M_\theta(\mathcal{Q}, \alpha_n)$ is isomorphic to a $\mathbb{P}^1$-bundle over $M_\theta(\mathcal{Q}, \alpha_{n-1})$ (see Lemma (3.7) for explicit definition of the bundle structure). Moreover, $M_\theta(\mathcal{Q}, \alpha_n)$ is an irreducible smooth projective toric variety of dimension n (see Lemma (3.6) for precise description of the corresponding fan) and does not depend on $k_0, \dots, k_{n+1}$.*

The rest of the chapter is devoted to the proof of the above result.

Moving the origin of dimension vector α_n along the line $i = -j$ leaves the moduli space invariant but changes natural polarization (see Remark (3.9)). It is expected that any moduli space of homogeneous bundles on $\mathbb{P}^2$ with fixed $\operatorname{gr} F$ (for not necessarily self–dual F) is a (possibly non–normal) toric variety.

3 Semi–invariants via toric picture

For $n \geq 1$ consider α_n as above and fix a complex vector space $W = V_{j,-j}$ for some j, $1 \leq j \leq n$, such that $\dim(W) = k_j > 0$. By Lemma (2.5), we can write the value of θ at the vector spaces $V_{i,j}$ associated to the dimension vector α_n

$$\theta = (0, 1, -1, 0, 2, -2, \dots, 0, n+1, -n-1, 0) \tag{3.1}$$

Chosen stability parameter θ takes value 0 at the point corresponding to W, therefore we look for semi–invariants of weight zero at W i.e. for invariants under the $\mathsf{GL}(W) \subset \mathsf{GL}(\alpha)$ action. For convenience rename a_j, b_j, c_j and d_j of diagram (2.4) as a, b, c, d in order to draw the following picture

$$\begin{array}{ccccc} & & V_{\mathsf{t}\,a} & & \\ & & \downarrow{\scriptstyle a} & & \\ V_{\mathsf{h}\,b} & \xleftarrow{\;b\;} & W & \xleftarrow{\;c\;} & V_{\mathsf{t}\,c} \\ & & \downarrow{\scriptstyle d} & & \\ & & V_{\mathsf{h}\,d} & & \end{array}$$

Following theorem is a particular case of a classical result known already by Herman Weyl (see [Wey39], [dCP76] or [Has05] for alternative proofs).

Lemma 3.1. *Let f be a regular function on $V(\alpha_n)$ satisfying $f(g\,x) = f(x)$ for all $g \in \mathsf{GL}(W)$. Then f is a linear combination of forms of the type $f'\,h$ where h is a function independent of a, b, c, d and f' lies in the ring $\mathbb{C}[b\,a, d\,a, b\,c, d\,c]$.*

Proof. Write $K(\mathcal{Q}, V(\alpha_n)) = K^W(\alpha_n) \oplus K'(\alpha_n)$, where no arrow in $K'(\alpha_n)$ has tail or head in W and

$$K^W(\alpha_n) = \mathrm{Hom}(V_{\mathsf{t}\,a}, W) \oplus \mathrm{Hom}(V_{\mathsf{t}\,c}, W) \oplus \mathrm{Hom}(W, V_{\mathsf{h}\,b}) \oplus \mathrm{Hom}(W, V_{\mathsf{h}\,d})$$

A regular function f is represented by an element of the polynomial algebra $\mathrm{Sym}(K(\mathcal{Q}, V(\alpha_n))) \simeq \mathrm{Sym}(K^W(\alpha_n)) \otimes \mathrm{Sym}(K'(\alpha_n))$. Clearly any element in $\mathrm{Sym}(K'(\alpha_n))$ is independent of a, b, c, d. Put $S(W) := \mathrm{Sym}(K^W(\alpha_n))$. We need to identify the $\mathsf{GL}(W)$-invariant elements in $S(W)$.

Decomposing $S(W)$ with respect to $\mathsf{GL}(W)$-action we find

$$S(W) \simeq \bigoplus_{\eta,\xi} \mathrm{S}^\eta\, W \otimes \mathrm{S}^\xi\, W^\vee \otimes (\mathrm{S}^\eta(V_{\mathsf{t}\,a}^\vee \oplus V_{\mathsf{t}\,c}^\vee) \otimes \mathrm{S}^\xi(V_{\mathsf{h}\,b} \oplus V_{\mathsf{h}\,d}))$$

where η and ξ run through Young tableaux with at most $\dim(W)$ rows and S^η, S^ξ are the associated Schur functors.

Let $\mathsf{T}_{a,c} := V_{\mathsf{t}\,a}^\vee \oplus V_{\mathsf{t}\,c}^\vee$ and $\mathsf{H}_{b,d} := V_{\mathsf{h}\,b} \oplus V_{\mathsf{h}\,d}$. The spaces $\mathsf{T}_{a,c}$ and $\mathsf{H}_{b,d}$ are 2-dimensional, so η and ξ run through Young tableaux with at most 2 rows, respectively of length $p + q$ and p. Since we look for $\mathsf{GL}(W)$-invariant elements, Schur Lemma gives $\eta = \xi$ and we get

$$S(W)^{\mathsf{GL}(W)} \simeq \bigoplus_{p,q} \wedge^2(\mathsf{T}_{a,c})^{\otimes p} \otimes \wedge^2(\mathsf{H}_{b,d})^{\otimes p} \otimes \mathrm{S}^q(\mathsf{T}_{a,c}) \otimes \mathrm{S}^q(\mathsf{H}_{b,d})$$

Now the unique element in $\wedge^2(\mathsf{T}_{a,c})^{\otimes p} \otimes \wedge^2(\mathsf{H}_{b,d})^{\otimes p}$ represents the p-th power of the determinant D of the matrix

$$\mathsf{T}_{a,c} \xrightarrow{\begin{pmatrix} b\,a & b\,c \\ d\,a & d\,c \end{pmatrix}} \mathsf{H}_{b,d}$$

On other hand $\bigoplus_q \mathrm{S}^q(\mathsf{T}_{a,c})\otimes \mathrm{S}^q(\mathsf{H}_{b,d})$ can be identified with the subspace of degree 0 elements in the subring $\mathrm{Sym}(\mathsf{T}_{a,c}) \otimes \mathrm{Sym}(\mathsf{H}_{b,d})$, with gradation given by $\deg(a) = \deg(c) = -1$, $\deg(b) = \deg(d) = 1$.

It is immediate to check that $b\,a, d\,a, b\,c, d\,c$ have degree zero and that they generate this subring. Finally, we observe that $D = b\,a \cdot d\,c - b\,c \cdot d\,a$, so the generator D is actually redundant.

For a quiver representation as in diagram (2.4) with dimension vector α_n, we set

$$x_i = b_i\,c_i \cdot d_i\,a_i \qquad \text{for } i = 1 \dots n, \tag{3.2}$$
$$y_i = b_i\,a_i = d_{i-1}\,c_{i-1} \qquad \text{for } i = 1 \dots n+1, \tag{3.3}$$

where x_i, y_i are seen as regular functions on $V(\alpha_n)$.

Lemma 3.2. *Let α_n be as in Definition (2.6) and let θ denote stability condition as in expression (3.1). Then, there exists a $\mathbb{C}-$basis of the vector space $A_m(\alpha_n)$, independent of $k_0, \dots, k_{n+1}$, given by the elements*

$$x_1^{e_1} \cdot x_2^{e_2} \cdots x_n^{e_n} \cdot y_1^{m-e_1} \cdot y_2^{2m-e_1-e_2} \cdots y_{n+1}^{(n+1)m-e_n} \tag{3.4}$$

with the conditions

$$\begin{cases} 0 & \le e_1 & \le \ \ m, \\ 0 & \le e_1 + e_2 & \le 2m, \\ & \vdots & \\ 0 & \le e_{n-1} + e_n & \le nm, \\ 0 & \le e_i & \end{cases} \tag{3.5}$$

Proof. Let f be an element of $A_m(\alpha_n)$. By the choice of stability parameter θ (see equation (3.1)), f is an invariant element for the $\mathsf{GL}(V_{i,-i})$-action for every i (as $\theta_{i,-i} = 0$). By Lemma (3.1) and simple induction, any f as above is a linear combinations of functions being products of $d_0\,c_0$, $b_i\,a_i$, $d_i\,a_i$, $b_i\,c_i$, $d_i\,c_i$ and $b_{n+1}\,a_{n+1}$ where i goes through $1, \dots, n$. Fix some monomial $u \in A_m(\alpha_n)$ of this form, that is

$$u = (b_{n+1}\,a_{n+1})^{p_{n+1}} \prod_{i=1}^{n} (b_i\,a_i)^{p_i} \cdot (b_i\,c_i)^{q_i} \cdot (d_i\,a_i)^{t_i} \tag{3.6}$$

where, by the commutativity relations in $V(\alpha_n)$ (see Definition (2.1)), any occurrence of $d_i\,c_i$ was replaced by $b_{i+1}\,a_{i+1}$ for $i = 0, \dots, n$ and p_i, q_i, t_i are some non–negative integers. This will alow to express relations between p_i, q_i, t_i equivalent to the fact that u is a semi–invariant of degree m (i.e. $u \in A_m(\alpha_n)$).

By Remark (2.2), an element g of the group $\mathsf{G}(\alpha)$ acts on $b_i\,a_i$, $d_i\,a_i$, $b_i\,c_i$ as follows:

$$g \cdot (b_i\,a_i) = g_{\mathsf{hb}_i}^{-1} g_{\mathsf{ta}_i} (b_i\,a_i), \tag{3.7}$$

$$g \cdot (d_i\, a_i) = g^{-1}_{\mathsf{h}d_i} g_{\mathsf{t}a_i} (d_i\, a_i), \tag{3.8}$$

$$g \cdot (b_i\, c_i) = g^{-1}_{\mathsf{h}b_i} g_{\mathsf{t}c_i} (b_i\, c_i), \tag{3.9}$$

where $g_{i,j} \in \mathbb{C}^*$ for $(i,j) = \mathsf{h}b_i, \mathsf{h}d_i, \mathsf{t}a_i, \mathsf{t}d_i$. Therefore, function u belongs to $A_m(\alpha_n)$ if and only if the following system of linear equations is satisfied:

$$\left\{ \begin{array}{lllcll} p_1 & & +t_1 & = & m, & \\ p_i & +q_{i-1} & +t_i & = & mi, & \text{for } i = 2, \dots, n, \\ p_{n+1} & +q_n & & = & m(n+1), & \\ -p_1 & -q_1 & & = & -m, & \\ -p_i & -q_i & -t_{i-1} & = & -mi, & \text{for } i = 2, \dots, n, \\ -p_{n+1} & & -t_n & = & -m(n+1), & \end{array} \right. \tag{3.10}$$

where the first three equations count weights in vertices $\mathsf{t}a_i = \mathsf{t}c_{i-1} = (i, -i+1)$ and the three last in vertices $\mathsf{h}b_i = \mathsf{h}d_{i-1} = (i-1, -i)$ of quiver $\mathcal{Q}$.

First and fourth equations imply that $q_1 = t_1$. By second and fifth equations we got $q_i - q_{i-1} = t_i - t_{i-1}$, for $i = 2, \dots, n$. This implies that $q_i = t_i$ for $i = 1, \dots, n+1$ (with third and sixth equations). Setting $e_i = q_i = t_i$ the system of linear equations reduces to the form:

$$\left\{ \begin{array}{lcll} p_1 & = & m - e_1, & \\ p_i & = & mi - e_{i-1} - e_i, & \text{for } i = 2, \dots, n, \\ p_{n+1} & = & m(n+1) - e_n, & \end{array} \right. \tag{3.11}$$

We need to restrict to non–negative p_i and e_i, therefore we can write function u in the form $x_1^{e_1} \cdot x_2^{e_2} \cdots x_n^{e_n} \cdot y_1^{m-e_1} \cdot y_2^{2m-e_1-e_2} \cdots y_{n+1}^{(n+1)m-e_n}$, where e_i satisfy conditions:

$$\left\{ \begin{array}{llll} 0 & \leq e_1 & \leq m, & \\ 0 & \leq e_{i-1} + e_i & \leq mi, & \text{for } i = 2, \dots, n \\ 0 & \leq e_i & & \text{for } i = 1, \dots, n. \end{array} \right. \tag{3.12}$$

Note that inequality $e_n \leq m(n+1)$ is redundant and functions u of the form (3.6), satisfying conditions (3.12),(3.11), are a $\mathbb{C}-$basis of $A_m(\alpha_n)$ for any $m \geq 1$.

The conditions (3.5) of Lemma (3.2) cut a rational convex polyhedron in $M_n \otimes \mathbb{R}$, where $M_n := \mathbb{Z}^n$ is a lattice. For $m = 1$, denote this polyhedron by $\Delta(n)$. Note that by the above lemma we can identify $\mathbb{C}-$basis of $A_m(\alpha)$ with elements of $m\Delta(n) \cap M_n$. Moreover, polyhedron $\Delta(n)$ is compact (coordinates of points in $\Delta(n)$ are non–negative and not greater that n), which implies that $A_0(\alpha_n) = \mathbb{C}$.

Lemma 3.3. *The ring $\bigoplus_m A_m(\alpha_n)$ is generated by $A_1(\alpha_n)$.*

Proof. To proof the statement of lemma it is enough to show that any $n-$tuple $(e_1, \ldots, e_n) \in m\Delta(n) \cap M_n$ for any m is a sum of m $n-$tuples $(e_1^i, \ldots, e_n^i) \in \Delta(n) \cap M_n$, where $i = 1, \ldots, m$.

The proof of this fact goes by induction on n. Fix $n = 2$ and a $2-$tuple (e_1, e_2) satisfying (3.5), for any $m > 1$. Any number, not greater than $2m-e_1$, can be written in the form $2k + l$, where $0 \leq k \leq m - e_1$ and $0 \leq l \leq e_1$. Therefore

$$(e_1, e_2) = (m - e_1 - k)(0, 0) + k(0, 2) + (e_1 - l)(1, 0) + l(1, 1).$$

Fix $n > 2$ and let $n-$tuple $(e_1, \ldots, e_n)$ satisfy (3.5) for fixed m. By inductive step there are m $(n-1)-$tuples $(e_1^i, \ldots, e_{n-1}^i) \in \Delta(n-1)$, such that

$$(e_1, \ldots, e_{n-1}) = \sum_{i=1}^{m} (e_1^i, \ldots, e_{n-1}^i).$$

It is trivial that any number not greater that $mn - e_{n-1}$ can be expressed as a sum of m non–negative numbers $e_n^1, \ldots, e_n^m$ such that

$$e_n^i \leq n - e_{n-1}^i \quad \text{for} \quad i = 1, \ldots, m,$$

where

$$\sum_{i=1}^{m} e_{n-1}^i = e_{n-1}.$$

Note that $e_n \leq mn - e_{n-1}$, which finishes the proof.

Remark 3.4. *We say that polyhedron Δ in lattice M is normal if*

$$m\Delta = \Delta + \ldots + \Delta \ (m \text{ times})$$

for any natural m, where $+$ denotes Minkowski sum. A result from [BGT97] says that polyhedron Δ, which has an unimodular covering (i.e. covering by simplices with vertices in lattice M, having normalized volume equal to 1) is normal. Using this fact we could give an alternate proof that polyhedron $\Delta(n) \cap M_n$ is normal (that is $A_1(\alpha_n)$ generates $A_m(\alpha_m)$, cf. Lemma (3.3)).

Let $F(n)$ be the inner normal fan of polyhedron $\Delta(n)$ (cf. [Ful93, page 26]) in $N_n \otimes \mathbb{R}$, where $N_n := \mathrm{Hom}_{\mathbb{Z}}(M_n, \mathbb{Z}^n) \simeq \mathbb{Z}^n$ is a dual lattice to M_n. We identify N_n with a sublattice of N_{n+1} via function

$$\iota_n : N_n \longrightarrow N_{n+1},$$

given by natural inclusion $\mathbb{Z}^n \subset \mathbb{Z}^n \times \{0\}$. Let $\widehat{\iota}_n$ denote dual function to ι_n, that is projection onto first n coordinates

$$\widehat{\iota}_n : M_{n+1} \to M_n.$$

Function $\widehat{\iota}_n$ allows to identify M_n as a sublattice of M_{n+1} via $\mathbb{Z}-$linear function η_n

$$\eta_n : M_n \longrightarrow M_{n+1},$$

such that

$$M_{n+1} = \eta_n(M_n) \oplus \ker \widehat{\iota}_n.$$

We will omit $\iota_n, \widehat{\iota}_n$ and η_n where no confusion is possible.

Lemma 3.5. *The primitive vectors along rays of fan $F(n)$ (in the lattice N_n) are as follows:*

$$\begin{cases} r_1 = (1,0,0,\dots,0,0), & s_1 = (-1,\ \ 0,\ \ 0,\dots,0,\ \ 0,\ \ 0),\\ r_2 = (0,1,0,\dots,0,0), & s_2 = (-1,-1,\ \ 0,\dots,0,\ \ 0,\ \ 0),\\ r_3 = (0,0,1,\dots,0,0), & s_2 = (\ \ 0,-1,-1,\dots,0,\ \ 0,\ \ 0),\\ \quad\vdots & \quad\vdots \\ r_n = (0,0,0,\dots,0,1), & s_n = (\ \ 0,\ \ 0,\ \ 0,\dots,0,-1,-1). \end{cases} \tag{3.13}$$

The $n-$dimensional cones in fan $F(n)$ are generated by cones of $\iota_{n-1}(F(n-1))$ and ray r_n (resp. s_n). Moreover, fan $F(n)$ describes smooth toric variety.

Proof. The rays of the fan $F(n)$ correspond to inner normal vectors to facets of $\Delta(n)$. Therefore, up to the sign, they correspond to coefficients of inequalities describing $\Delta(n)$. This proves the first part of the theorem. The proof of second part goes by induction. Fan $F(2)$ consists of rays $r_1 = (1,0), r_2 = (0,1), s_1 = (-1,0), s_2 = (-1,-1)$ and describes a blow–up of $\mathbb{P}^2$, which is a smooth variety. The description of $\Delta(n)$ by the set of inequalities (3.5) implies that facets of $\Delta(n)$ are as follows:

(1) facet $\eta_{n-1}(\Delta(n-1))$, with inner normal vector equal to r_n,

(2) facet corresponding to inequality $-e_{n-1} - e_n \geq -n$, with inner normal vector equal to s_n,

(3) facets whose supporting functions are equal to images of supporting functions of facets of $\Delta(n-1)$ via ι_{n-1}.

Facets from point (3) have the same inner normal vectors as facets of $\Delta(n-1)$ (seen via ι_{n-1}). Therefore, the $n-$dimensional cones of fan $F(n)$, corresponding to vertices of $\Delta(n)$, are spanned by images of $(n-1)-$dimensional cones of $F(n-1)$ via ι_{n-1} and by rays $\mathbb{R}_+ r_n$ or $\mathbb{R}_+ s_n$. Moreover, any $\mathbb{Z}-$basis of $\iota_{n-1}(N_{n-1})$ is completed to a $\mathbb{Z}-$basis of N_n by adding vector r_n (resp. s_n). Therefore variety described by fan $F(n)$ in lattice N_n is smooth.

Using previous lemma we can give even more concrete description of fan $F(n)$.

Lemma 3.6. *The $n-$dimensional cones σ of fan $F(n)$ are precise of the form:*

$$\sigma = \mathbb{R}_+ t_1 + \mathbb{R}_+ t_2 + \ldots + \mathbb{R}_+ t_n,$$

where $t_i = r_i$ or $t_i = s_i$ for $i = 1, \ldots, n$. In particular, there are 2^n of them.

Proof. The cones in $F(2)$ are $\mathbb{R}_+ r_1 + \mathbb{R}_2 r_2$, $\mathbb{R}_+ r_1 + \mathbb{R}_2 s_2$, $\mathbb{R}_+ s_1 + \mathbb{R}_2 r_2$, $\mathbb{R}_+ s_1 + \mathbb{R}_2 s_2$. By Lemma (3.5) $n-$dimensional cones in $F(n)$ are generated by $(n-1)-$dimensional cones of $F(n-1)$ and either $\mathbb{R}_+ r_n$ or $\mathbb{R}_+ s_n$. The proof follows by induction.

The total space of a projective bundle over a toric variety is again a toric variety. The following description of $\mathbb{P}^1-$bundles over toric varieties can be found in general form in [Oda88, p. 58–59]. Let L, L' be two line bundles on toric variety given by fan F in $N \otimes \mathbb{R}$, where N is a lattice. Denote by h and h' piecewise linear functions on support of F corresponding to L and L', respectively. Denote by $N_{\mathbb{P}}$ lattice $N_{\mathbb{P}} := \mathbb{Z}e \oplus N$ and define an $\mathbb{R}-$linear function $g : N \otimes \mathbb{R} \longrightarrow N_{\mathbb{P}} \otimes \mathbb{R}$ by setting $g(v) := ((-h(v) + h'(v))e, v) \in N_{\mathbb{P}}$ for any $v \in N \otimes \mathbb{R}$. Then fan of the total space of the bundle $\mathbb{P}(\mathcal{O}(L) \oplus \mathcal{O}(L'))$ is given in $N_{\mathbb{P}} \otimes \mathbb{R}$ by sums of the images of cones in fan F via g and rays $\mathbb{R}_+ e$ or $\mathbb{R}_+(-e)$.

Lemma 3.7. *The fan $F(n)$, corresponding to $\Delta(n)$ (as defined in Lemma (3.5)), is associated to the toric variety P_n recursively defined by*

$$P_n = \mathbb{P}(\mathcal{O}_{P_{n-1}} \oplus \mathcal{O}_{P_{n-1}}(S_1)) \to P_{n-1}$$
$$P_0 = \{\text{pt}\},$$

where S_1 is a closure of codimension 1 orbit corresponding to ray $\mathbb{R}_+ s_1$. The variety P_n is naturally equipped with the very ample divisor

$$L_n = \sum_{i=1}^{n} i S_i,$$

where S_i for $i = 2, \ldots, n$ is defined as S_1 above (see Lemma (3.5)).

Proof. The coefficients of the very ample divisor follow easily from the inequalities describing the polyhedron $\Delta(n)$ (see [Ful93, p. 66]).

Set $N_{n,\mathbb{P}} := \mathbb{Z}e \oplus N_{n-1}$. We will construct fan of $\mathbb{P}(\mathcal{O}_{P_{n-1}} \oplus \mathcal{O}_{P_{n-1}}(S_1))$ in $N_{n,\mathbb{P}} \otimes \mathbb{R}$ and show that it is isomorphic to the fan $F(n)$ in $N_n \otimes \mathbb{R}$. Linear function corresponding to trivial line bundle on P_{n-1} vanishes on fan $F(n-1)$ and the linear function corresponding to line bundle $\mathcal{O}(S_1)$ admits value -1 on point s_1 and vanishes on all other primitive points along rays of $F(n-1)$. Define isomorphism $j : N_{n,\mathbb{P}} \longrightarrow N_n$ of lattices $N_{n,\mathbb{P}}$ and N_n by setting $j(e, 0) = r_1$ and $j(0, r_i) = r_{i+1}$, for $i = 1, \ldots, n-1$. Following construction of $\mathbb{P}^1-$bundles given above we see that

$$(j_{\mathbb{R}} \circ g)(r_i) = j_{\mathbb{R}}(0, r_i) = r_{i+1},$$

$$(j_{\mathbb{R}} \circ g)(s_i) = j_{\mathbb{R}}(-e, s_1) = s_2,$$
$$(j_{\mathbb{R}} \circ g)(s_i) = j_{\mathbb{R}}(0, s_i) = s_{i+1}, \quad \text{for } i > 1,$$

where $j_{\mathbb{R}} = j \otimes \mathbb{R}$ and the vectors r_i, s_i are defined in Lemma (3.5).

Moreover $j_{\mathbb{R}}(e, 0) = r_1$ and $j_{\mathbb{R}}(-e, 0) = s_1$. By Lemma (3.6) $\mathbb{R}-$linear function $j_{\mathbb{R}}$ induces isomorphism of fan of $\mathbb{P}(\mathcal{O}_{P_{n-1}} \oplus \mathcal{O}_{P_{n-1}}(S_1))$ with the fan $F(n)$. Namely, denote by $t^1, \ldots, t^{n-1}$ sequence of $n-1$ symbols, where $t^i = r$ or $t^i = s$. Then $n-$dimensional cone $\mathbb{R}_+ e + \mathbb{R}_+ t^1_1 + \ldots + \mathbb{R}_+ t^{n-1}_{n-1}$ in fan of $\mathbb{P}(\mathcal{O}_{P_{n-1}} \oplus \mathcal{O}_{P_{n-1}}(S_1))$ is isomorphic via $j_{\mathbb{R}}$ to the $n-$dimensional cone $\mathbb{R}_+ r_1 + \mathbb{R}_+ t^1_2 + \ldots + \mathbb{R}_+ t^{n-1}_n$ in $F(n)$. Respectively, cone $\mathbb{R}_+(-e) + \mathbb{R}_+ t^1_1 + \ldots + \mathbb{R}_+ t^{n-1}_{n-1}$ is isomorphic via $j_{\mathbb{R}}$ to the cone $\mathbb{R}_+ s_1 + \mathbb{R}_+ t^1_2 + \ldots + \mathbb{R}_+ t^{n-1}_n$.

Note that by [Har77, Ex. II.7.10.(d)] projective bundles correspond to locally free sheaves up to twist by a line bundle, that is

$$\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(S_1)) \simeq \mathbb{P}(\mathcal{O} \oplus (-S_1)).$$

Proof of the main result. By Lemmas (3.2),(3.3),(3.5),(3.7) the graded ring $\bigoplus_{m=0}^{\infty} \mathrm{H}^0(P_n, \mathcal{O}(mL_n))$ is isomorphic to graded ring $\bigoplus_{m=0}^{\infty} A_m(\alpha_n)$, therefore the moduli space $M_\theta(\mathcal{Q}, \alpha_n)$ is isomorphic to the variety P_n i.e. it is identified with a $\mathbb{P}^1$-bundle over $M_\theta(\mathcal{Q}, \alpha_{n-1})$. Hence it is clearly an irreducible smooth n-dimensional variety. This concludes the proof of Theorem (2.8).

Remark 3.8. *The natural projection $P_n \longrightarrow P_{n-1}$ interpreted in terms of moduli spaces of representations of quiver $\mathcal{Q}$ "forgets" matrices denoted by $a_{n+1}, b_{n+1}, c_n, d_n$ in representation of $\mathcal{Q}$ (cf. Definition (2.6)).*

It known by works of Dolgachev, Hu and Thaddeus [DH98], [Tha96] that varying the stability parameter θ induces birational change in the corresponding moduli. Following two remarks give examples of such behavior.

Remark 3.9. *Denote by α^p_n the dimension vector α_n as in Definition (2.6) but with the origin at point $(p, -p)$ for some $p > 0$ (that is let α^p_n be a composition of translation by vector $(p, -p)$ and a function given by α_n on vertices of quiver $\mathcal{Q}$). From the point of view of representation theory, going from α_n to α^p_n changes the stability parameter. Repeating the proofs of Lemmas (3.2),(3.3),(3.5),(3.7), one can show that $M_\theta(\mathcal{Q}, \alpha^p_n)$ is isomorphic to $M_\theta(\mathcal{Q}, \alpha_n)$ but comes with different polarization, namely*

$$L^p_n = \sum_{i=1}^{n} (i + p) S_i.$$

For example, let $\alpha^p_1 = (1, 1, 1, 2, 1, 1, 1)$ have origin at $(p, -p)$. Then, the coordinate ring of the space $M_\theta(\mathcal{Q}, \alpha^p_1)$ is isomorphic to

$$\mathbb{C}[y_p^p\, y_{p+1}^{p+1}, x_p\, y_p^{p-1}\, y_{p+1}^{p}, \ldots, x_p^p\, y_{p+1}],$$

that is $M_\theta(\mathcal{Q}, \alpha_1^p)$ *is a rational normal curve of degree* p*. Recall that* $M_\theta(\mathcal{Q}, \alpha_1) \simeq \mathbb{P}^1$.

Remark 3.10. *Let* α_n *be a dimension vector as in Definition (2.6). Let* $\overline{\theta}$ *be the* $\mathsf{G}(\alpha_n)$*-character defined as follows:*

$$\overline{\theta}_{i,-i+1} = \quad 1,$$

$$\overline{\theta}_{i,-i-1} = -1.$$

Then, again reading through the proofs of Lemmas (3.2),(3.3),(3.5),(3.7) one can show that the coordinate ring of $M_{\overline{\theta}}(\mathcal{Q}, \alpha_n)$ *is isomorphic to*

$$\mathbb{C}[x^{e_1}x^{e_2}\cdots x^{e_n} \cdot y_1^{1-e_1}y_2^{1-(e_1+e_2)} \cdot y_n^{1-(e_{n-1}+e_n)}y_{n+1}^{1-e_n}],$$

where $0 \le e_1, e_n \le 1,\ 0 \le e_i + e_{i+1} \le 1$ *for* $i = 1, \dots, n-1$.

For example, if $\alpha_3 = (1,1,1,2,1,1,2,1,1,2,1,1,1)$*, then the space* $M_{\overline{\theta}}(\mathcal{Q}, \alpha_3)$ *is isomorphic to the cone in* $\mathbb{P}^4$ *cut by equations* $z_0\, z_4 = z_1\, z_3$*, where* $z_0 = y_1y_2y_3y_4$*,* $z_1 = x_3\, y_1\, y_2$*,* $z_2 = x_2\, y_1\, y_4$*,* $z_3 = x_1\, y_3\, y_4$*,* $z_4 = x_1\, x_3$*. In particular,* $M_{\overline{\theta}}(\mathcal{Q}, \alpha_3)$ *is singular in this case, though birational to* $M_\theta(\mathcal{Q}, \alpha_3)$*.*

Remark 3.11. *For an explicit discussion of moduli space* $M_\theta(\mathcal{Q}, \alpha_1)$*, for* $\alpha_1 = (1,1,1,2,1,1,1)$ *see [OR06, Example 7.3].*

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