A SIMPLE VANISHING THEOREM FOR TWISTED HOLOMORPHIC FORMS ON HERMITIAN SYMMETRIC VARIETIES

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Abstract. We find an upper bound for the degree of nonvanishing cohomology of twisted holomorphic q-forms on an irreducible Hermitian symmetric variety X as a function of q depending linearly on the twist, the dimension and the index of X, making use of simple Euclidean geometry.

1. Introduction

Let $X = \mathsf{G}/\mathsf{P}$ be an irreducible Hermitian symmetric variety embedded equivariantly into a projective space $\mathbb{P}(V)$ by a very ample line bundle $\mathcal{O}_X(1)$ and consider the bundle of twisted holomorphic q-forms $\Omega^q_X(\ell) = \Omega^q_X \otimes \mathcal{O}_X(1)^{\otimes \ell}$. Some vanishing theorems for the cohomology of $\Omega^q_X(\ell)$ were first obtained by Bott in [Bot57] in the case $X \simeq \mathbb{P}^n$ and extended to Grassmannians by Le Potier in [LP75] and to quadrics by Shiffman and Sommese in [SS85].

Making use of work by Kostant [Kos61], Snow [Sno86b] in [Sno86a], developed an algorithm to determine wether a given cohomology group $\mathrm{H}^p(X, \Omega^q_X(\ell))$ vanishes for any X. These results were extended by Manivel and Snow to arbitrary homogeneous varieties in [MS96]. Applications to ample vector bundles on any projective variety can be found in [Man96], [Man97] and [Cha03].

However Snow's algorithm is of combinatoric nature and leads to nontrivial computations even on simple examples. In this paper we present a graphic and geometric approach to the problem, making use of Bott theorem and Kostant's work plus undergraduate Euclidean geometry. We find an upper bound for the degree of nonvanishing cohomology $\mathrm{H}^p(X, \Omega^q_X(\ell))$ as a function of q, ℓ depending linearly on the dimension and the index of X.

For basic definitions and results on homogeneous varieties and representations we refer to [FH91], [Ott95], [LM03], [Tit67].

We would like to thank Giorgio Ottaviani for suggesting us these ideas.

2. A vanishing theorem for $\mathrm{H}^p(X, \Omega^q_X(\ell))$

In this section we let X be an irreducible Hermitian symmetric variety i.e. a rational homogeneous variety $X = \mathsf{G}/\mathsf{P}$ where G is a simple algebraic group over $\mathbb{C}$ and $\mathsf{P} = \mathsf{P}(\alpha_k)$ is a parabolic subgroup of G such that Ω^1_X is an irreducible P-module. Let $\Delta = \{\alpha_1 \dots \alpha_n\}$ be the set of fundamental roots of $\mathfrak{g} = \mathrm{Lie}(\mathsf{G})$, $\Lambda = \{\lambda_1 \dots \lambda_n\}$ be

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the associated set of fundamental weights, $(-,-)$ be the Killing form on $\mathfrak{g}$ and Φ^+ be the set of positive roots of $\mathfrak{g}$. We have $\mathrm{Pic}(X) \cong \mathbb{Z} = \mathbb{Z} \cdot H$ and $-K_X = \iota_X H$ where ι_X is a positive integer called the *index* of X. Define also $g = \sum \lambda_i$. Finally, for a vector μ in the weight lattice of $\mathfrak{g}$ denote by E_μ the homogeneous bundle associated to the irreducible representation of maximal weight μ.

Theorem 2.1. *Let $X = \mathsf{G}/\mathsf{P}(\alpha_k)$ be an irreducible Hermitian symmetric variety and let l be an integer. Define*

$$p_{\max} = \max\{p \mid \exists q \text{ such that } \mathrm{H}^p(\Omega^q_X(\ell)) \neq 0\}$$

If $\ell \leq 0$ then $p_{\max} = 0$. If $\ell > \iota_X$ then $p_{\max} = \dim(X)$. If $0 < \ell \leq \iota_X$ then

$$p_{\max} \leq \frac{\iota_X - \ell}{2\,\iota_X} \dim(X)$$

3. Proofs and figures

Lemma 3.1. *Let $X = \mathsf{G}/\mathsf{P}(\alpha_k)$ be an irreducible Hermitian symmetric variety. Then the expression of any weight of Ω^1_X in the fundamental roots contains $-\alpha_k$ with multiplicity* 1.

Proof. Since T_X is the bundle associated to $\mathfrak{g}/\mathfrak{p}(\alpha_k)$, all the weights of T_X are precisely the roots of $\mathfrak{g}$ that contain α_k with multiplicity at least 1. So any weight of Ω_X contains $-\alpha_k$ with multiplicity at least 1.
Now since Ω^1_X is indecomposable, it is given by an irreducible representation of $ss(\mathfrak{p}(\alpha_k))$, twisted by a character $\ell\lambda_k$ and $-\alpha_k$ belongs to the weight space of Ω^1_X. Moreover the root system of $ss(\mathfrak{p}(\alpha_k))$ is generated by $\alpha_1, \ldots, \hat{\alpha_k}, \ldots, \alpha_n$. So any weight having a coefficient grater than 1 for $-\alpha_k$ does not lie in the $ss(\mathfrak{p}(\alpha_k))$-orbit of $-\alpha_k$. □

Lemma 3.2. *Let $X = \mathsf{G}/\mathsf{P}(\alpha_k)$ be an irreducible Hermitian symmetric variety, with $\iota_X = \iota$ and $\dim_X = d$. Then the maximal weights of Ω^q_X lie*

- *on the sphere $S(0)$ centered in $-g$ and passing through* 0.
- *on the hyperplane orthogonal to λ_k passing through the point $-\frac{q\iota}{d}\lambda_k$*

Proof. Denote by H_α the hyperplane orthogonal to $\alpha \in \Phi$ and passing through $-g$ and by w_α the reflection in the weight lattice of $\mathfrak{g}$ along the hyperplane H_α. The maximal weights of Ω^q_X are then obtained as $w_\alpha(\mu)$ where $\alpha \in \Phi^+$ contains α_k in its expression with respect to Δ and μ is a maximal weight of Ω^{q-1}_X. All such weights then lie on the sphere centered in $-g$ and since $\Omega^0_X \cong \mathcal{O}$ the sphere passes through 0. Now $(\alpha_i, \lambda_k) = 0$ for $i \neq k$, so the coefficient (α, μ) is a constant c for all such α's by lemma 3.1. The weights of Ω^q_X then lie on the hyperplane orthogonal to λ_k. Notice that after d consecutive reflections 0 is mapped to $\Omega^d_X \cong K_X$ whose weight is $-\iota\,\lambda_k$, so the constant is $c = \iota/d$ as required. □

Proof of theorem 2.1. Recall that $E_\mu \otimes \mathcal{O}_X(\ell) \cong E_{\mu+\ell\lambda_k}$. The maximal weights of $\Omega^q_X(\ell)$ then lie on the sphere $S(\ell)$ centered in $-g + \ell\lambda_k$ and of radius $|g|$. Consider the intersection of the two spheres $S(\ell)$ and $S(0)$. As one can read off from figure 1, or 2 these two spheres meet along the hyperplane H_z orthogonal to λ_k and passing

through $z = \frac{-\iota_X+\ell}{2}\lambda_k$ i.e. the dashed line in figure 1 or 2. The figures refer respectively to the weight lattice of $\mathbb{P}^2$ and Q_3.
From the proof of lemma 3.2 recall the constant $c = \iota_X / \dim(X)$. Whenever $p > \frac{\iota_X-\ell}{2c}$, the vertex of a p-th Bott Chamber μ_p (i.e. a maximal weight of Ω^p_X) lies beyond the hyperplane H_z i.e. $(\mu_p, \lambda_k) < (z, \lambda_k)$. Then, for $p > \frac{\iota_X-\ell}{2c}$ any weight of $\Omega^q_X(\ell)$ meets no Bott chamber of order p, so $\mathrm{H}^p(\Omega^q_X(\ell)) = 0$. So the statement is proved. □

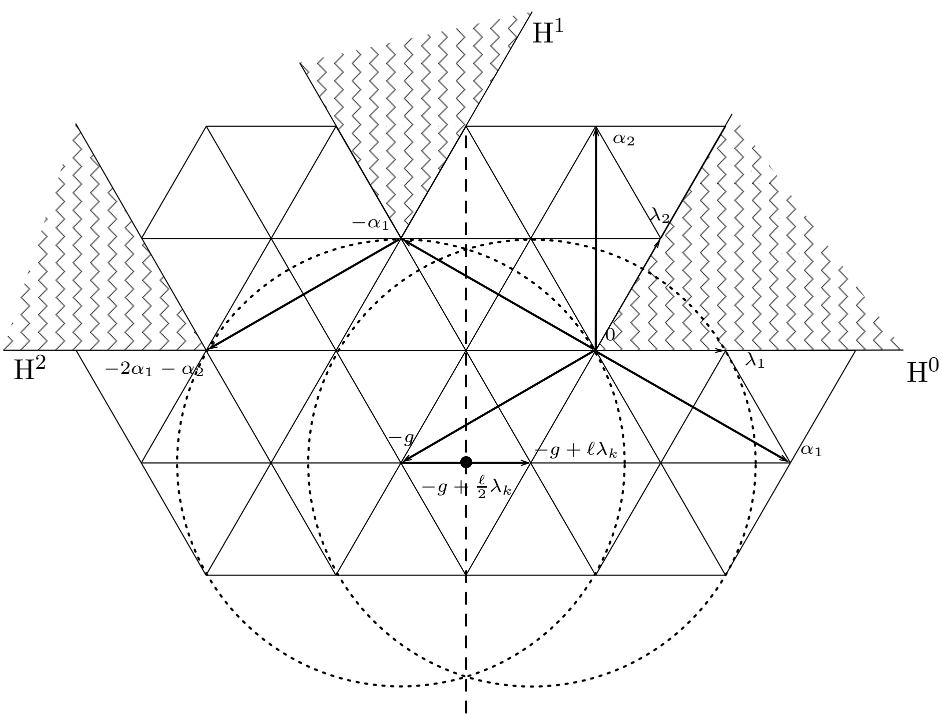

FIGURE 1. Twisting weights of Ω^q over $\mathbb{P}^2$ by $\mathcal{O}(\ell)$

Since irreducible Hermitian symmetric varieties are completely classified, see for instance [Ram66], one can comprehend the possible cases in the following corollary.

Corollary 3.3. *Let X be in irreducible Hermitian symmetric variety, let $0 < \ell \leq \iota_X$ be an integer and $\mathrm{p}_{\max}$ be defined as in theorem 2.1. Then one of the following alternatives takes place*

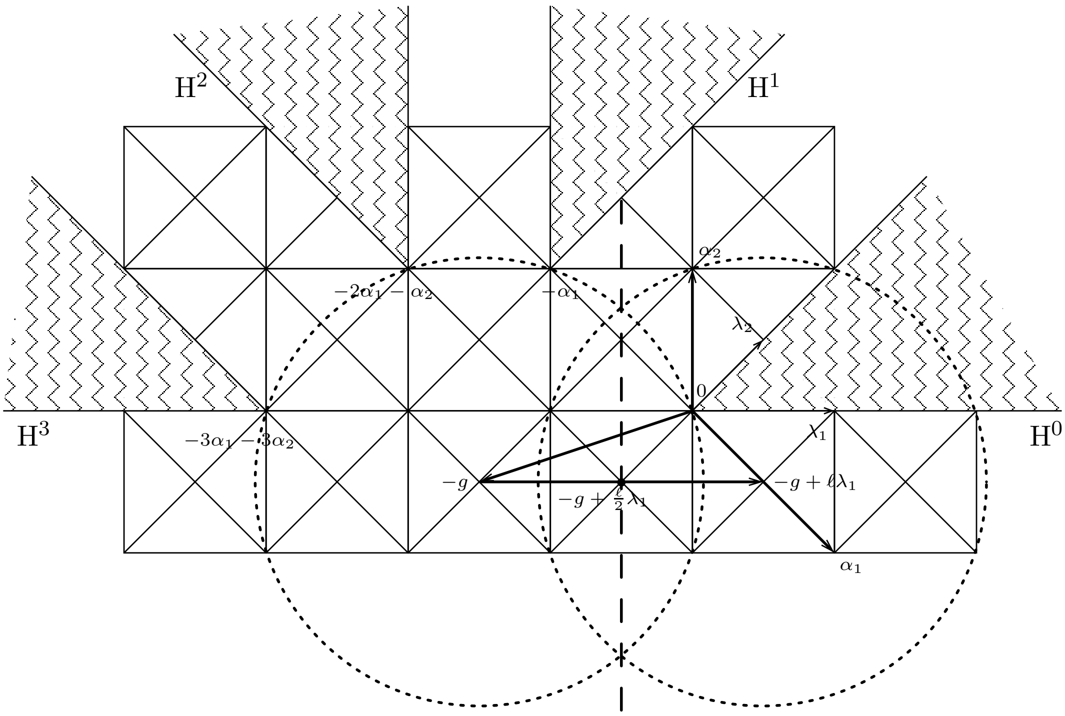

FIGURE 2. Twisting weights of Ω^q over Q_3 by $\mathcal{O}(\ell)$

Group	X	$\dim(X)$	ι_X	$p_{\max} \leq$
A_n	$\mathsf{SL}(n+1)/\mathsf{P}(\alpha_{k+1})$	$(n-k)(k+1)$	$n+1$	$\frac{(n-k)(k+1)(n+1-\ell)}{2\,(n+1)}$
B_n	$\mathsf{Spin}(2n+1)/\mathsf{P}(\alpha_1)$	$2n-1$	$2n-1$	$\frac{2n-1-\ell}{2}$
C_n	$\mathsf{Sp}(2n)/\mathsf{P}(\alpha_n)$	$\frac{n(n+1)}{2}$	$n+1$	$\frac{n(n+1-\ell)}{4}$
D_n	$\mathsf{Spin}(2n)/\mathsf{P}(\alpha_1)$	$2n-2$	$2n-2$	$\frac{2n-2-\ell}{2}$
D_n	$\mathsf{Spin}(2n)/\mathsf{P}(\alpha_n)$	$\frac{n(n-1)}{2}$	$2n-2$	$\frac{n(2n-2-\ell)}{8}$
E_6	$\mathsf{E}_6/\mathsf{P}(\alpha_6)$	16	12	$\frac{2(12-\ell)}{3}$
E_7	$\mathsf{E}_7/\mathsf{P}(\alpha_7)$	27	18	$\frac{3(18-\ell)}{4}$

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