SKEW-SYMMETRIC MATRICES AND PALATINI SCROLL

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ABSTRACT. We prove that, for m greater than 3 and k greater than $m-2$, the Grassmannian of m-dimensional subspaces of the space of skew-symmetric forms over a vector space of dimension $2k$ is birational to the Hilbert scheme of Palatini scrolls in $\mathbf{P}^{2k-1}$.

For $m = 3$ and $k > 3$, this Grassmannian is proved to be birational to the set of pairs $(\mathcal{E}, Y)$, where Y is a smooth plane curve of degree k and $\mathcal{E}$ is a stable rank-2 bundle on Y whose determinant is $\mathscr{O}_Y(k-1)$. Scrolls, ACM sheaves, pfaffian hypersurfaces, skew-symmetric forms, Hilbert scheme 14E05, 14M12, 14N15, 14J40

1. INTRODUCTION

Let $\mathbf{k}$ be a field, $n \geq 3$ be an integer, and V be a $\mathbf{k}$-vector space of dimension $n + 1$. Degeneracy loci of general morphisms $\phi : \mathscr{O}_{\mathbf{P}(V)}^m \to \Omega_{\mathbf{P}(V)}(2)$, have been considered by several authors, for example M. C. Chang [12] and G. Ottaviani [31]. Relying on a nice interpretation due to Ottaviani, we identify the global sections of $\Omega_{\mathbf{P}(V)}(2)$ with $\wedge^2 V^*$, the space of skew-symmetric forms on V, or of skew-symmetric matrices of size $n + 1$ with coefficients in the base field $\mathbf{k}$. Thus a morphism $\phi : \mathscr{O}_{\mathbf{P}(V)}^m \to \Omega_{\mathbf{P}(V)}(2)$ can be written in coordinates by means of m skew-symmetric matrices $A_1, \ldots, A_m$.

In classical terminology, each matrix A_i corresponds to a linear line complex Γ_i of $\mathbf{P}(V)$, so that the degeneracy locus X_ϕ of ϕ, can be thought of as the set of centers of complexes belonging to the linear system spanned by $\Gamma_1, \ldots, \Gamma_m$. Under this point of view, these varieties were considered already by classical algebraic geometers. For instance, in 1891 G. Castelnuovo in [11] considered the case $m = 3$ and $n = 4$, namely the case of nets of linear complexes in $\mathbf{P}^4$. The locus of lines which are centers of the linear complexes belonging to a general net of linear complexes in $\mathbf{P}^4$, or, in modern language, the degeneracy locus of a general morphism $\phi : \mathscr{O}_{\mathbf{P}^4}^3 \to \Omega_{\mathbf{P}^4}(2)$, is the projected Veronese surface in $\mathbf{P}^4$.

In 1901 F. Palatini in [32] and [33] considered the case of linear systems of linear complexes in $\mathbf{P}^5$ of dimension > 1. The case $m = 3$ leads to the elliptic scroll surface of degree 6, which was further studied by G. Fano [20]. The case $m = 4$ gives a 3-fold of degree 7 which is a scroll over a cubic surface of $\mathbf{P}^3$, also called Palatini scroll.

In the present paper we will consider the Hilbert scheme $\mathcal{H}_m(V)$ of these degeneracy loci. Given a general map $\phi : \mathscr{O}_{\mathbf{P}(V)}^m \to \Omega_{\mathbf{P}(V)}(2)$, we consider ϕ as the point $[\phi]$ of the Grassmann variety $\mathbf{G}(m, \wedge^2 V^*)$ parametrizing m-dimensional vector subspaces of $\wedge^2 V^*$. Since a linear change of coordinate on U does not change

the degeneracy locus X_ϕ, we have a rational map:

$$\rho : \mathbf{G}(m, \wedge^2 V^*) \dashrightarrow \mathcal{H}_m(V)$$
$$[\phi] \mapsto X_\phi,$$

where $\mathcal{H}_m(V)$ denotes the union of components of the Hilbert scheme of $\mathbf{P}(V)$ containing the degeneracy locus of a general map of the form ϕ.

As an instance of classical results in this direction, let us mention that, if $m = 3$ and $n = 4$, from the results contained in [11], one can prove that the component of $\mathcal{H}_m(V)$ containing Veronese surfaces in $\mathbf{P}^4$ is birational to $\mathbf{G}(3, \wedge^2 V^*)$. A similar statement holds for the Palatini scroll in $\mathbf{P}^5$: the main result of [19] states that ρ is a birational map if $m = 4$, $n = 5$. On the other hand, it was proved in [4], and in fact classically known to Fano (see [20]), that the map ρ is generically $4 : 1$ in case $m = 3$, $n = 5$.

We focus here on the case when n is odd (say $n = 2k - 1$), in which case we call a scroll of the form X_ϕ a *Palatini scroll*, including also the case of surface scrolls over a curve. This last case is particularly well-studied, in the framework of surface scrolls in $\mathbf{P}(V)$ which are non-special. In particular we mention the results on the Hilbert schemes of non-special scrolls due to [8].

In the present paper we generalise the result of [19] by proving that it holds for all $m \geq 4$ and all $k \geq m - 1$. Moreover, we prove that, for $m = 3$, the map ρ is generically injective for all $k \geq 4$. In other words, we show that the case studied in [4] is the only exception to injectivity of ρ. More precisely, we prove:

Theorem 1. *Let m, k be integers and let ρ be the map introduced above.*

A) *The map ρ is birational for all $m \geq 4$, $k \geq m - 1$, in particular the Hilbert scheme $\mathcal{H}_m(V)$ is generically smooth of dimension $m(k(2k - 1) - m)$.*
B) *For $m = 3$ and for all $k \geq 4$, the map ρ is generically injective. Moreover, it is dominant on the closed subscheme of $\mathcal{H}_3(V)$ whose general element is a general plane curve Y of degree k equipped with a general stable rank-2 bundle whose determinant is $\mathscr{O}_Y(k - 1)$.*

Remark 1. For $m \geq 4$ the general element X of $\mathcal{H}_m(V)$ is a smooth subvariety X of $\mathbf{P}(V)$, whose normal bundle $\mathcal{N}$ satisfies $\dim \mathrm{H}^0(X, \mathcal{N}) = m(k(2k - 1) - m)$. For $m = 4$, $k \geq 5$, we have $\mathrm{H}^1(X, N) \neq 0$, nevertheless this space gives no obstructions to $\mathcal{H}_m(V)$.

For $m = 3$, the general element of $\mathcal{H}_m(V)$ is a scroll of the form $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a stable bundle of rank 2 and degree $k(k - 1)$ over a smooth curve of genus $\binom{k-1}{2}$. In other words, the curve is not planar in general, and only the degree of $\mathcal{E}$ is fixed, not its determinant.

2. BASIC CONSTRUCTIONS

In this section we review some classical constructions related to Palatini scrolls. All the material in this section is essentially well-known.

Let $\mathbf{k}$ be a field. Given a $\mathbf{k}$-vector space A of finite dimension, we set $\mathbf{P}(A)$ for the projective space of 1-dimensional subspaces of A. We use the notation $\mathbf{G}(a, A)$ for the Grassmannian of a-dimensional vector subspaces of A. Given a torsion-free coherent sheaf $\mathcal{E}$ on an integral scheme Y, let $\mathcal{S} = \mathrm{Sym}(\mathcal{E})$ be the symmetric algebra of $\mathcal{E}$. With $\mathbb{P}(\mathcal{E})$, the associated projective space bundle, we mean $\mathbb{P}(\mathcal{E}) := \mathrm{Proj}(\mathcal{S})$

and we let $p : \mathbb{P}(\mathcal{E}) \to Y$ denote the projection morphism. Note that there is a natural map $\mathbb{P}(\mathcal{E}) \to \mathbf{P}(\mathrm{H}^0(Y, \mathcal{E})^*)$.

2.1. **Skew-symmetric forms and scrolls.** Let m and k be two integers, with $2k \geq m + 1$, and let V be a $\mathbf{k}$-vector space of dimension $2k$ and U be a $\mathbf{k}$-vector space of dimension m. Let us introduce the projective spaces $\mathbf{P}(V)$ and $\mathbf{P}(U)$. We consider the space $\wedge^2 V^*$ of skew-symmetric linear forms on V. This is well-known to be canonically isomorphic to the space $\mathrm{H}^0(\mathbf{P}(V), \Omega_{\mathbf{P}(V)}(2))$ of twisted 1-forms on $\mathbf{P}(V)$.

Let ϕ be an injective map $U \to \wedge^2 V^*$. We denote by $[\phi]$ the point of the Grassmannian $\mathbf{G}(m, \wedge^2 V^*)$ of m-dimensional vector subspaces of $\wedge^2 V^*$ corresponding to ϕ. In view of the identification $\mathrm{H}^0(\mathbf{P}(V), \Omega_{\mathbf{P}(V)}(2)) \cong \wedge^2 V^*$, we may consider ϕ as a map:

$$(2.1) \qquad U \otimes \mathcal{O}_{\mathbf{P}(V)} \to \Omega_{\mathbf{P}(V)}(2),$$

which we still denote by ϕ.

Definition 1. Given a map $\phi : U \to \wedge^2 V^*$ as in (2.1), we define the subscheme $X_\phi \subset \mathbf{P}(V)$ as the degeneracy locus $\mathrm{D}_{m-1}(\phi)$ of ϕ. The underlying set of X_ϕ consists of the points of $\mathbf{P}(V)$ where ϕ is not of maximal rank. If the codimension of $X = X_\phi$ is $2k - m$, we say that X_ϕ is the *Palatini scroll* in $\mathbf{P}(V)$ associated to ϕ. In turn, the Hilbert scheme $\mathcal{H}_m(V)$ of Palatini scrolls is defined as the (union of) component(s) of the Hilbert scheme of subschemes of $\mathbf{P}(V)$ containing Palatini scrolls.

We refer for instance to [34] for basic facts on degeneracy loci. Observe that the variety X can be regarded as the degeneracy locus of the transpose map:

$$(2.2) \qquad {}^{\mathrm{t}}\phi : \mathcal{T}_{\mathbf{P}(V)}(-2) \to U^* \otimes \mathcal{O}_{\mathbf{P}(V)}.$$

Notice that, composing ${}^{\mathrm{t}}\phi$ with the (twisted) Euler exact sequence on $\mathbf{P}(V)$:

$$0 \to \mathcal{O}_{\mathbf{P}(V)}(-2) \to V \otimes \mathcal{O}_{\mathbf{P}(V)}(-1) \to \mathcal{T}_{\mathbf{P}(V)}(-2) \to 0,$$

we obtain a map:

$$F_\phi : V \otimes \mathcal{O}_{\mathbf{P}(V)}(-1) \to U^* \otimes \mathcal{O}_{\mathbf{P}(V)},$$

whose image is just $\mathrm{Im}({}^{\mathrm{t}}\phi)$. Thus, a Palatini scroll X defined as the degeneracy locus of a map ϕ can be seen as well as the degeneracy locus of the map F_ϕ. This map is an $m \times 2k$ matrix of linear forms, and X is defined by the vanishing of all its $m \times m$ minors. In coordinates, the map F_ϕ looks like:

$$F_\phi = \begin{pmatrix} \sum_{j=1}^{2k} a_{1,j}^1 x_j & \cdots & \sum_{j=1}^{2k} a_{2k,j}^1 x_j \\ \vdots & & \vdots \\ \sum_{j=1}^{2k} a_{1,j}^m x_j & \cdots & \sum_{j=1}^{2k} a_{2k,j}^m x_j \end{pmatrix},$$

for some constants $a_{i,j}^\ell$ satisfying:

$$a_{i,j}^\ell = -a_{j,i}^\ell, \qquad \text{for all } i, j, \ell.$$

Note that a point $[v]$ of $\mathbf{P}(V)$ corresponding to a vector $v = (v_1, \dots, v_{2k}) \in V$ lies in $X_\phi = \mathrm{D}_{m-1}(\phi) = \mathrm{D}_{m-1}(F_\phi)$ if and only if there is a vector $u = (u_1, \dots, u_m) \in U$

such that v lies in $\ker({}^{\mathrm{t}}F_{[u]})$, i.e. such that:

$$\begin{pmatrix} \sum_{j=1}^{2k} a_{1,j}^1 v_j & \cdots & \sum_{j=1}^{2k} a_{1,j}^m v_j \\ \vdots & & \vdots \\ \sum_{j=1}^{2k} a_{2k,j}^1 v_j & \cdots & \sum_{j=1}^{2k} a_{2k,j}^m v_j \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = 0.$$

This happens if and only if:

$$(2.3) \qquad\qquad \sum_{j,\ell} a_{i,j}^{\ell} v_j u_{\ell} = 0, \qquad \text{for all } i.$$

Remark 1. Let $\phi = \phi_m$ be a map $\mathscr{O}_{\mathbf{P}^{2k-1}}^m \to \Omega_{\mathbf{P}^{2k-1}}(2)$. Then the degree of $X_m = X_{\phi_m}$ is:

$$(2.4) \qquad\qquad \deg(X_m) = \sum_{i=0}^{2k-m} (-1)^i \binom{2k-1-i}{m-1},$$

see [4]. Indeed, since X_m has the expected codimension, it suffices to compute the $(2k-m)$-th Chern class of $\Omega_{\mathbf{P}^{2k-1}}(2)$.

2.2. **Skew-symmetric forms and pfaffian hypersurfaces.** Let ϕ be as in (2.1). Note that we can consider ϕ also as a map $U \to V^* \otimes V^*$, i.e. as an element of $\mathrm{Hom}_{\mathbf{k}}(U, V^* \otimes V^*)$. In turn, this vector space is naturally isomorphic to $\mathrm{Hom}_{\mathbf{k}}(U \otimes V, V^*)$. So, in view of the canonical isomorphism:

$$\mathrm{Hom}_{\mathbf{k}}(U \otimes V, V^*) \cong \mathrm{Hom}_{\mathbf{P}(U)}(V \otimes \mathscr{O}_{\mathbf{P}(U)}(-1), V^* \otimes \mathscr{O}_{\mathbf{P}(U)}),$$

the map ϕ gives a matrix of linear forms:

$$M_\phi : V \otimes \mathscr{O}_{\mathbf{P}(U)}(-1) \to V^* \otimes \mathscr{O}_{\mathbf{P}(U)}.$$

The fact that the map ϕ lies in $\mathrm{Hom}_{\mathbf{k}}(U, \wedge^2 V^*)$ implies that the matrix M_ϕ lies in $\mathrm{H}^0(\mathbf{P}(U), \wedge^2 V^* \otimes \mathscr{O}_{\mathbf{P}(U)}(1))$, that is, the matrix M_ϕ is skew-symmetric. This means that the transpose ${}^{\mathrm{t}}M_\phi$, once twisted by $\mathscr{O}_{\mathbf{P}(U)}(-1)$, equals $-M_\phi$ (we still write ${}^{\mathrm{t}}M_\phi = -M_\phi$). Therefore, the determinant of the matrix M_ϕ is the square of a homogeneous polynomial of degree k, which is called the pfaffian $\mathrm{Pf}(M_\phi)$ of M_ϕ. If the form $\mathrm{Pf}(M_\phi)$ is non-zero, we set Y_ϕ for the hypersurface in $\mathbf{P}(U)$, defined by $\mathrm{Pf}(M_\phi)$. So the degree of Y_ϕ is k. Set $Y = Y_\phi$, and define the hyperplane divisor class H_Y as $c_1(\mathscr{O}_Y(1))$, where $\mathscr{O}_Y(1)$ is the restriction to Y of $\mathscr{O}_{\mathbf{P}(U)}(1)$. A different way to see the hypersurface $\mathrm{Pf}(M_\phi)$ is as the intersection of $\mathbf{P}(U)$ and the pfaffian Pf in $\mathbf{P}(\wedge^2 V^*)$, where $\mathbf{P}(U)$ is embedded in $\mathbf{P}(\wedge^2 V^*)$ by ϕ and Pf is the hypersurface $\check{\mathbf{G}}(2, V) \subset \mathbf{P}(\wedge^2 V^*)$ dual to $\mathbf{G}(2, V) \subset \mathbf{P}(\wedge^2 V)$.

Let us now assume that $Y = Y_\phi$ is integral. Note that the singular locus of Y contains the further degeneracy locus $\mathrm{D}_{2k-4}(M_\phi)$ of the matrix M_ϕ. Thus the assumption that Y is integral implies that M_ϕ is generically of corank 2 over Y. We define thus the rank 2 sheaf $\mathcal{E}_\phi = \mathrm{Cok}(M_\phi)$ supported over Y. In view of [5, Theorem B, Corollary 2.4], one can consider the sheaf $\mathcal{E}_\phi$ as a rank-2 torsion-free sheaf supported on Y, such that:

$$c_1(\mathcal{E}_\phi) = (k-1)H_Y,$$

$$(2.5) \qquad \mathrm{H}^i(Y, \mathcal{E}_\phi(t)) = 0, \qquad\qquad \text{for all } t \in \mathbb{Z} \text{ and } 0 < i < m-2.$$

The sheaf $\mathcal{E}_\phi$ is ACM (which stands for arithmetically Cohen-Macaulay), hence reflexive, see [9, Proposition 2.3]. We have $V^* \cong \mathrm{H}^0(Y, \mathcal{E}_\phi)$, hence a natural map $\mathbb{P}(\mathcal{E}_\phi) \to \mathbf{P}(V)$, which we denote by q.

The scroll X_ϕ is the image in $\mathbf{P}(V)$ of the map q. Therefore, a Palatini scroll is generically ruled over the pfaffian hypersurface Y_ϕ of $\mathbf{P}(U)$. The hypersurface $Y = Y_\phi$ is called the *base* of the scroll $X = X_\phi$. We have thus a diagram:

$$Y \xleftarrow{p} \mathbb{P}(\mathcal{E}_\phi) \xrightarrow{q} X \subset \mathbf{P}(V),$$

where p is the scroll map.

Let us rephrase the situation in coordinates. Setting $M = M_\phi$, we note that a point $[u]$ of Y corresponds to a vector $u = (u_1, \ldots, u_m) \in U$ such that $M_{[u]}$ has not maximal rank. This means that there is a vector $v = (v_1, \ldots, v_{2k})$ in V such that (2.3) holds. In other words, an element $[v, u] \in \mathbb{P}(\mathcal{E}_\phi)$, is the class of a pair (v, u) where v is a vector lying in the space $\mathrm{Cok}(M_{[u]})$. The maps p and q are respectively the two projections of $[v, u]$ to u and v.

Given a Palatini scroll X, we denote by H the divisor class $c_1(\mathcal{O}_X(1))$, where $\mathcal{O}_X(1)$ is obtained restricting $\mathcal{O}_{\mathbf{P}(V)}(1)$ to X. Note that H is the first Chern class of the tautological line bundle on the variety $\mathbb{P}(\mathcal{E}_\phi)$, namely we have:

$$p_*(q^*(\mathcal{O}_X(H))) \cong \mathcal{E}_\phi.$$

Lemma 1. *If the locus $\mathrm{D}_{m-2}(\phi)$ is empty, then $q : \mathbb{P}(\mathcal{E}_\phi) \to X_\phi$ is an isomorphism.*

Proof. Set $M = M_\phi$, $\mathcal{E} = \mathcal{E}_\phi$. Recall that a point of $\mathbb{P}(\mathcal{E})$ can be written as the class $[v, u]$ of a pair (v, u), where the class $[u]$ of the vector u of U lies in Y (so that $M_{[u]}$ is degenerate) and the vector v of V lies in $\mathrm{Cok}(M_{[u]})$. The map q associates to $[v, u]$ the class $[v]$. An inverse to the map q can be thus constructed as follows. Given a point $[v]$ of X, we must have $0 \neq \ker(\phi_{[v]}) \subset U$, for X consists of the points of $\mathbf{P}(V)$ where ϕ is degenerate. Now, since the degeneracy locus $\mathrm{D}_{m-2}(\phi)$ is empty, this kernel must be generated by a single vector u. Note that v naturally lies in $\mathrm{Cok}(M_{[u]})$ since $M_{[u]}(v) = \phi_{[v]}(u) = 0$. Therefore we can associate the pair $[v, u]$ to $[v]$, and this gives an inverse to q. $\qquad\square$

Remark 2. Note that the hypersurface $Y = Y_\phi \subset \mathbf{P}(U)$ which is the base of the scroll X_ϕ is singular as soon as $m \geq 7$. Indeed, the singular locus of Y contains the degeneracy locus $\mathrm{D}_{2k-4}(M_\phi)$ of the matrix M_ϕ, i.e. the subscheme of $\mathbf{P}(U)$ where the rank of M_ϕ drops at least by four. This locus has codimension at most 6, so it is not empty as soon as $m \geq 7$.

By the same reason, the sheaf $\mathcal{E} = \mathcal{E}_\phi$ over the hypersurface Y_ϕ is not a vector bundle as soon as $m \geq 7$. Indeed, the locus where $\mathcal{E}$ is not locally free contains $\mathrm{D}_{2k-4}(M_\phi)$. Recall that anyway $\mathcal{E}$ is ACM (hence reflexive). In fact, if X is smooth then $\mathrm{D}_{m-2}(\phi) = \emptyset$, so $X \cong \mathbb{P}(\mathcal{E})$ by Lemma 1. So $\mathcal{E}$ is a Bănică sheaf in the sense of [3].

Remark 3. One can ask which hypersurfaces of degree $k > 1$ of $\mathbf{P}(U)$ arise as the pfaffian of a matrix of the form ϕ (we say that Y is a linear pfaffian in this case). Let us collect some well-known answers to this question.

$m = 3$: Any smooth curve $Y \subset \mathbf{P}(U)$ is a linear pfaffian by [5], see also [7].

$m = 4$: A general surface $Y \subset \mathbf{P}(U)$ is a linear pfaffian if and only if k is at most 15. Otherwise, surfaces of degree k which are a linear pfaffian fill in only a closed subset of $\mathbf{P}(\mathrm{Sym}^k U^*)$. We refer to [5]. In particular

when $m = 4, k = 3$, one knows that every cubic surface is a linear pfaffian. We refer again to [5] and to [19] for the singular case. The cases with $m = 4, k \le 5$ are also treated in [18], [13], [14].

$m = 5$: A general threefold $Y \subset \mathbf{P}(U)$ is a linear pfaffian if and only if $k \le 5$. We refer to [6], [28], [26] for the case $k = 3$, [27] and [24] for the case $k = 4$, [15] for $k = 5$, [30], [16] for $k \ge 6$.

$m = 6$: A general fourfold of degree k is not pfaffian unless $k = 2$, see [29].

$m \ge 7$: No smooth hypersurface in $\mathbf{P}(U)$ is a linear pfaffian, see [25].

2.3. **Kernel, cokernel and normal bundles.** Consider a Palatini scroll $X = X_\phi$ defined by a map ϕ as in (2.1), and assume that the degeneracy locus $\mathrm{D}_{m-2}(\phi)$ is empty (which is the case if X is smooth). Then it is well-known (see for instance [34]) that restricting ϕ to X we obtain two locally free sheaves on X, $\ker(\phi_{|X})$ and $\mathrm{Cok}(\phi_{|X})$, respectively of rank 1 and $2k - m$. We restrict ${}^{\mathrm{t}}\phi$ to X and we set set $\mathscr{L}_\phi = \mathrm{Cok}({}^{\mathrm{t}}\phi_{|X})$. Then $\mathrm{Cok}({}^{\mathrm{t}}\phi)$ is the extension by zero of $\mathscr{L}_\phi$ to $\mathbf{P}(V)$. We get $\ker({}^{\mathrm{t}}\phi_{|X})^* \cong \mathrm{Cok}(\phi_{|X})$ and $\mathscr{L}_\phi$ is a line bundle on X isomorphic to $\ker(\phi_{|X})^*$.

Moreover, the normal bundle $\mathcal{N}$ of X in $\mathbf{P}(V)$ is isomorphic to $\ker({}^{\mathrm{t}}\phi_{|X})^* \otimes \mathscr{L}_\phi$. We have thus the exact sequence defined on X:

$$(2.6) \qquad 0 \to \mathscr{O}_X \to U \otimes \mathscr{L}_\phi \xrightarrow{\alpha} \Omega_{\mathbf{P}(V)}(2) \otimes \mathscr{L}_\phi \to \mathcal{N} \to 0,$$

where the map α is simply $\phi_{|X}$ tensored with $\mathscr{L}_\phi$. This gives rise to the short exact sequences:

$$(2.7) \qquad 0 \to \mathscr{O}_X \to U \otimes \mathscr{L}_\phi \to \mathrm{Im}(\alpha) \to 0,$$

$$(2.8) \qquad 0 \to \mathrm{Im}(\alpha) \to \Omega_{\mathbf{P}(V)}(2) \otimes \mathscr{L}_\phi \to \mathcal{N} \to 0.$$

Note that, under the natural isomorphism $\mathrm{Hom}_{\mathbf{k}}(U, V^* \otimes V^*) \cong \mathrm{Hom}_{\mathbf{k}}(V, U^* \otimes V^*)$, the map ϕ gives rise to a linear map

$$(2.9) \qquad f_\phi : V \to U^* \otimes V^*.$$

Note also that the map f_ϕ agrees with the map obtained by taking global sections of (2.2) twisted by $\mathscr{O}_{\mathbf{P}(V)}(1)$.

2.4. **Assumptions on skew-symmetric forms.** We summarize here the hypothesis that we will need to formulate our results, and some of their basic consequences. Let again U and V be $\mathbf{k}$-vector spaces respectively of dimension m and $2k$, with $k \ge 3$, and let ϕ be a map $U \to \wedge^2 V^*$, or in other words let ϕ be as in (2.1). The assumptions that we will need on the map ϕ are the following.

(1) The subscheme $X \subset \mathbf{P}(V)$ defined as $\mathrm{D}_{m-1}(\phi)$ is a smooth irreducible variety of codimension $2k - m$ in $\mathbf{P}(V)$, i.e. $\dim(X) = m - 1$. This is an open condition, and it implies $\mathrm{D}_{m-2}(\phi) = \emptyset$. This in turn forces:

$$m \le k + 1.$$

In the above range, the condition takes place for general ϕ.

As a consequence of this we get the following two conditions:

(2) the matrix M_ϕ is generically of maximal rank, so that $\mathrm{Pf}(M_\phi)$ does define a hypersurface Y_ϕ of $\mathbf{P}(U)$. Equivalently, the image of $\mathbf{P}(U)$ in $\mathbf{P}(\wedge^2 V^*)$ is not contained in the pfaffian hypersurface;

(3) the map f_ϕ of (2.9) is injective.

We further assume:

(4) the hypersurface $Y = Y_\phi$ which is the base of X is smooth in codimension 4. This is an open condition that takes place for general ϕ. In view of a result of Grothendieck, [21, Exposé XI, Corollaire 3.14], this condition implies that Y is locally factorial (see [22, Corollary 14]). Clearly the hypersurface Y is also integral in this case.

Remark 4. In the assumption that X has expected codimension, we have $X \neq \emptyset$ as soon as $m \geq 2$. The scheme X is defined by the minors of order m of the matrix F_ϕ, so X is cut scheme-theoretically by homogeneous forms of degree m. In particular we have:

$$(2.10) \qquad \mathrm{H}^0(\mathbf{P}(V), \mathcal{I}_X(m-1)) = 0,$$

if $m \geq 2$.

3. Image and cokernel sheaves

In this section we work with the image and cokernel sheaves associated to a map ϕ as in (2.1). Our goal is to show that we can read off $\mathscr{O}_{Y_\phi}(1)$ as the cokernel of ${}^t\phi$, where $Y_\phi \subset \mathbf{P}(U)$ is the base of the scroll X_ϕ. This will hold under the assumption that $m \geq 4$ or that $k \geq 4$ if $m = 3$. We first need a vanishing result (proved by induction) on the image of ${}^t\phi$. We denote by I_ϕ the torsion-free coherent sheaf defined as the image of ${}^t\phi$.

Lemma 2. *Let $m \geq 2$ and $k \geq 2$ be integers, V and U be vector spaces of dimension respectively $2k$, and m. Let X_ϕ be the degeneracy locus $\mathrm{D}_{m-1}(\phi)$ of a map $\phi : U \otimes \mathscr{O}_{\mathbf{P}(V)} \to \Omega_{\mathbf{P}(V)}(2)$ satisfying the conditions listed in Section 2.4. Then we have:*
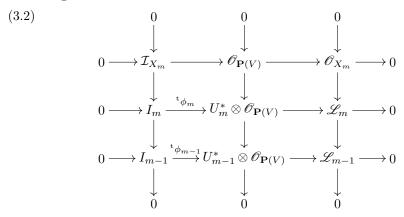
$$(3.1) \qquad \mathrm{H}^0(\mathbf{P}(V), I_\phi) = 0.$$

Proof. For the sake of this lemma, we let ϕ_m be a map $\phi : U_m \otimes \mathscr{O}_{\mathbf{P}(V)} \to \Omega_{\mathbf{P}(V)}(2)$, where U_m has dimension m. Correspondingly, we denote $X_m = \mathrm{D}_{m-1}(\phi_m)$, and $\mathscr{L}_m = \mathrm{Cok}({}^t\phi_m)$, $I_m = \mathrm{Im}({}^t\phi_m)$, $F_m = F_{\phi_m}$. Restricting ϕ_m to an $(m-1)$-dimensional subspace U_{m-1} of U_m we obtain a map ϕ_{m-1}. Transposing ϕ_{m-1} we obtain thus the degeneracy locus X_{m-1} and the sheaves $\mathscr{L}_{m-1}$ and I_{m-1}. We also define the pfaffian hypersurface $Y_m = Y_{\phi_m} \subset \mathbf{P}^{m-1}$ and the rank 2 sheaf $\mathcal{E}_m = \mathcal{E}_{\phi_m}$.

We first claim that, for a general choice of the subspace U_{m-1}, we may suppose that X_{m-1} is still of the expected codimension. Indeed, since $X_m \cong \mathbb{P}(\mathcal{E}_m)$, and Y_m contains Y_{m-1} as a hyperplane section, the variety $\mathbb{P}(\mathcal{E}_{m-1}) \cong X_{m-1}$ has dimension $m-2$, hence expected codimension in $\mathbf{P}(V)$.

The surjective map $U^* \otimes \mathscr{O}_{\mathbf{P}(V)} \to \mathscr{L}_m$ provides nonzero maps $\mathscr{O}_{\mathbf{P}(V)} \to \mathscr{L}_m$, and, since $\mathscr{L}_m$ is supported on X_m, such map factors through a (nonzero) map $\mathscr{O}_{X_m} \to \mathscr{L}_m$. This map is thus injective, and we have the following commutative

exact diagram:

(3.2)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{X_m} & \longrightarrow & \mathscr{O}_{\mathbf{P}(V)} & \longrightarrow & \mathscr{O}_{X_m} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_m & \xrightarrow{{}^{\mathrm{t}}\phi_m} & U_m^* \otimes \mathscr{O}_{\mathbf{P}(V)} & \longrightarrow & \mathscr{L}_m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_{m-1} & \xrightarrow{{}^{\mathrm{t}}\phi_{m-1}} & U_{m-1}^* \otimes \mathscr{O}_{\mathbf{P}(V)} & \longrightarrow & \mathscr{L}_{m-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Recall that the ideal sheaf of $X_m = \mathrm{D}_{m-1}(\phi_m)$, is generated by the minors of order m of the matrix F_m. Note that the same holds for all $2 \leq m' \leq m$. Recall also that X_m is not empty if $m \geq 2$, hence we may use the vanishing (2.10). Therefore, taking global sections in the leftmost column of (3.2) and using induction, we obtain the vanishing (3.1) as soon as we prove it for $m = 2$.

We prove now the vanishing (3.1) for $m = 2$. Note that (2.4) implies $\deg(X_1) = 0$, so that $X_1 = \emptyset$ i.e. ϕ_1 vanishes nowhere. This implies that I_1 is isomorphic to $\mathscr{O}_{\mathbf{P}(V)}$. The leftmost column of (3.2) reads in this case:

$$0 \to \mathcal{I}_{X_2} \to I_2 \to \mathscr{O}_{\mathbf{P}(V)} \to 0.$$

Since clearly $\mathrm{H}^0(\mathbf{P}(V), \mathcal{I}_{X_2}) = 0$, we obtain $\mathrm{h}^0(\mathbf{P}(V), I_2) \leq 1$. Note that the value 1 is attained if and only if I_2 splits as $\mathscr{O}_{\mathbf{P}(V)} \oplus \mathcal{I}_2$. But, since I_2 is the image of the map F_ϕ, this is contradicted by $\mathrm{H}^0(\mathbf{P}(V), \mathcal{I}_{X_2}(1)) = 0$. In turn, the last vanishing takes place in view of (2.10). $\qquad\square$

Lemma 3. *Fix the assumptions as above and set $\mathscr{L}_\phi = \mathrm{Cok}({}^{\mathrm{t}}\phi)$. Assume moreover $k \geq 4$ in case $m = 3$. Then we have an isomorphism:*

$$\mathscr{L}_\phi \cong q_*(p^*(\mathscr{O}_{Y_\phi}(1))).$$

Proof. Set $\mathscr{L} = \mathscr{L}_\phi$, and recall that $\mathscr{L}$ is an invertible sheaf on X. Restricting to X the dual Euler exact sequence on $\mathbf{P}(V)$ and twisting by $\mathscr{L} \otimes \mathscr{O}_X(2)$, we obtain the equality $c_1(\Omega_{\mathbf{P}(V)|X}(2) \otimes \mathscr{L}) = (2k-2)H + (2k-1)c_1(\mathscr{L})$. Further, the sequence (2.7) gives $c_1(\mathrm{Im}(\alpha)) = mc_1(\mathscr{L})$ while the sequence (2.8) gives $c_1(\mathcal{N}) = c_1(\Omega_{\mathbf{P}(V)|X}(2) \otimes \mathscr{L}) - c_1(\mathrm{Im}(\alpha))$. Combining them we get:

$$c_1(\mathcal{N}) = (2k - 2)H + (2k - m - 1)c_1(\mathscr{L}).$$

On the other hand, from adjunction we know that $c_1(\mathcal{N}) = 2kH - c_1(X)$. Because X is generically a scroll over Y, we know that $c_1(X) = 2H + p^*(c_1(Y) - c_1(\mathcal{E})) = 2H + p^*((m+1-2k)H_Y)$ and thus $c_1(\mathcal{N}) = (2k-2)H + p^*((2k-m-1)H_Y)$. Hence:

(3.3) $(2k - m - 1)c_1(\mathscr{L}) = p^*((2k - m - 1)H_Y).$

This implies that $c_1(\mathscr{L}) = p^*(c_1(\mathscr{M}))$ for some line bundle $\mathscr{M} \in \mathrm{Pic}(Y)$, and, by (3.3), we get that:

$$(2k - m - 1)(c_1(\mathscr{M}) - H_Y) = 0 \qquad \text{in } \mathrm{Pic}(Y).$$

Now recall that, for $m \geq 5$, the group $\mathrm{Pic}(Y)$ is generated by H_Y by the theorem Grothendieck-Lefschetz, while, for $m = 4$, the cokernel of the natural restriction map $\mathrm{Pic}(\mathbf{P}(U)) \to \mathrm{Pic}(Y)$ is torsion-free (we refer for instance to [2]). It follows that $c_1(\mathscr{M}) = H_Y$, and we deduce $\mathscr{L} \cong p^*(\mathscr{O}_Y(1))$.

It remains to prove the statement for $m = 3$. Note that we have $\mathrm{h}^0(Y, \mathscr{M}) = \mathrm{h}^0(X, p^*(\mathscr{M})) = \mathrm{h}^0(X, \mathscr{L})$, and Lemma 2 easily implies that $\mathrm{h}^0(X, \mathscr{L}) \geq 3$. Note also that $\deg(\mathscr{M}) = \deg(\mathscr{L}) = k$, by (3.3).

Hence the linear system $|\mathscr{M}|$ is a g_k^s on the curve Y, with $s \geq 2$. Now we use a well-known result of Castelnuovo (see [10], we refer to [17, Theorem 2.11] and [1] for a modern treatment). Namely, any linear series g_k^s on a smooth plane curve Y of degree $k \geq 4$ (here we need our hypothesis) and with $s \geq 2$ must coincide with the hyperplane linear system $|\mathscr{O}_Y(1)|$ (in particular $s = 2$). This gives $c_1(\mathscr{M}) = H_Y$, and we have proved $\mathscr{L} = p^*(\mathscr{O}_Y(1))$. Note that we have also proved that, if $m = 3$ and $k \geq 4$, we must have $\mathrm{h}^0(\mathbf{P}(V), \mathscr{L}) = 3$. $\square$

The previous proof suggests the following.

Problem 1. Let $Y = Y_\phi$ be a general pfaffian surface, i.e. Y is given as the pfaffian of a general $2k \times 2k$ skew-symmetric matrix of linear forms over $\mathbf{P}^3$. Then, is the Picard group of Y generated by the hyperplane divisor H_Y, for $k \geq 16$?

Lemma 4. *Fix the hypothesis as in Lemma 2. Then we have:*
$$\mathrm{H}^1(\mathbf{P}(V), I_\phi) = 0, \qquad \text{for all } m \geq 3.$$

Proof. We borrow the notation from the proof of Lemma 2. Let us recall that, by [4, Proposition 1], we have:

$$\mathrm{H}^1(\mathbf{P}(V), \mathcal{I}_{X_m}(t)) = 0, \qquad \text{unless } m \text{ is even and } t \text{ equals } m - 2,$$
$$\mathrm{H}^2(\mathbf{P}(V), \mathcal{I}_{X_m}) = 0, \qquad \text{if } m \neq 3.$$

Therefore, taking cohomology in the leftmost column of (3.2) and using induction on m, we easily get $\mathrm{H}^1(\mathbf{P}(V), I_m) = 0$ for all $m \geq 4$ once we prove $\mathrm{H}^1(\mathbf{P}(V), I_3) = 0$.

To check the last vanishing, recall that, at the end of the proof of the previous lemma we have shown $\mathrm{h}^0(\mathbf{P}(V), \mathscr{L}_3) = 3$ if $k \geq 4$. However, the equality $\mathrm{h}^0(\mathbf{P}(V), \mathscr{L}_3) = 3$ holds even if $k = 3$ for $\mathscr{L}$ in this case is a line bundle of degree 3 supported on Y which is a smooth elliptic curve. In any case, by (3.1), we immediately deduce $\mathrm{H}^1(\mathbf{P}(V), I_3) = 0$, and the lemma is proved. $\square$

4. NORMAL BUNDLE

Let again ϕ be as in (2.1) and set $X = X_\phi$. Assume that X is smooth. We have noticed that the normal bundle $\mathcal{N}$ of X in $\mathbf{P}(V)$ fits in the exact sequence (2.6).

The goal of this section is to compute the dimension of the Hilbert scheme of Palatini scroll $\mathcal{H}_m(V)$ at the point represented by X, by calculating the dimension of $\mathrm{H}^0(X, \mathcal{N})$. The idea is to compute this dimension by pushing down the exact sequence (2.6) to the base variety Y. The key argument relies on a technical result of Mohan Kumar-Rao-Ravindra, see [29].

Theorem 1. *Assume that the map ϕ satisfies the conditions of Section 2.4, and let $\mathcal{N}$ be the normal bundle of X in $\mathbf{P}(V)$. Then we have, for all $m \geq 4$ and $k \geq m-1$:*

$$(4.1) \qquad \mathrm{h}^0(X, \mathcal{N}) = m\left(k(2k - 1) - m\right).$$

Furthermore, if $m \geq 5$ or if $m = 4, k \leq 4$, we have:

$$(4.2) \qquad\qquad \mathrm{H}^1(X, \mathcal{N}) = 0,$$

while, if $m = 4$ and $k \geq 5$, we have:

$$(4.3) \qquad\qquad \mathrm{h}^1(X, \mathcal{N}) = \frac{2(k-2)(k-3)(k-4)}{3}.$$

Finally, if $m = 3$, $k \geq 4$, we have:

$$\mathrm{h}^0(X, \mathcal{N}) = \frac{3k(5k-7)}{2},$$
$$\mathrm{H}^1(X, \mathcal{N}) = 0.$$

In order to prove the theorem, we will need a little more material. Let us introduce it in the next subsection.

4.1. **The exterior square of a skew-symmetric matrix of linear forms.**
Consider again the matrix $M = M_\phi$ of linear forms on $\mathbf{P}(U)$ associated to $\phi :$
$U \otimes \mathscr{O}_{\mathbf{P}(V)} \to \Omega_{\mathbf{P}(V)}(2)$, and the pfaffian hypersurface $Y = Y_\phi$ defined by M in
$\mathbf{P}(U)$. The restriction of the map M to Y provides an exact sequence:

$$(4.4) \qquad\qquad 0 \to G \to V^* \otimes \mathscr{O}_Y \to \mathcal{E} \to 0,$$

where G is a sheaf of rank $2(k-1)$ on Y defined by the above sequence.

We consider the symmetric square of (4.4), and we twist the result by $\mathscr{O}_Y(1)$,
whereby obtaining a four-term exact sequence of the form:

$$(4.5) \qquad 0 \to \wedge^2 G(1) \xrightarrow{\gamma} \wedge^2 V^* \otimes \mathscr{O}_Y(1) \xrightarrow{\delta} V^* \otimes \mathcal{E}(1) \xrightarrow{\eta} \mathrm{Sym}^2 \mathcal{E}(1) \to 0.$$

We will make use of the setup of [29]. Taking the exterior square of the matrix
M and twisting by $\mathscr{O}_{\mathbf{P}(U)}(1)$, we obtain the exact sequence:

$$(4.6) \qquad 0 \to \wedge^2 V \otimes \mathscr{O}_{\mathbf{P}(U)}(-1) \xrightarrow{\wedge^2 M} \wedge^2 V^* \otimes \mathscr{O}_{\mathbf{P}(U)}(1) \to \mathcal{F}(1) \to 0,$$

for some cokernel sheaf $\mathcal{F}$ supported on Y.

Lemma 5. *Fix the setup as above. Then, for all $m \geq 4$ we have:*

$$\mathrm{h}^0(Y, \mathrm{Im}(\delta)) = mk(2k-1) - 1.$$

Moreover, if $m \neq 4$ we have the vanishing:

$$\mathrm{H}^1(Y, \mathrm{Im}(\delta)) = 0,$$

while for $m = 4$, we have the vanishing $\mathrm{H}^2(Y, \mathrm{Im}(\delta)) = 0$ and the equality:

$$\mathrm{h}^1(Y, \mathrm{Im}(\delta)) = \mathrm{h}^0(Y, \omega_Y),$$

*which equals zero if $k \leq 3$ and equals $\frac{(k-1)(k-2)(k-3)}{6}$ if $k \geq 4$. Finally, if $m = 3$ we
have:*

$$\mathrm{h}^0(Y, \mathrm{Im}(\delta)) = mk(2k-1) + \frac{(k-1)(k-2)}{2} - 1,$$
$$\mathrm{H}^1(Y, \mathrm{Im}(\delta)) = 0.$$

Proof. Looking at the diagram in the proof of [29, Lemma 2.1], we see that the image of the map δ, (which would be denoted in the notation of that paper by $\overline{\mathcal{F}}(1)$) fits into the exact sequence:

$$(4.7) \qquad 0 \to \mathcal{L}(1) \to \mathcal{F}(1) \to \mathrm{Im}(\delta) \to 0,$$

where $\mathcal{L}$ is the line bundle on Y provided by [*op. cit.*, Lemma 2.1.] Note that, even though the hypersurface Y is not smooth in general, we can still use this lemma since Y is integral and locally factorial in our assumptions (see the hypothesis (4)), and the sheaf $\mathcal{L}$ on Y is also a line bundle in this case. In fact this will follow by [23, Propositions 1.1 and 1.9], once we prove the following claim.

Claim 1. The above sheaf $\mathcal{L}$ on Y is reflexive of rank 1.

of the claim. We dualize (4.6). Since the matrix M is skew-symmetric, the matrix $\wedge^2 M$ is symmetric, so we have an isomorphism:

$$\tau : \mathrm{Cok}(\wedge^2 M) \cong \mathrm{Cok}(^{\mathrm{t}} \wedge^2 M).$$

Grothendieck duality implies:

$$\mathrm{Cok}(^{\mathrm{t}} \wedge^2 M) \cong \mathscr{E}xt^1_{\mathbf{P}(U)}(\mathcal{F}, \mathscr{O}_{\mathbf{P}(U)}) \cong \mathcal{F}^*(k-1).$$

We also know that $\mathrm{Cok}(\wedge^2 M) = \mathcal{F}(1)$ so that:

$$(4.8) \qquad \mathcal{F}(1) \cong \mathcal{F}^*(k-1).$$

By restricting (4.6) to Y we get an exact sequence

$$(4.9) \qquad 0 \to \mathcal{F}^*(-1) \to \wedge^2 V \otimes \mathscr{O}_Y(-1) \xrightarrow{\overline{\wedge^2 M}} \wedge^2 V^* \otimes \mathscr{O}_Y(1) \to \mathcal{F}(1) \to 0$$

Define the torsion-free sheaf P as the image of $\overline{\wedge^2 M}$. Twisting (4.9) by $\mathscr{O}_Y(k)$ and using (4.8) we see that $\mathcal{F}(1)$ is contained in $\wedge^2 V \otimes \mathscr{O}_Y(k-1)$ with cokernel $P(k)$. Combining this with the sequence (4.7) we get an exact sequence:

$$(4.10) \qquad 0 \to \mathcal{L}(1) \to \wedge^2 V \otimes \mathscr{O}_Y(k-1) \to P' \to 0,$$

where P' sits in the following exact sequence:

$$0 \to \mathrm{Im}(\delta) \to P' \to P(k) \to 0.$$

Hence P' is torsion-free. Moreover $\wedge^2 V \otimes \mathscr{O}_Y(k-1)$ is locally free and thus by [23, Proposition 1.1] it follows that $\mathcal{L}(1)$ is reflexive and of rank 1. $\square$

Claim 2. Fix the setup as above. Then we have an isomorphism:

$$\mathcal{L} \cong \mathscr{O}_Y(-1).$$

of the claim. Recall that $c_1(\mathcal{E}) = (k-1)H_Y$. To calculate $c_1(\mathcal{L})$, first note that:

$$(4.11) \qquad c_1(\mathcal{L}) = c_1(\mathcal{F}) - c_1(\mathrm{Im}(\delta)(-1)),$$

and splitting (4.5) into short exact sequences we easily compute:

$$c_1(\mathrm{Im}(\delta)(-1)) = (k-1)(2k-3)H_Y.$$

From (4.8) we deduce that:

$$c_1(\mathcal{F}(1)) = c_1(\mathcal{F}^*(k-1)),$$

which implies, since $\mathrm{rk}(\mathcal{F}) = 2(2k-1)$, the equality:

$$c_1(\mathcal{F}) = (2k-1)(k-2)H_Y.$$

Now (4.11) implies $c_1(\mathcal{L}) = -H_Y$ and we are done. $\square$

We can now conclude the proof of the lemma. We take cohomology of the exact sequence (4.6). For $m \geq 3$, we get:

$$\mathrm{h}^0(Y, \mathcal{F}(1)) = m\binom{2k}{2} = mk(2k-1),$$

$$\mathrm{h}^i(Y, \mathcal{F}(1)) = 0, \quad \text{for } i \geq 1.$$

In order to get the desired formulas, we take now cohomology of (4.7) and we use Claim (2). Then the result follows, once we note that $\mathrm{H}^1(Y, \mathscr{O}_Y) = 0$ for $m \geq 4$ and $\mathrm{h}^1(Y, \mathscr{O}_Y) = \frac{(k-1)(k-2)}{2}$ for $m = 3$. $\qquad\square$

4.2. **Projecting the normal bundle on the pfaffian hypersurface.** Let $X = X_\phi$ be a scroll defined by a map ϕ as in (2.1), and satisfying the conditions of Section 2.4. The idea to prove Theorem 1 is to push the exact sequence defining the normal bundle $\mathcal{N}$ of the scroll $X = X_\phi$ in $\mathbf{P}(V)$ down to the pfaffian hypersurface $Y = Y_\phi$. We do this in the next lemma.

Lemma 6. *Fix the setup as in Theorem 1. Then the following equality holds for all $m \geq 3$ and $k \geq 4$:*

$$(4.12) \qquad \mathrm{h}^0(X, \mathrm{Im}(\alpha)) = m^2 - 1,$$

and, if $m \geq 5$, and $(m, k) \neq (5, 5)$ we also have:

$$(4.13) \qquad \mathrm{H}^1(X, \mathrm{Im}(\alpha)) = 0,$$

$$(4.14) \qquad \mathrm{H}^2(X, \mathrm{Im}(\alpha)) = 0.$$

If $(m, k) = (5, 5)$, then the vanishing (4.13) holds and:

$$(4.15) \qquad \mathrm{h}^2(X, \mathrm{Im}(\alpha)) = 1.$$

Assume now $m = 4$. Then the same vanishing results hold if $k \leq 3$. If $(m, k) = (4, 4)$, then (4.14) holds and:

$$(4.16) \qquad \mathrm{h}^1(X, \mathrm{Im}(\alpha)) = 1,$$

while if $m = 4$ and $k \geq 5$, then (4.13) holds and:

$$(4.17) \qquad \mathrm{h}^2(X, \mathrm{Im}(\alpha)) = \frac{(k-2)(k-3)(k-5)}{2}.$$

Proof. Note that, since $\mathscr{L} = \mathrm{Cok}(^{\mathrm{t}}\phi) \cong q_*(p^*(\mathscr{O}_Y(1)))$ by Lemma 3, in the sequence (2.6) we may identify the inclusion $\mathscr{O}_X \to U \otimes \mathscr{L}$ with the map $q_*(p^*(\beta))$, where β fits in the Euler exact sequence on $\mathbf{P}(U)$, restricted to Y:

$$(4.18) \qquad 0 \to \mathscr{O}_Y \xrightarrow{\beta} U \otimes \mathscr{O}_Y(1) \to (\mathcal{T}_{\mathbf{P}(U)})_{|Y} \to 0.$$

Therefore the image of α is isomorphic to $q_*(p^*((\mathcal{T}_{\mathbf{P}(U)})_{|Y}))$ and we deduce, for all i:

$$(4.19) \qquad \mathrm{H}^i(X, \mathrm{Im}(\alpha)) \cong \mathrm{H}^i(Y, (\mathcal{T}_{\mathbf{P}(U)})_{|Y}).$$

We use now the exact sequence defining Y in $\mathbf{P}(U)$, tensored with $\mathcal{T}_{\mathbf{P}(U)_{|Y}}$:

$$0 \to \mathcal{T}_{\mathbf{P}(U)}(-k) \to \mathcal{T}_{\mathbf{P}(U)} \to (\mathcal{T}_{\mathbf{P}(U)})_Y \to 0.$$

Note that $h^0(\mathbf{P}(U), \mathcal{T}_{\mathbf{P}(U)}) = m^2 - 1$, while $H^i(\mathbf{P}(U), \mathcal{T}_{\mathbf{P}(U)}) = 0$ for all $0 < i < m - 1$. By Bott's theorem, we have, for such i:

$$h^i(\mathbf{P}(U), \mathcal{T}_{\mathbf{P}(U)}(-k)) = \begin{cases} 1 & \text{if } i = m - 2 \text{ and } k = m, \\ 0 & \text{otherwise.} \end{cases}$$

This implies (4.12) (indeed $m \geq 3$ and $k \geq 4$). We also get (4.13) for $m \geq 5$ and for $m = 4, k \neq 4$, while $(m, k) = (4, 4)$ gives (4.16). Likewise we get (4.14) for $m \geq 4$, except $(m, k) = (5, 5)$, in which case we get (4.15).

To conclude the proof, we look at the case $m = 4$. We note that by Serre duality:

$$(4.20) \qquad H^2(X, \mathrm{Im}(\alpha)) = H^0(Y, \Omega_{\mathbf{P}(U)|Y} \otimes \omega_Y)^*.$$

It is easy to see that $h^0(Y, \Omega_{\mathbf{P}(U)|Y} \otimes \omega_Y)$ takes value $(k-2)(k-3)(k-5)/2$ if $k \geq 5$, or zero if $k \leq 4$. This finishes the proof. $\qquad\square$

Set the notation as in Lemma 5. We have then:

Lemma 7. *Let $m \geq 3$ and $k \geq m - 1$ be integers, and fix the setup as above. Then, for all integers $i \geq 0$, there is a natural isomorphism:*

$$(4.21) \qquad H^i(X, \Omega_{\mathbf{P}(V)}(2) \otimes \mathcal{L}) \cong H^i(Y, \mathrm{Im}(\delta)).$$

Proof. We first write the dual Euler sequence on $\mathbf{P}(V)$, twisted by $\mathcal{L} \otimes \mathcal{O}_{\mathbf{P}(V)}(2)$:

$$0 \to \Omega_{\mathbf{P}(V)}(2) \otimes \mathcal{L} \to V^* \otimes \mathcal{O}_{\mathbf{P}(V)}(1) \otimes \mathcal{L} \xrightarrow{\xi} \mathcal{O}_{\mathbf{P}(V)}(2) \otimes \mathcal{L} \to 0.$$

Recall now the natural embedding $q : \mathbb{P}(\mathcal{E}) \to \mathbf{P}(V)$ and the canonical projection $p : \mathbb{P}(\mathcal{E}) \to Y$. For all $t \geq 0$ and all ℓ, we have the natural isomorphism:

$$p_* q^* (\mathcal{O}_{\mathbf{P}(V)}(t) \otimes \mathcal{L}^\ell) \cong \mathrm{Sym}^t \mathcal{E}(\ell).$$

Moreover, applying the functor $p_* q^*$ to the map ξ, we find the natural projection:

$$\eta : V^* \otimes \mathcal{E}(1) \to \mathrm{Sym}^2 \mathcal{E}(1),$$

appearing in the exact sequence (4.5). Therefore we have, for all integers i:

$$H^i(X, \Omega_{\mathbf{P}(V)}(2) \otimes \mathcal{L}) \cong H^i(X, \ker(\xi)) \cong H^i(Y, \ker(\eta)) \cong H^i(Y, \mathrm{Im}(\delta)).$$

$$\square$$

We prove now the main result of this section.

of Theorem 1. By the above claim, taking cohomology of (2.8), we get the long exact sequence:

$$(4.22) \quad \cdots \to H^i(X, \mathrm{Im}(\alpha)) \to H^i(Y, \mathrm{Im}(\delta)) \to H^i(X, \mathcal{N}) \to H^{i+1}(X, \mathrm{Im}(\alpha)) \to \cdots$$

Having all this set up, we use Lemma 6 and we apply Lemma 5 to obtain, for all $(m, k) \neq (4, 4)$:

$$h^0(X, \mathcal{N}) = mk(2k - 1) - 1 - (m^2 - 1) = m(k(2k - 1) - m).$$

So (4.1) is proved except for $(m, k) = (4, 4)$. Likewise we get (4.2) for $m \neq 4, 5$ and for $m = 4, k \leq 3$ as well as for $m = 5, k \neq 5$. We also have (4.3) for $m = 4$ and

$k \geq 5$, in view of:

$$\mathrm{h}^1(X, \mathcal{N}) = \mathrm{h}^1(X, \mathrm{Im}(\delta)) + \mathrm{h}^2(X, \mathrm{Im}(\alpha)) =$$
$$= \frac{(k-1)(k-2)(k-3)}{6} + \frac{(k-2)(k-3)(k-5)}{2} =$$
$$= \frac{2(k-2)(k-3)(k-4)}{3}.$$

It remains to take care of the cases $(m, k) \in \{(4, 4), (5, 5)\}$ and to treat the case $m = 3$. Let us accomplish the first task. We will need the following:

Claim 3. Assume $m \geq 4$ and consider the exact sequences (4.7) and (4.18). Then the inclusion $\mathrm{Im}(\alpha) \to \Omega_{\mathbf{P}(V)}(2) \otimes \mathscr{L}$ given by (2.6) can be identified with $q_*(p^*(\zeta))$, where the map ζ fits in the following exact commutative diagram:

(4.23)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{O}_Y & \longrightarrow & U \otimes \mathscr{O}_Y(1) & \longrightarrow & (\mathcal{T}_{\mathbf{P}(U)})_{|Y} & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow{\scriptstyle \zeta} & & \\
0 & \longrightarrow & \mathscr{O}_Y & \longrightarrow & \mathcal{F}(1) & \longrightarrow & \mathrm{Im}(\delta) & \longrightarrow & 0
\end{array}$$

of the claim. First recall the natural isomorphisms $\Omega_{\mathbf{P}(V)}(2) \otimes \mathscr{L} \cong q_*(p^*(\mathrm{Im}(\delta)))$ (see the proof of Lemma 7) and $\mathrm{Im}(\alpha) \cong q_*(p^*((\mathcal{T}_{\mathbf{P}(U)})_{|Y}))$ (see Claim 6). Then the map $\mathrm{Im}(\alpha) \to \Omega_{\mathbf{P}(V)}(2) \otimes \mathscr{L}$ is induced by some nonzero map $\zeta : (\mathcal{T}_{\mathbf{P}(U)})_{|Y} \to \mathrm{Im}(\delta)$. Composing the natural surjection $U \otimes \mathscr{O}_Y(1) \to (\mathcal{T}_{\mathbf{P}(U)})_{|Y}$ with ζ, we get a nonzero map $U \otimes \mathscr{O}_Y(1) \to \mathrm{Im}(\delta)$. Now we can lift this map to a nonzero map $U \otimes \mathscr{O}_Y(1) \to \mathcal{F}(1)$, indeed the group $\mathrm{Ext}^1(U \otimes \mathscr{O}_Y(1), \mathscr{O}_Y)$ vanishes in the range $m \geq 4$. We have thus an induced map $\mathscr{O}_Y \to \mathscr{O}_Y$, which must be a nonzero multiple of the identity. This gives the diagram (4.23). □

Let us now prove Theorem 1 in case $(m, k) = (4, 4)$. We look at the map $\mathrm{H}^1(X, \mathrm{Im}(\alpha)) \to \mathrm{H}^1(X, \Omega_{\mathbf{P}(V)}(2) \otimes \mathscr{L})$ induced by the inclusion $\mathrm{Im}(\alpha) \to \Omega_{\mathbf{P}(V)}(2) \otimes \mathscr{L}$. Under the isomorphism (4.21) and (4.20), this map is induced by the map ζ introduced in the previous claim. Thus it suffices to show that ζ induces an isomorphism of $\mathrm{H}^1(Y, (\mathcal{T}_{\mathbf{P}(U)})_{|Y})$ to $\mathrm{H}^1(Y, \mathrm{Im}(\delta))$. In order to show this, we look at the cohomology of the diagram (4.23) provided by the above claim. We get a commutative diagram of the form:

$$\begin{array}{ccc}
\mathrm{H}^1(Y, (\mathcal{T}_{\mathbf{P}(U)})_{|Y}) & \longrightarrow & \mathrm{H}^2(Y, \mathscr{O}_Y) \\
\downarrow & & \| \\
\mathrm{H}^1(Y, \mathrm{Im}(\delta)) & \longrightarrow & \mathrm{H}^2(Y, \mathscr{O}_Y).
\end{array}$$

Note that both horizontal arrows are isomorphisms, and the rightmost vertical arrow is an isomorphism too. Therefore the map $\mathrm{H}^1(Y, (\mathcal{T}_{\mathbf{P}(U)})_{|Y}) \to \mathrm{H}^1(Y, \mathrm{Im}(\delta))$ induced by ζ is an isomorphism, and we are done. Similarly, in the case $(m, k) = (5, 5)$, we get a diagram of the form

$$\begin{array}{ccc}
\mathrm{H}^2(Y, (\mathcal{T}_{\mathbf{P}(U)})_{|Y}) & \longrightarrow & \mathrm{H}^3(Y, \mathscr{O}_Y) \\
\downarrow & & \| \\
\mathrm{H}^2(Y, \mathrm{Im}(\delta)) & \longrightarrow & \mathrm{H}^3(Y, \mathscr{O}_Y).
\end{array}$$

and as before we can conclude that the map $\mathrm{H}^2(Y, (\mathcal{T}_{\mathbf{P}(U)})_{|Y}) \to \mathrm{H}^2(Y, \mathrm{Im}(\delta))$ induced by ζ is an isomorphism.

To complete the proof of Theorem 1, it only remains to check the case $m = 3$. Let us compute $\mathrm{h}^0(X, \mathcal{N})$ in this case. Using Lemma 7, in view of the exact sequences (4.7) and (4.6), we calculate:

$$(4.24) \qquad \mathrm{h}^0(X, \Omega_{\mathbf{P}(V)}(2) \otimes \mathcal{L}) = \mathrm{h}^0(Y, \mathrm{Im}(\delta)) =$$
$$= \mathrm{h}^0(Y, \mathcal{F}(1)) + \mathrm{h}^1(Y, \mathcal{O}_Y) - 1$$
$$= \frac{k(13k - 9)}{2},$$

and $\mathrm{H}^i(X, \Omega_{\mathbf{P}(V)}(2) \otimes \mathcal{L}) = 0$ for $i \geq 1$. Moreover, the isomorphism (4.19) still holds for $m = 3$. So, we easily get $\mathrm{h}^0(Y, \mathrm{Im}(\alpha)) = 8$ and:

$$(4.25) \qquad \mathrm{h}^1(Y, \mathrm{Im}(\alpha)) = \mathrm{h}^1(Y, (\mathcal{T}_{\mathbf{P}(U)})_{|Y}) = (k - 2)(k - 4).$$

Therefore, using (4.24) and (4.25), we can calculate $\mathrm{H}^i(X, \mathcal{N})$ by (4.22), obtaining:

$$\mathrm{h}^0(X, \mathcal{N}) = \frac{k(13k - 9)}{2} + (k - 2)(k - 4) - 8 = \frac{3k(5k - 7)}{2},$$

and $\mathrm{H}^i(X, \mathcal{N}) = 0$ for $i \geq 1$. This completes the proof of Theorem 1. $\qquad\square$

Our theorem asserts that the cohomology of the normal bundle of X in $\mathbf{P}(V)$ behaves *as expected*. However, the following problem remains open.

Problem 2. Let ϕ be a general morphism $\mathcal{O}_{\mathbf{P}(V)}^4 \to \Omega_{\mathbf{P}(V)}(2)$, where $\dim(V) = 2k$, and let $\mathcal{E} = \mathcal{E}_\phi$ the associated bundle on $Y = Y_\phi$. Then do we have, for all $k \geq 3$, $\mathrm{H}^2(Y, \mathrm{Sym}^2 \mathcal{E}(1 - k)) = 0$? In other words, is the moduli space of stable sheaves containing $\mathcal{E}$ smooth at $[\mathcal{E}]$?

Up to the authors' knowledge, the above question has a well-known answer (in the affirmative sense) only for $k \leq 15$, due to a result of Schreyer-Beauville [5].

5. Injectivity of the map ρ

The results of this section are intended to achieve the proof of our main result. Once we have computed the dimension of the Hilbert scheme in the previous section, it only remains to check that ρ is generically injective (hence birational onto its image).

The idea to prove this is that, in view of Lemma 3, two maps ϕ and ϕ' defining same scroll X will have isomorphic cokernel sheaves $\mathrm{Cok}(^t\phi) \cong \mathrm{Cok}(^t\phi')$. We prove thus some suitable cohomology vanishing in order to lift this isomorphism to the image sheaves I_ϕ and $I_{\phi'}$ and then to the whole tangent bundle $\mathcal{T}_{\mathbf{P}(V)}(-2)$. This will show that ϕ and ϕ' give the same point in $\mathbf{G}(m, \wedge^2 V^*)$. This will prove the theorem, except in case $m = 3$, where the image of ρ is not dense in the Hilbert scheme $\mathcal{H}_3(V)$ containing Palatini scrolls. We will thus conclude by computing the codimension of the image of ρ in $\mathcal{H}_3(V)$.

Let us now consider a map ϕ as in the hypothesis of Section 2.4, and set $K_\phi = \ker(^t\phi)$, $I_\phi = \mathrm{Im}(^t\phi)$, $\mathcal{L}_\phi = \mathrm{Cok}(^t\phi)$.

Lemma 8. *Fix the hypothesis as in Section 2.4. Then we have:*

$$\mathrm{H}^1(\mathbf{P}(V), K_\phi \otimes \Omega_{\mathbf{P}(V)}(2)) = 0, \qquad \textit{for all } m \geq 3.$$

Proof. Recall the exact sequences:

(5.1) $0 \to I_\phi \to U^* \otimes \mathscr{O}_{\mathbf{P}(V)} \to \mathscr{L}_\phi \to 0,$

(5.2) $0 \to K_\phi \to \mathcal{T}_{\mathbf{P}(V)}(-2) \to I_\phi \to 0.$

We consider the dual Euler sequence on $\mathbf{P}(V)$, twisted by $K_\phi(2)$, and take global sections. We obtain an inclusion:

$$\mathrm{H}^0(\mathbf{P}(V), K_\phi \otimes \Omega_{\mathbf{P}(V)}(2)) \subset V^* \otimes \mathrm{H}^0(\mathbf{P}(V), K_\phi(1)).$$

Note that, taking cohomology of (5.2) and (5.1) twisted by $\mathscr{O}_{\mathbf{P}(V)}(1)$ we obtain, respectively, a map $V \to \mathrm{H}^0(\mathbf{P}(V), I_\phi(1))$ and a map $\mathrm{H}^0(\mathbf{P}(V), I_\phi(1)) \to U^* \otimes V^*$. The composition of these two maps is the linear map f_ϕ in (2.9), which is injective in our hypothesis. Therefore we have $\mathrm{H}^0(\mathbf{P}(V), K_\phi(1)) = 0$ and we deduce the vanishing:

(5.3) $\mathrm{H}^0(\mathbf{P}(V), K_\phi \otimes \Omega_{\mathbf{P}(V)}(2)) = 0.$

Tensoring the exact sequence (5.2) by $\Omega_{\mathbf{P}(V)}(2)$, since $\mathrm{H}^0(\mathbf{P}(V), \mathcal{T}_{\mathbf{P}(V)} \otimes \Omega_{\mathbf{P}(V)}) \cong \mathbf{k}$ and $\mathrm{H}^1(\mathbf{P}(V), \mathcal{T}_{\mathbf{P}(V)} \otimes \Omega_{\mathbf{P}(V)}) = 0$, using (5.3) we get the exact sequence:

$$0 \to \mathbf{k} \to \mathrm{H}^0(\mathbf{P}(V), I_\phi \otimes \Omega_{\mathbf{P}(V)}(2)) \to \mathrm{H}^1(\mathbf{P}(V), K_\phi \otimes \Omega_{\mathbf{P}(V)}(2)) \to 0.$$

In order to conclude the proof of the lemma, we have thus to prove:

(5.4) $\mathrm{h}^0(\mathbf{P}(V), I_\phi \otimes \Omega_{\mathbf{P}(V)}(2)) = 1.$

The rest of the proof is devoted to show the above equality. First note that, tensoring the dual Euler sequence on $\mathbf{P}(V)$ by the ideal sheaf $\mathcal{I}_X(2)$ and taking cohomology, we obtain:

(5.5) $\mathrm{H}^i(\mathbf{P}(V), \mathcal{I}_X \otimes \Omega_{\mathbf{P}(V)}(2)) = 0, \qquad \text{for } i = 0, 1.$

Indeed, the vanishing $\mathrm{H}^0(\mathbf{P}(V), \mathcal{I}_X(2)) = 0$ (see our hypothesis (2.10)) implies $\mathrm{H}^0(\mathbf{P}(V), \mathcal{I}_X \otimes \Omega_{\mathbf{P}(V)}(2)) = 0$, while $\mathrm{H}^1(\mathbf{P}(V), \mathcal{I}_X(1)) = 0$ (see [4, Proposition 1]) implies $\mathrm{H}^1(\mathbf{P}(V), \mathcal{I}_X \otimes \Omega_{\mathbf{P}(V)}(2)) = 0$.

We will use the equality (5.5) in order to prove (5.4). We have shown in Lemma 3 that the sheaf $\mathscr{L}_\phi$ is the extension by zero of the line bundle $q_*(p^*(\mathscr{O}_Y(1)))$ on X. So, restricting (5.1) to X we obtain an exact commutative diagram:

(5.6)

where the sheaf J is defined as the kernel of the map $U^* \otimes \mathscr{O}_X \to q_*(p^*(\mathscr{O}_Y(1)))$, so that it is supported on X. We note that, applying p_*q^* to the bottom row of

the above diagram, we obtain the dual twisted Euler exact sequence restricted to Y, so that we have an identification:

$$(5.7) \qquad p_*(q^*(J)) \cong \Omega_{\mathbf{P}(U)|Y}(1).$$

Tensoring the leftmost column of (5.6) by $\Omega_{\mathbf{P}(V)}(2)$ and taking cohomology, in view of (5.5) we find:

$$\mathrm{H}^0(\mathbf{P}(V), I_\phi \otimes \Omega_{\mathbf{P}(V)}(2)) \cong \mathrm{H}^0(X, J \otimes \Omega_{\mathbf{P}(V)}(2)).$$

So our final goal is now to show:

$$\mathrm{h}^0(X, J \otimes \Omega_{\mathbf{P}(V)}(2)) = 1.$$

In order to prove the above equality, we tensor the lowest row of diagram (5.6) by $\Omega_{\mathbf{P}(V)}(2)$ and we use the dual Euler sequence on $\mathbf{P}(V)$ twisted by $\mathscr{O}_{\mathbf{P}(V)}(2)$. We get the following exact commutative diagram :
(5.8)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J \otimes \Omega_{\mathbf{P}(V)}(2) & \longrightarrow & U^* \otimes \Omega_{\mathbf{P}(V)}(2) \otimes \mathscr{O}_X & \longrightarrow & \mathscr{L}_\phi \otimes \Omega_{\mathbf{P}(V)}(2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & J \otimes V^* \otimes \mathscr{O}_{\mathbf{P}(V)}(1) & \longrightarrow & U^* \otimes V^* \otimes \mathscr{O}_X(1) & \longrightarrow & \mathscr{L}_\phi \otimes V^* \otimes \mathscr{O}_X(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J \otimes \mathscr{O}_{\mathbf{P}(V)}(2) & \longrightarrow & U^* \otimes \mathscr{O}_X(2) & \longrightarrow & \mathscr{L}_\phi \otimes \mathscr{O}_X(2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Because the sheaf $J \otimes \Omega_{\mathbf{P}(V)}(2)$ is supported on X, we have the natural isomorphisms:

$$\mathrm{H}^0(\mathbf{P}(V), J \otimes \Omega_{\mathbf{P}(V)}(2)) \cong \mathrm{H}^0(X, q^*(J \otimes \Omega_{\mathbf{P}(V)})(2))) \cong$$
$$\cong \mathrm{H}^0(Y, p_* q^*(J \otimes \Omega_{\mathbf{P}(V)}(2))).$$

Recall that our aim is to show that the above vector space is one-dimensional. To show this, in (5.8) we pull-back by q and push-forward by p. By the argument of Lemma 7, we can write $p_*(q^*(\mathscr{L}_\phi \otimes \Omega_{\mathbf{P}(V)}(2)))$ as $\mathrm{Im}(\delta)$, where δ is defined by (4.5). Using (5.7), we can thus write the following exact commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_{\mathbf{P}(U)|Y} \otimes \mathrm{Im}(\delta) & \longrightarrow & U^* \otimes \mathrm{Im}(\delta)(-1) & \longrightarrow & \mathrm{Im}(\delta) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_{\mathbf{P}(U)|Y} \otimes V^* \otimes \mathcal{E}(1) & \longrightarrow & U^* \otimes V^* \otimes \mathcal{E} & \longrightarrow & V^* \otimes \mathcal{E}(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_{\mathbf{P}(U)|Y} \otimes \mathrm{Sym}^2 \mathcal{E}(1) & \longrightarrow & U^* \otimes \mathrm{Sym}^2 \mathcal{E}(1) & \longrightarrow & \mathrm{Sym}^2 \mathcal{E}(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

This gives:

$$p_*(q^*(J \otimes \Omega_{\mathbf{P}(V)}(2))) \cong \mathrm{Im}(\delta) \otimes \Omega_{\mathbf{P}(U)|Y}.$$

So we obtain the following isomorphism:

$$\mathrm{H}^0(X, J \otimes \Omega_{\mathbf{P}(V)}(2)) \cong \mathrm{H}^0(Y, \mathrm{Im}(\delta) \otimes \Omega_{\mathbf{P}(U)|Y}).$$

We have thus reduced the problem to show:

$$\mathrm{h}^0(Y, \mathrm{Im}(\delta) \otimes \Omega_{\mathbf{P}(U)|Y}) = 1.$$

We use now Claim 2 and we twist (4.6) by $\Omega_{\mathbf{P}(U)}$. Taking cohomology we obtain $\mathrm{H}^p(Y, \mathcal{F} \otimes \Omega_{\mathbf{P}(U)}(1)) = 0$ for $p = 0, 1$ since $\mathrm{H}^p(Y, \Omega_{\mathbf{P}(U)}(-1)) = 0$ and $\mathrm{H}^p(Y, \Omega_{\mathbf{P}(U)}(1)) = 0$. In view of the vanishing just obtained, tensoring (4.7) by $\Omega_{\mathbf{P}(U)}$ and taking cohomology, we get, for $m \geq 4$:

$$\mathrm{h}^0(Y, \mathrm{Im}(\delta) \otimes \Omega_{\mathbf{P}(U)|Y}) = \mathrm{h}^1(Y, \Omega_{\mathbf{P}(U)|Y}) = 1.$$

In order to complete the proof of the lemma we need to take care of the case $m = 3$.

Let us use the notation of the proof of Lemma 2. For every m we write the exact sequence (5.2)

$$0 \to K_m \to \mathcal{T}_{\mathbf{P}(V)}(-2) \to I_m \to 0.$$

The leftmost exact sequence in the exact diagram (3.2)

$$0 \to \mathcal{I}_{X_m} \to I_m \to I_{m-1} \to 0$$

induces, using the snake lemma, the following exact sequence

$$0 \to K_m \to K_{m-1} \to \mathcal{I}_m \to 0$$

which for $m = 4$ gives

(5.9) $$0 \to K_4 \to K_3 \to \mathcal{I}_4 \to 0.$$

Tensoring (5.9) with $\Omega_{\mathbf{P}(V)}(2)$ and taking cohomology we see that $\mathrm{H}^1(\mathbf{P}(V), K_3 \otimes \Omega_{\mathbf{P}(V)}(2)) = 0$ since we have already proved $\mathrm{H}^1(\mathbf{P}(V), K_4 \otimes \Omega_{\mathbf{P}(V)}(2)) = 0$ and $\mathrm{H}^1(\mathbf{P}(V), \mathcal{I}_X \otimes \Omega_{\mathbf{P}(V)}(2)) = 0$ by (5.5).   □

Lemma 9. *Let $m, k \geq 3$ be integers, V and U be vector spaces of dimension respectively $2k$ and m. Let $\phi_1, \phi_2 : U \otimes \mathcal{O}_{\mathbf{P}(V)} \to \Omega_{\mathbf{P}(V)}(2)$ be two morphisms such that $X_1 = X_{\phi_1}$ and $X_2 = X_{\phi_2}$ are smooth. Assume that the hypothesis of Section 2.4 are satisfied at least by the morphism ϕ_2. If the two sheaves $\mathscr{L}_1 = \mathrm{Cok}({}^{\mathrm{t}}\phi_1)$ and $\mathscr{L}_2 = \mathrm{Cok}({}^{\mathrm{t}}\phi_2)$ are isomorphic, then there is an invertible matrix $\Phi \in \mathsf{GL}(U^*)$ and a nonzero scalar λ such that the following diagram commutes:*

(5.10)
$$
\begin{array}{ccc}
\mathcal{T}_{\mathbf{P}(V)}(-2) & \xrightarrow{\ {}^{\mathrm{t}}\phi_1\ } & U^* \otimes \mathcal{O}_{\mathbf{P}(V)} \\
\Big\| {\scriptstyle \lambda\,\mathsf{id}} & & \Big\downarrow {\scriptstyle \Phi} \\
\mathcal{T}_{\mathbf{P}(V)}(-2) & \xrightarrow{\ {}^{\mathrm{t}}\phi_2\ } & U^* \otimes \mathcal{O}_{\mathbf{P}(V)}
\end{array}
$$

Proof. Consider the composition σ of the surjection $U^* \otimes \mathcal{O}_{\mathbf{P}(V)} \to \mathscr{L}_1$ with the isomorphism $\mathscr{L}_1 \to \mathscr{L}_2$. In order to lift σ to a map $\Phi : U^* \otimes \mathcal{O}_{\mathbf{P}(V)} \to U^* \otimes \mathcal{O}_{\mathbf{P}(V)}$, we have to check the vanishing of the group:

$$\mathrm{Ext}^1_{\mathbf{P}(V)}(\mathcal{O}_{\mathbf{P}(V)}, I_{\phi_2}) \cong \mathrm{H}^1(\mathbf{P}(V), I_{\phi_2}).$$

Since this is guaranteed by Lemma 2, we get the map Φ. The kernel of Φ is isomorphic to a direct sum of copies of $\mathcal{O}_{\mathbf{P}(V)}$. Moreover, since the map $\mathscr{L}_1 \to \mathscr{L}_2$ is an isomorphism, $\ker(\Phi)$ fits as a sub-sheaf of I_{ϕ_1}, and thus provides a nonzero

element in $\mathrm{Hom}_{\mathbf{P}(V)}(\mathscr{O}_{\mathbf{P}(V)}, I_{\phi_1}) \cong \mathrm{H}^0(\mathbf{P}(V), I_{\phi_1})$. But this group is zero by Lemma 2, so Φ is an isomorphism.

So we have an induced isomorphism $I_{\phi_1} \to I_{\phi_2}$ and we consider the composition σ' of the projection $\mathcal{T}_{\mathbf{P}(V)}(-2) \to I_{\phi_1}$ with this isomorphism. We would like to lift σ' to an automorphism Φ' of $\mathcal{T}_{\mathbf{P}(V)}(-2)$. So we have to check the vanishing of:

$$\mathrm{Ext}^1_{\mathbf{P}(V)}(\mathcal{T}_{\mathbf{P}(V)}(-2), K_{\phi_2}) \cong \mathrm{H}^1(\mathbf{P}(V), K_{\phi_2} \otimes \Omega_{\mathbf{P}(V)}(2)).$$

This is provided by Lemma 8, so we have Φ'. Clearly the map Φ' is not zero, hence it is a (nonzero) multiple of the identity since $\mathcal{T}_{\mathbf{P}(V)}$ is a simple sheaf. $\qquad\square$

Up to multiplying ${}^{\mathrm{t}}\phi_2$ by a nonzero scalar, we may assume that Φ' is in fact the identity.

of the main theorem. Let us show that the map $\rho : \mathbf{G}(m, \wedge^2 V^*) \to \mathcal{H}_m(V)$ is generically injective. By contradiction, we take two maps ϕ_1 and ϕ_2 satisfying the hypothesis of Section 2.4, and such that $X_{\phi_1} = X_{\phi_2} = X$. Then the scroll X is smooth, and the maps ϕ_1 and ϕ_2 define two line bundles on X. Namely, the cokernel sheaves $\mathscr{L}_i = \mathrm{Cok}({}^{\mathrm{t}}\phi_i)$ satisfy $\mathscr{L}_i \cong q_*(p^*(\mathscr{M}_i))$ for some line bundles $\mathscr{M}_i$ on the base Y of the scroll X. But by Lemma 3, both line bundles $\mathscr{M}_1$ and $\mathscr{M}_2$ are isomorphic to $\mathscr{O}_Y(1)$. So we have an isomorphism $\mathscr{L}_1 \cong \mathscr{L}_2$, and we can thus apply Lemma 9. In view of diagram (5.10), the maps ${}^{\mathrm{t}}\phi_1$ and ${}^{\mathrm{t}}\phi_2$ are thus conjugate by an isomorphism $\Phi \in \mathsf{GL}(m, \mathbf{k})$. Therefore, the maps ϕ_1 and ϕ_2 correspond to the same point in the Grassmann variety $\mathbf{G}(m, \wedge^2 V^*)$. This proves that ρ is generically injective.

To complete the proof for $m \geq 4$, recall that we have computed the dimension of $\mathrm{H}^0(X, \mathcal{N})$ in Theorem 1. For $m \geq 4$, this equals the dimension of $\mathbf{G}(m, \wedge^2 V^*)$. This implies that the map ρ is dominant onto its image, and in fact a birational map of $\mathbf{G}(m, \wedge^2 V^*)$ onto $\mathcal{H}_m(V)$ for $m \geq 4$.

Let us now discuss the case $m = 3$. We have proved that ρ is birational onto its image. Thus an open dense subset of the image of ρ is immersed in the subscheme $\mathcal{P} \subset \mathcal{H}_3(V)$ whose general element consists of a general smooth plane curve Y of degree k equipped with a general stable rank-2 vector bundle $\mathcal{E}$, with $c_1(\mathcal{E}) = K_Y + 2H_Y$ (see [5]).

In order to finish the proof, we check now that $\mathcal{P}$ is a subscheme of $\mathcal{H}_3(V)$ whose dimension equals $3(k(2k - 1) - 3) = 3(2k^2 - k - 3) = \dim(\mathbf{G}(3, \wedge^2 V^*))$. To do this, let Y be a general projective curve of genus $g = \binom{k-1}{2}$, equipped with a general stable rank-2 bundle of degree $k(k - 1)$ defined on Y. Then $\mathbb{P}(\mathcal{E})$ lies in $\mathcal{H}_3(V)$ (see for instance [8]). The condition that the curve is planar is of codimension $k^2 - 6k + 8$ in $\mathcal{H}_3(V)$. Moreover, the rank-2 bundle $\mathcal{E}(-H_Y)$ should have determinant equal to K_Y, that is it should lie on the fiber of K_Y under the determinant map $\det : \mathscr{M}_Y(2, 2g - 2) \to \mathrm{Pic}(Y)$, where $\mathscr{M}_Y(2, 2g - 2)$ denotes the moduli space of stable rank-2 bundles of degree $2g - 2$ on Y. This is a condition of codimension $\dim(\mathrm{Pic}(Y)) = \binom{k-1}{2}$, so $\mathcal{P}$ has codimension $3/2k^2 - 15/2k + 9$ in $\mathcal{H}_3(V)$. Since $\dim(\mathcal{H}_3(V)) = 3k(5k - 7)/2$ by Theorem 1, this implies that $\mathcal{P}$ has dimension $3(2k^2 - k - 3)$. This completes the proof. $\qquad\square$

Our theorem asserts that ρ is generically injective onto the set of Palatini scrolls. However, one could consider the following related question. Namely, we consider the pfaffian map which associates to $\phi \in W = U^* \otimes \wedge^2 V^*$ the pfaffian of M_ϕ as an element of $|\mathscr{O}_{\mathbf{P}(U)}(k)|$. This factors through a map $\overline{\mathrm{Pf}}$ defined on the quotient of

W by $\mathsf{GL}(V)$ acting by congruence. Beauville proved in [5] that $\overline{\mathrm{Pf}}$ is generically injective for $m = 6$, $k = 3$.

Problem 3. Is the map $\overline{\mathrm{Pf}}$ generically injective for $m \geq 7$, $k \geq 3$? For $m = 6$, $k \geq 4$? For $m = 5$, $k \geq 6$? For $m = 4, k \geq 16$?

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