A SIMPLE VANISHING THEOREM FOR TWISTED
HOLOMORPHIC FORMS ON HERMITIAN SYMMETRIC
VARIETIES

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Abstract. We find an upper bound for the degree of nonvanishing cohomology of
twisted holomorphic \(q\)-forms on an irreducible Hermitian symmetric variety \(X\) as
a function of \(q\) depending linearly on the twist, the dimension and the index of \(X\),
making use of simple Euclidean geometry.

1. Introduction

Let \(X = G/P\) be an irreducible Hermitian symmetric variety embedded equivariantly
into a projective space \(\mathbb{P}(V)\) by a very ample line bundle \(\mathcal{O}_X(1)\) and consider
the bundle of twisted holomorphic \(q\)-forms \(\Omega^q_X(\ell) = \Omega^q_X \otimes \mathcal{O}_X(1)^{\otimes \ell}\). Some vanishing
theorems for the cohomology of \(\Omega^q_X(\ell)\) were first obtained by Bott in [Bot57] in the
case \(X \simeq \mathbb{P}^n\) and extended to Grassmannians by Le Potier in [LP75] and to quadrics
by Shiffman and Sommese in [SS85].

Making use of work by Kostant [Kos61], Snow [Sno86b] in [Sno86a], developed an
algorithm to determine whether a given cohomology group \(H^p(X, \Omega^q_X(\ell))\) vanishes for
any \(X\). These results were extended by Manivel and Snow to arbitrary homogeneous
varieties in [MS96]. Applications to ample vector bundles on any projective variety
can be found in [Man96], [Man97] and [Cha03].

However Snow’s algorithm is of combinatoric nature and leads to nontrivial com-
putations even on simple examples. In this paper we present a graphic and geometric
approach to the problem, making use of Bott theorem and Kostant’s work plus under-
graduate Euclidean geometry. We find an upper bound for the degree of nonvanishing
cohomology \(H^p(X, \Omega^q_X(\ell))\) as a function of \(q, \ell\) depending linearly on the dimension
and the index of \(X\).

For basic definitions and results on homogeneous varieties and representations we
refer to [FH91], [Ott95], [LM03], [Tit67].

We would like to thank Giorgio Ottaviani for suggesting us these ideas.

2. A vanishing theorem for \(H^p(X, \Omega^q_X(\ell))\)

In this section we let \(X\) be an irreducible Hermitian symmetric variety i.e. a
rational homogeneous variety \(X = G/P\) where \(G\) is a simple algebraic group over \(\mathbb{C}\)
and \(P = P(\alpha_k)\) is a parabolic subgroup of \(G\) such that \(\Omega^1_X\) is an irreducible \(P\)-module.
Let \(\Delta = \{\alpha_1, \ldots, \alpha_n\}\) be the set of fundamental roots of \(\mathfrak{g} = \text{Lie}(G)\), \(\Lambda = \{\lambda_1, \ldots, \lambda_n\}\) be

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the associated set of fundamental weights, \((-,-)\) be the Killing form on \(g\) and \(\Phi^+\) be the set of positive roots of \(g\). We have \(\text{Pic}(X) \cong \mathbb{Z} = \mathbb{Z} \cdot H\) and \(-K_X = \iota_X H\) where \(\iota_X\) is a positive integer called the index of \(X\). Define also \(g = \sum \lambda_i\). Finally, for a vector \(\mu\) in the weight lattice of \(g\) denote by \(E_\mu\) the homogeneous bundle associated to the irreducible representation of maximal weight \(\mu\).

**Theorem 2.1.** Let \(X = G/P(\alpha_k)\) be an irreducible Hermitian symmetric variety and let \(l\) be an integer. Define
\[
 p_{\max} = \max \{ p \mid \exists g \text{ such that } H^p(\Omega_X^q(\ell)) \neq 0 \}
\]
If \(\ell \leq 0\) then \(p_{\max} = 0\). If \(\ell > \iota_X\) then \(p_{\max} = \dim(X)\). If \(0 < \ell \leq \iota_X\) then
\[
 p_{\max} \leq \frac{\iota_X - \ell}{2 \iota_X} \dim(X)
\]

### 3. Proofs and figures

**Lemma 3.1.** Let \(X = G/P(\alpha_k)\) be an irreducible Hermitian symmetric variety. Then the expression of any weight of \(\Omega_X^1\) in the fundamental roots contains \(-\alpha_k\) with multiplicity 1.

**Proof.** Since \(T_X\) is the bundle associated to \(g/p(\alpha_k)\), all the weights of \(T_X\) are precisely the roots of \(g\) that contain \(\alpha_k\) with multiplicity at least 1. So any weight of \(\Omega_X\) contains \(-\alpha_k\) with multiplicity at least 1.

Now since \(\Omega_X^1\) is indecomposable, it is given by an irreducible representation of \(ss(p(\alpha_k))\), twisted by a character \(\ell \lambda_k\) and \(-\alpha_k\) belongs to the weight space of \(\Omega_X^1\). Moreover the root system of \(ss(p(\alpha_k))\) is generated by \(\alpha_1, \ldots, \alpha_k, \ldots, \alpha_n\). So any weight having a coefficient grater than 1 for \(-\alpha_k\) does not lie in the \(ss(p(\alpha_k))\)-orbit of \(-\alpha_k\).

**Lemma 3.2.** Let \(X = G/P(\alpha_k)\) be an irreducible Hermitian symmetric variety, with \(\iota_X = \iota\) and \(\dim X = d\). Then the maximal weights of \(\Omega_X^q\) lie
- on the sphere \(S(0)\) centered in \(-g\) and passing through 0.
- on the hyperplane orthogonal to \(\lambda_k\) passing through the point \(-\frac{\iota_k}{\iota} \lambda_k\)

**Proof.** Denote by \(H_\alpha\) the hyperplane orthogonal to \(\alpha \in \Phi\) and passing through \(-g\) and by \(w_\alpha\) the reflection in the weight lattice of \(g\) along the hyperplane \(H_\alpha\). The maximal weights of \(\Omega_X^q\) are then obtained as \(w_\alpha(\mu)\) where \(\alpha \in \Phi^+\) contains \(\alpha_k\) in its expression with respect to \(\Delta\) and \(\mu\) is a maximal weight of \(\Omega_X^{q-1}\). All such weights then lie on the sphere centered in \(-g\) and since \(\Omega_X^q \cong \mathbb{O}\) the sphere passes through 0. Now \((\alpha_i, \lambda_k) = 0\) for \(i \neq k\), so the coefficient \((\alpha, \mu)\) is a constant \(c\) for all such \(\alpha\)’s by lemma 3.1. The weights of \(\Omega_X^q\) then lie on the hyperplane orthogonal to \(\lambda_k\). Notice that after \(d\) consecutive reflections 0 is mapped to \(\Omega_X^d \cong K_X\) whose weight is \(-\iota \lambda_k\), so the constant is \(c = \iota/d\) as required.

**Proof of theorem 2.1.** Recall that \(E_\mu \otimes \mathcal{O}_X(\ell) \cong E_{\mu + \ell \lambda_k}\). The maximal weights of \(\Omega_X^q(\ell)\) then lie on the sphere \(S(\ell)\) centered in \(-g + \ell \lambda_k\) and of radius \(|g|\). Consider the intersection of the two spheres \(S(\ell)\) and \(S(0)\). As one can read off from figure 1, or 2 these two spheres meet along the hyperplane \(H_z\) orthogonal to \(\lambda_k\) and passing
through \( z = \frac{-i_X + \ell}{2} \lambda_k \) i.e. the dashed line in figure 1 or 2. The figures refer respectively to the weight lattice of \( P^2 \) and \( Q_3 \).

From the proof of lemma 3.2 recall the constant \( c = \frac{i_X}{\dim(X)} \). Whenever \( p > \frac{i_X - \ell}{2c} \), the vertex of a \( p \)-th Bott Chamber \( \mu_p \) (i.e. a maximal weight of \( \Omega^p_X \)) lies beyond the hyperplane \( H_z \) i.e. \( (\mu_p, \lambda_k) < (z, \lambda_k) \). Then, for \( p > \frac{i_X - \ell}{2c} \) any weight of \( \Omega^q_X(\ell) \) meets no Bott chamber of order \( p \), so \( \mathbb{H}(\Omega^q_X(\ell)) = 0 \). So the statement is proved. □

\[ \begin{align*}
\text{Figure 1. Twisting weights of } \Omega^q \text{ over } P^2 \text{ by } O(\ell) 
\end{align*} \]

Since irreducible Hermitian symmetric varieties are completely classified, see for instance [Ram66], one can comprehend the possible cases in the following corollary.

**Corollary 3.3.** Let \( X \) be an irreducible Hermitian symmetric variety, let \( 0 < \ell \leq i_X \) be an integer and \( p_{\text{max}} \) be defined as in theorem 2.1. Then one of the following alternatives takes place
Figure 2. Twisting weights of $\Omega^q$ over $Q_3$ by $Ω(ℓ)$

<table>
<thead>
<tr>
<th>Group</th>
<th>$X$</th>
<th>$\dim(X)$</th>
<th>$\ell_X$</th>
<th>$p_{\text{max}} \leq$</th>
</tr>
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<tbody>
<tr>
<td>$A_n$</td>
<td>$\text{SL}(n+1)/\mathcal{P}(\alpha_{k+1})$</td>
<td>$(n-k)(k+1)$</td>
<td>$n+1$</td>
<td>$\frac{2(n+1)(n+1-\ell)}{2n-1}$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$\text{Spin}(2n+1)/\mathcal{P}(\alpha_1)$</td>
<td>$2n-1$</td>
<td>$2n-1$</td>
<td>$\frac{n(n+1)}{2}$</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$\text{Sp}(2n)/\mathcal{P}(\alpha_n)$</td>
<td>$\frac{n(n+1)}{2}$</td>
<td>$n+1$</td>
<td>$\frac{n(n+1-\ell)}{2}$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$\text{Spin}(2n)/\mathcal{P}(\alpha_1)$</td>
<td>$2n-2$</td>
<td>$2n-2$</td>
<td>$\frac{2n-2-\ell}{2}$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$\text{Spin}(2n)/\mathcal{P}(\alpha_n)$</td>
<td>$\frac{n(n-1)}{2}$</td>
<td>$2n-2$</td>
<td>$\frac{n(n-2-\ell)}{2}$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$E_6/\mathcal{P}(\alpha_6)$</td>
<td>16</td>
<td>12</td>
<td>$\frac{2(12-\ell)}{3}$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$E_7/\mathcal{P}(\alpha_7)$</td>
<td>27</td>
<td>18</td>
<td>$\frac{3(18-\ell)}{4}$</td>
</tr>
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</table>

References


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