RANK TWO ACM BUNDLES ON THE DEL PEZZO THREEFOLD WITH PICARD NUMBER 3

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Abstract. A del Pezzo threefold F with maximal Picard number is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In the present paper we completely classify locally free sheaves $\mathcal{E}$ of rank 2 such that $h^i\big(F, \mathcal{E}(t)\big) = 0$ for $i = 1, 2$ and $t \in \mathbb{Z}$. Such a classification extends similar results proved by E. Arrondo and L. Costa regarding del Pezzo threefolds with Picard number 1.

1. Introduction and Notation

Let k be an algebraically closed field of characteristic 0 and $\mathbb{P}^n$ the n-dimensional projective space over k. A well–known theorem of Horrocks (see [32] and the references therein) states that a locally free sheaf $\mathcal{E}$ on $\mathbb{P}^n$ splits as direct sum of invertible sheaves if and only if it has no intermediate cohomology, i.e. $h^i\big(\mathbb{P}^n, \mathcal{E}(t)\big) = 0$ for $0 < i < n$ and $t \in \mathbb{Z}$.

It is thus natural to ask for the meaning of such a vanishing on other kind of algebraic varieties F. Obviously such a vanishing makes sense only if a natural polarization is defined on F. For instance, if there is a natural embedding $F \subseteq \mathbb{P}^n$, then one can consider $\mathcal{O}_F(h) := \mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{O}_F$ and we can ask for locally free sheaves $\mathcal{E}$ on F such that $H^i_*\big(F, \mathcal{E}\big) := \bigoplus_{t\in\mathbb{Z}} H^i\big(F, \mathcal{E}(th)\big) = 0$ for $0 < i < \dim(F)$ which are called arithmetically Cohen-Macaulay (aCM for short) bundles. Obviously we are particularly interested in characterizing indecomposable aCM bundles, i.e. bundles of rank $r \geq 2$ which do not split as sum of invertible sheaves. Among aCM bundles, there are bundles $\mathcal{E}$ such that $H^0_*\big(F, \mathcal{E}\big)$ has the highest possible number of generators in degree 0. After [34] such bundles are simply called Ulrich bundles. Ulrich bundles have many good properties, thus their description is of particular interest.

There are a lot of classical and recent papers devoted to the aforementioned topics. For example, in the case of smooth quadrics and Grassmannians, there are some classical result generalizing the above Horrocks' criterion (see [1], [6], and [33]). More recently, in the same setup, several authors dealt with the case of hypersurfaces (see e.g. [26], [27], [28], [13], [29], [14], [12], [25]) and of Segre products (see [8]). Nevertheless the Ulrich property has been recently object of deep inspection (see, e.g. [10], [11], [16], [17]).

Another case which could be worth of a particular attention is the case of Fano and del Pezzo n–folds. We recall that a smooth n–fold F is Fano if its anticanonical sheaf ω_F^{-1} is ample (see [24] for results about Fano and del Pezzo varieties). The

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greatest positive integer r such that $\omega_F \cong \mathcal{O}_F(-rh)$ for some ample class $h \in \mathrm{Pic}(F)$ is called the index of F and one has $1 \leq r \leq n+1$. If $r = n-1$, then F is called del Pezzo. For such an F, the group $\mathrm{Pic}(F)$ is torsion–free so that $\mathcal{O}_F(h)$ is uniquely determined. By definition, the degree of F is the integer $d := h^n$.

Let us restrict to the case $n = 3$. It is well–known that $1 \leq d \leq 8$. When $d \geq 3$ the sheaf $\mathcal{O}_F(h)$ is actually very ample so h is the hyperplane class of a natural embedding $F \subseteq \mathbb{P}^{d+1}$. If $d = 8$, then F is $\mathbb{P}^3$, and $\mathcal{O}_F(h)$ gives the second Veronese embedding.

If $1 \leq d \leq 5$, the complete classification of aCM bundles of rank 2 on F can be found in [3], [5], [9], [26], [19], [4].

A fundamental hypothesis in all the aforementioned papers, both on hypersurfaces and on Fano threefolds, which considerably simplifies the proofs, is that the varieties always have Picard number $\varrho(F) := \mathrm{rk}(\mathrm{Pic}(F)) = 1$. Indeed in this case both c_1 and c_2 can be handled as integral numbers. In the cases we are interested in this is no longer possible.

When $d = 6$, then $F \subseteq \mathbb{P}^7$ is either the Segre product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, thus $\varrho(F) = 3$, or a hyperplane section of the Segre product $\mathbb{P}^2 \times \mathbb{P}^2 \subseteq \mathbb{P}^8$, thus $\varrho(F) = 2$. If $d = 7$, then F is the blow up of $\mathbb{P}^3$ at a point p embedded in $\mathbb{P}^8$ via the linear system of quadrics through p, thus again $\varrho(F) = 2$.

In the present paper we focus our attention on the del Pezzo threefold with the highest Picard number, namely the Segre embedding $F := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^7$. In [17] it was proved that F (and any other Segre product except $\mathbb{P}^1 \times \mathbb{P}^1$) is of wild representation type i.e. there are ℓ-dimensional families of non-isomorphic indecomposable aCM sheaves, for arbitrarily large $\ell \in \mathbb{Z}$.

Let $A(F)$ be the Chow ring of F, so that $A^r(F)$ denotes the set of cycles of codimension r. We have three different projections $\pi_i \colon F \to \mathbb{P}^1$ and we denote by h_i the pull–back in $A^1(F)$ of the class of a point in the i–th copy of $\mathbb{P}^1$. The exterior product morphism $A(\mathbb{P}^1) \otimes A(\mathbb{P}^1) \otimes A(\mathbb{P}^1) \to A(F)$ is an isomorphism (see [21], Example 8.3.7), thus we finally obtain

$$A(F) \cong \mathbb{Z}[h_1, h_2, h_3]/(h_1^2, h_2^2, h_3^2).$$

In particular $\mathrm{Pic}(F) \cong \mathbb{Z}^{\oplus 3}$, whence $\varrho(F) = 3$. Such an equality characterizes F among del Pezzo threefolds.

The aim of this paper is to classify the rank two aCM bundles on F. In Section 2 we first recall some general definitions and facts on locally free sheaves. The next three Sections, 3, 4 and 5, are devoted to the proof of our first main result.

Theorem A. *Let $\mathcal{E}$ be an indecomposable aCM bundle of rank 2 on F and let $c_1(\mathcal{E}) = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3$. Assume that $h^0\big(F, \mathcal{E}\big) \neq 0$ and $h^0\big(F, \mathcal{E}(-h)\big) = 0$ (we briefly say that $\mathcal{E}$ is initialized). Then:*

1. *the zero locus $E := (s)_0$ of a general section $s \in H^0\big(F, \mathcal{E}\big)$ has codimension 2 inside F;*
2. *either $0 \leq \alpha_i \leq 2$, $i = 1, 2, 3$, or $c_1 = h_1 + 2h_2 + 3h_3$ up to permutations of the h_i's;*
3. *if $c_1 = 2h$, or $c_1 = h_1 + 2h_2 + 3h_3$ up to permutations of the h_i's, then $\mathcal{E}$ is Ulrich.*

Going back to the classification given in [3] of indecomposable initialized aCM bundles of rank 2 on del Pezzo threefolds of degree $d = 3, 4, 5$, let us note two things. First, such a characterization can be easily generalized also to the cases

$d = 1, 2$, and the bound $0 \leq c_1 \leq 2$ remains true in these cases. Second, from such a viewpoint the statement of Theorem A can be read as follows: either c_1 satisfies the aforementioned "standard " bound, or $\mathcal{E}$ is a "sporadic"bundle.

In Section 6 we provide a complete classification of bundles satisfying the standard bound. Then we examine the sporadic case in Section 7. Finally, we will answer to the natural question of determining which intermediate cases are actually admissible in Section 8. We summarize this second main result in the following simplified statement (see Theorems 6.1, 6.6, 7.1, 8.1 for an expanded and detailed statement).

Theorem B. *There exists an indecomposable and initalized aCM bundle $\mathcal{E}$ of rank 2 on F with $c_1(\mathcal{E}) = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3$ if and only if $(\alpha_1, \alpha_2, \alpha_3)$ is one of the following, up to permutations:*

$$(0,0,0), \quad (2,2,2), \quad (1,2,3), \quad (0,0,1), \quad (1,1,1), \quad (1,2,2).$$

Moreover, denote by E the zero–locus of a general section of such an $\mathcal{E}$. Then

(1) if $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then E is a line and any line on F can be obtained in such a way;
(2) if $\alpha_1 = \alpha_2 = \alpha_3 = 2$, then E is an elliptic normal curve of degree 8 in $\mathbb{P}^7$;
(3) if $\alpha_i = i$ up to permutation, then E is a rational normal curve of degree 7 in $\mathbb{P}^7$;
(4) if $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$, then E is a line and any line on F can be obtained in such a way;
(5) if $\alpha_1 = \alpha_2 = \alpha_3 = 1$, then E is a, possibly reducible, reduced conic;
(6) if $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 2$, then E is a, possibly reducible, quintic with arithmetic genus 0.

In particular, for each line $E \subseteq F$ we show the existence of exactly four indecomposable non–isomorphic aCM bundles of rank 2 whose general section vanishes exactly along E.

The case of del Pezzo threefolds with Picard number 2 and the study of moduli spaces of aCM bundles of rank 2 on del Pezzo threefolds will be the object of forthcoming papers.

1.1. **Notation.** Let R be a Noetherian local ring. We denote by $d(R)$ its depth and by $\dim(R)$ its Krull dimension. R is Cohen–Macaulay (reps. Gorenstein) if and only if $d(R) = \dim(R)$ (reps. its injective dimension as R–module is finite). Let R be a Noetherian ring. We say that R is Cohen–Macaulay (resp Gorenstein) if its localization $R_{\mathfrak{M}}$ is Cohen–Macaulay (resp. Gorenstein) for all the maximal ideals $\mathfrak{M} \subseteq R$.

If X and Y are schemes over k and X is a closed subscheme of Y we will denote by $\mathcal{I}_{X|Y}$ the sheaf of ideals of X inside Y. The sheaves $\mathcal{C}_{X|Y} := \mathcal{I}_{X|Y}/\mathcal{I}^2_{X|Y}$ and its $\mathcal{O}_X$ dual $\mathcal{N}_{X|Y} := \check{\mathcal{C}}_{X|Y}$ are respectively called the conormal and the normal sheaf of X inside Y.

Let $X \subseteq \mathbb{P}^N$ be a subvariety, i.e. an integral closed subscheme. We set $\mathcal{O}_X(h) := \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_X$. X is arithmetically Cohen–Macaulay (resp. arithmetically Gorenstein) if its homogeneous coordinate ring is Cohen–Macaulay (resp. Gorenstein). In this case we will say that X is aCM (resp. aG).

The variety X is aCM if and only if the natural restriction maps $H^0\big(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t)\big) \to H^0\big(X, \mathcal{O}_X(th)\big)$ are surjective and $h^i\big(X, \mathcal{O}_X(th)\big) = 0$, $1 \leq i \leq \dim(X) - 1$. X is

aG if and only if it is aCM and α–subcanonical, i.e. its dualizing sheaf satisfies $\omega_X \cong \mathcal{O}_X(\alpha h)$ for some $\alpha \in \mathbb{Z}$.

A vector bundle on X is locally free sheaf on X of finite rank. If X is a scheme, then $A^r(X)$ denotes the Chow group of codimension r cycles on X up to rational equivalence and $A(X) := \bigoplus_{r=0}^{+\infty} A^r(X)$. By a vector bundle, we mean a coherent locally free sheaf. For all the other unmentioned definitions, notations and results see [24] and [23].

2. Some facts on aCM locally free sheaves

In what follows F will denote an aCM, integral and smooth subvariety of $\mathbb{P}^N$ of positive dimension n. Moreover $\mathcal{O}_F(h) := \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_F$.

Definition 2.1. Let $\mathcal{E}$ be a vector bundle on F. We say that $\mathcal{E}$ is arithmetically Cohen–Macaulay (aCM) if $H^i_*\big(F, \mathcal{E}\big) = 0$ for each $i = 1, \dots, n-1$.

If $\mathcal{E}$ is an aCM bundle, then the minimal number of generators $m(\mathcal{E})$ of $H^0_*\big(F, \mathcal{E}\big)$ as a module over the graded coordinate ring of F is $\mathrm{rk}(\mathcal{E})\deg(F)$ at most (e.g. see [10]). The aCM bundles for which the maximum is attained are worth of particular interest. They are called Ulrich bundles in the sequel. We recall the following definition (see [11], Definition 2.1 and Lemma 2.2: see also [10], Definition 3.4 and which is slightly different).

Definition 2.2. Let $\mathcal{E}$ be a vector bundle on F. We say that $\mathcal{E}$ is initialized if

$$\min\{\ t \in \mathbb{Z} \mid h^0\big(F, \mathcal{E}(th)\big) \neq 0\ \} = 0.$$

We say that $\mathcal{E}$ is Ulrich if it is initialized, aCM and $h^0\big(F, \mathcal{E}\big) = \mathrm{rk}(\mathcal{E})\deg(F)$.

Let $\mathcal{E}$ be an Ulrich bundle. On one hand we know that $m(\mathcal{E}) = \mathrm{rk}(\mathcal{E})\deg(F)$. On the other hand the generators of $H^0\big(F, \mathcal{E}\big)$ are minimal generators of $H^0_*\big(F, \mathcal{E}\big)$ due to the vanishing of $H^0\big(F, \mathcal{E}(-h)\big)$. We conclude that $\mathcal{E}$ is necessarily globally generated. Several other results are known for Ulrich bundles (e.g. see [18], [10], [11], [16]).

Assume now $n = 3$, and let $\mathcal{E}$ be a bundle of rank r on F. We denote by ω_i the Chern classes of the sheaf $\Omega^1_{F|k}$. In this case Riemann–Roch theorem is

$$\begin{aligned}\chi(\mathcal{E}) = -\frac{\mathrm{rk}(\mathcal{E})}{24}\omega_1\omega_2 + \frac{1}{6}(c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}) + 3c_3(\mathcal{E})) - \\ - \frac{1}{4}(\omega_1 c_1(\mathcal{E})^2 - 2\omega_1 c_2(\mathcal{E})) + \frac{1}{12}(\omega_1^2 c_1(\mathcal{E}) + \omega_2 c_1(\mathcal{E})).\end{aligned} \tag{1}$$

Let s be a global section of $\mathcal{E}$. In general its zero–locus $(s)_0 \subseteq F$ is either empty or its codimension is at most 2. Thus, in this second case, we can always write $(s)_0 = E \cup D$ where E has codimension 2 (or it is empty) and D has pure codimension 1 (or it is empty). In particular $\mathcal{E}(-D)$ has a section vanishing on E, thus we can consider its Koszul complex

$$0 \longrightarrow \mathcal{O}_F(D) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|F}(c_1 - D) \longrightarrow 0. \tag{2}$$

Moreover we also have the following exact sequence

$$0 \longrightarrow \mathcal{I}_{E|F} \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{O}_E \longrightarrow 0. \tag{3}$$

The above construction can be reversed. Indeed on a del Pezzo threefold the following particular case of the more general Hartshorne–Serre correspondence holds (for further details about the statement in the general case see [35], [22], [2]).

Theorem 2.3. *Let $E \subseteq F$ be a local complete intersection subscheme of codimension 2 and $\mathcal{L}$ an invertible sheaf on F such that $H^2\big(F, \check{\mathcal{L}}\big) = 0$. If $\det(\mathcal{N}_{E|F}) \cong \mathcal{L} \otimes \mathcal{O}_E$, then there exists a vector bundle $\mathcal{E}$ of rank 2 on F such that:*

(1) $\det(\mathcal{E}) \cong \mathcal{L}$;

(2) *$\mathcal{E}$ has a section s such that E coincides with the zero locus $(s)_0$ of s.*

Moreover, if $H^1\big(F, \check{\mathcal{L}}\big) = 0$, the above two conditions determine $\mathcal{E}$ up to isomorphism.

From now on, $F := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We recall that the canonical sheaf satisfies $\omega_F \cong \mathcal{O}_F(-2h)$ where h is the hyperplane class on F given by the natural embedding $F \subseteq \mathbb{P}^7$. With the notation of the Introduction we have $h = h_1 + h_2 + h_3$.

We have $\deg(F) = h^3 = 6$, $\omega_1 = \omega_F = -2h$, $\omega_1\omega_2 = -24$ in Formula (1). More in general, if $D \in \mathrm{Pic}(F)$, then there are $\delta_1, \delta_2, \delta_3 \in \mathbb{Z}$ such that $\mathcal{O}_F(D) = \mathcal{O}_F(\delta_1 h_1 + \delta_2 h_2 + \delta_3 h_3)$. The following Künneth's formulas will be repeatedly used without explicit mention throughout the paper

$$h^i\big(F, \mathcal{O}_F(D)\big) = \sum_{\substack{(i_1,i_2,i_3)\in\mathbb{N}^3,\\ i_1+i_2+i_3=i}} h^{i_1}\big(F, \mathcal{O}_{\mathbb{P}^1}(\delta_1)\big) h^{i_2}\big(F, \mathcal{O}_{\mathbb{P}^1}(\delta_2)\big) h^{i_3}\big(F, \mathcal{O}_{\mathbb{P}^1}(\delta_3)\big).$$

We will write $D \geq 0$ to denote that D has sections, i.e. $\delta_j \geq 0$, $j = 1, 2, 3$.

In particular we immediately have the following result.

Lemma 2.4. *The initialized aCM bundle of rank 1 on F are:*

$$\begin{gathered}\mathcal{O}_F, \quad \mathcal{O}_F(h_1), \quad \mathcal{O}_F(h_2), \quad \mathcal{O}_F(h_3),\\ \mathcal{O}_F(h_1+h_2), \quad \mathcal{O}_F(h_1+h_3), \quad \mathcal{O}_F(h_1+h_2),\\ \mathcal{O}_F(2h_1+h_2), \quad \mathcal{O}_F(2h_1+h_3), \quad \mathcal{O}_F(h_1+2h_2),\\ \mathcal{O}_F(2h_2+h_3), \quad \mathcal{O}_F(h_1+2h_3), \quad \mathcal{O}_F(h_2+2h_3).\end{gathered}$$

Proof. Straightforward. □

We now turn our attention on rank 2 vector bundles $\mathcal{E}$ on F. In what follows, we will denote its Chern classes by

$$c_1 := c_1(\mathcal{E}) = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3,$$
$$c_2 := c_2(\mathcal{E}) = \beta_1 h_2 h_3 + \beta_2 h_1 h_3 + \beta_3 h_1 h_2.$$

3. A lower bound on the first Chern class

In this section we will find a bound from below for the first Chern class c_1 of an indecomposable initialized aCM bundle $\mathcal{E}$ of rank 2 on F.

The following lemma will be useful.

Lemma 3.1. *Let $\mathcal{E}$ be an initialized aCM bundle of rank 2 on F. Then $\check{\mathcal{E}}(2h)$ is globally generated.*

Proof. We have $h^i\big(F, \check{\mathcal{E}}((2-i)h)\big) = h^{3-i}\big(F, \mathcal{E}((i-4)h)\big) = 0$, $i = 1, 2, 3$. It follows that $\mathcal{E}$ is 2–regular in the sense of Castelnuovo–Mumford (see [31]) whence it follows the first assertion. □

An immediate consequence of Lemma 3.1 is that $4h - c_1 = c_1(\check{\mathcal{E}}(2h))$ is effective, whence $\alpha_i \leq 4$.

If $\alpha_i \geq 3$, $i = 1, 2, 3$, then there would be an injective morphism $\mathcal{O}_F(2h - c_1) \to \mathcal{O}_F(-h)$, which would induce an injective morphism $H^0\big(F, \mathcal{E}(2h - c_1)\big) \to$

$H^0\big(F,\mathcal{E}(-h)\big)$. On the one hand we know that the target space is zero. On the other hand $H^0\big(F,\mathcal{E}(2h-c_1)\big)\neq 0$ since $\mathcal{E}(2h-c_1)\cong\check{\mathcal{E}}(2h)$ is globally generated, a contradiction. It follows that at least one of the α_i is at most 2.

We now concentrate our attention in the proof that each non–zero section of an indecomposable initialized aCM bundle of rank 2 vanishes on F exactly along a curve. We first check that its zero–locus is non–empty.

Lemma 3.2. *Let $\mathcal{E}$ be an indecomposable initialized aCM bundle of rank 2 on F. Then the zero locus $(s)_0\subseteq F$ of a section of $\mathcal{E}$ is non–empty.*

Proof. Assume that $\alpha_1\leq\alpha_2\leq\alpha_3$. If $(s)_0=\emptyset$, then sequence (2) becomes

$$0\longrightarrow\mathcal{O}_F\longrightarrow\mathcal{E}\longrightarrow\mathcal{O}_F(c_1)\longrightarrow 0.$$

Such a sequence corresponds to an element of $\mathrm{Ext}^1\big(\mathcal{O}_F(c_1),\mathcal{O}_F\big)=H^1\big(F,\mathcal{O}_F(-c_1)\big)$. Since $\mathcal{E}$ is indecomposable, it follows that the last space must be non–zero. Thus $\alpha_3\geq 2$ and $\alpha_1\leq\alpha_2\leq 0$.

If either $\alpha_1\leq -1$ or $\alpha_3\geq 3$, then the cohomology of the above sequence twisted by $-2h$ gives

$$0\longrightarrow H^2\big(F,\mathcal{E}(-2h)\big)\longrightarrow H^2\big(F,\mathcal{O}_F(c_1-2h)\big)\longrightarrow H^3\big(F,\mathcal{O}_F(-2h)\big)\cong k.$$

Due to the hypothesis on c_1 we have $1\leq h^2\big(F,\mathcal{E}(-2h)\big)$, contradicting that $\mathcal{E}$ is aCM. We conclude that $\alpha_1=\alpha_2=0$, i.e. $c_1=2h_3$.

Since $h^1\big(F,\mathcal{O}_F(-2h_3)\big)=1$ there exists exactly one non–trivial exact sequence of the form

$$0\longrightarrow\mathcal{O}_F(-2h_3)\longrightarrow\mathcal{E}(-2h_3)\longrightarrow\mathcal{O}_F\longrightarrow 0.$$

Notice that we always have on F the pull–back via π_3^* of the Euler exact sequence, i.e.

$$0\longrightarrow\mathcal{O}_F(-2h_3)\longrightarrow\mathcal{O}_F(-h_3)\oplus\mathcal{O}_F(-h_3)\longrightarrow\mathcal{O}_F\longrightarrow 0.$$

We conclude that the two sequences above are isomorphic, whence $\mathcal{E}\cong\mathcal{O}_F(h_3)\oplus\mathcal{O}_F(h_3)$. It follows that there are no indecomposable initialized aCM vector bundles of rank 2 on F with $c_1=2h_3$. □

Lemma 3.2 implies that for each $s\in H^0\big(F,\mathcal{E}\big)$ we have $(s)_0=E\cup D$ where E has codimension 2 (or it is empty) and $D\in|\delta_1h_1+\delta_2h_2+\delta_3h_3|$ has codimension 1 (or it is empty). We will assume that such a decomposition holds for the general section s from now on.

On the one hand, twisting Sequence (2) by $\mathcal{O}_F(-h)$ and taking its cohomology, the vanishing of $h^0\big(F,\mathcal{E}(-h)\big)$ implies $h^0\big(F,\mathcal{O}_F(D-h)\big)=0$. In particular we know that at least one of the δ_i is zero.

On the other hand, twisting the same sequence by $\mathcal{O}_F(2h-c_1)$, using the isomorphism $\mathcal{E}(-c_1)\cong\check{\mathcal{E}}$ and taking into account that $\check{\mathcal{E}}(2h)$ is globally generated (see Lemma 3.1), we obtain that $\mathcal{I}_{E|F}(2h-D)$ is globally generated too. Thus

$$0\neq H^0\big(F,\mathcal{I}_{E|F}(2h-D)\big)\subseteq H^0\big(F,\mathcal{O}_F(2h-D)\big),$$

hence $\delta_i\leq 2$, $i=1,2,3$. Thus, up to permutations,

$$(\delta_1,\delta_2,\delta_3)\in\{\ (0,0,0),(0,0,1),(0,1,1),(0,0,2),(0,1,2),(0,2,2)\ \}. \tag{4}$$

We now go to prove that c_1-D, hence c_1, is effective. Assume c_1-D is non–effective on F: thanks to Sequence (3), $h^0\big(F,\mathcal{I}_{E|F}(c_1-D)\big)\leq h^0\big(F,\mathcal{O}_F(c_1-D)\big)=$

0. Taking into account that $\mathcal{E}$ is aCM and $\delta_i \geq 0$, the cohomology of Sequence (2) yields also the vanishings

$$h^1\big(F, \mathcal{I}_{E|F}(c_1 - D)\big) = h^2\big(F, \mathcal{I}_{E|F}(c_1 - D)\big) = 0.$$

Twisting Sequence (2) by $\mathcal{O}_F(-h)$, the same argument also gives

$$h^0\big(F, \mathcal{I}_{E|F}(c_1 - D - h)\big) = h^1\big(F, \mathcal{I}_{E|F}(c_1 - D - h)\big) = h^2\big(F, \mathcal{I}_{E|F}(c_1 - D - h)\big) = 0.$$

Finally, the cohomology of Sequence (2) twisted by $\mathcal{O}_F(-2h)$ gives

$$0 \longrightarrow H^1\big(F, \mathcal{I}_{E|F}(c_1 - D - 2h)\big) \longrightarrow H^2\big(F, \mathcal{O}_F(D - 2h)\big) \longrightarrow 0$$

because $\mathcal{E}$ is aCM. Due to the restrictions on the δ_i's, we immediately have

$$h^1\big(F, \mathcal{I}_{E|F}(c_1 - D - 2h)\big) = h^2\big(F, \mathcal{O}_F(D - 2h)\big) = h^1(D) := \begin{cases} 0 & \text{if } D \notin |2h_j|, \\ 1 & \text{if } D \in |2h_j|, \end{cases}$$

We also have the exact sequence

$$\begin{aligned} 0 \longrightarrow H^2\big(F, \mathcal{I}_{E|F}(c_1 - D - 2h)\big) \longrightarrow H^3\big(F, \mathcal{O}_F(D - 2h)\big) \longrightarrow \\ \longrightarrow H^3\big(F, \mathcal{E}(-2h)\big) \longrightarrow H^3\big(F, \mathcal{I}_{E|F}(c_1 - D - 2h)\big) \longrightarrow 0. \end{aligned}$$

If $D \neq 0$, then $h^3\big(F, \mathcal{O}_F(D - 2h)\big) = 0$, thus $h^2\big(F, \mathcal{I}_{E|F}(c_1 - D - 2h)\big) = 0$ too. Assume $D = 0$: then $E \neq \emptyset$ (see Lemma 3.2), so that $h^0\big(F, \mathcal{I}_{E|F}\big) = 0$, whence

$$\begin{aligned} h^3\big(F, \mathcal{E}(-2h)\big) = h^0\big(F, \mathcal{E}(-c_1)\big) = h^0\big(F, \mathcal{O}_F(-c_1)\big) = \\ = h^3\big(F, \mathcal{O}_F(c_1 - 2h)\big) = h^3\big(F, \mathcal{O}_F(c_1 - D - 2h)\big). \end{aligned}$$

Sequence (3) twisted by $\mathcal{O}_F(c_1 - D - 2h)$, also yields $h^3\big(F, \mathcal{I}_{E|F}(c_1 - D - 2h)\big) = h^3\big(F, \mathcal{O}_F(c_1 - D - 2h)\big)$. Thus the last map in the above sequence is an isomorphism. We conclude that

$$h^2\big(F, \mathcal{I}_{E|F}(c_1 - D - 2h)\big) = h^3\big(F, \mathcal{O}_F(D - 2h)\big) = h^2(D) := \begin{cases} 0 & \text{if } D \neq 0, \\ 1 & \text{if } D = 0. \end{cases}$$

in any case.

Lemma 3.3. *Let $\mathcal{E}$ be an indecomposable initialized aCM bundle of rank 2 on F. Assume that $s \in H^0\big(F, \mathcal{E}\big)$ is such that $(s)_0 = E \cup D$ where E has codimension 2 (or it is empty) and D has codimension 1. If $c_1 - D$ is not effective, then $E = \emptyset$.*

Proof. Assume $E \neq \emptyset$, whence $\deg(E) \geq 1$. Let H be a general hyperplane in $\mathbb{P}^7$. Define $S := F \cap H$ and $Z := E \cap H$, so that $\dim(Z) = 0$. We have the following exact sequence

$$0 \longrightarrow \mathcal{I}_{E|F}(c_1 - D - h) \longrightarrow \mathcal{I}_{E|F}(c_1 - D) \longrightarrow \mathcal{I}_{Z|S}(c_1 - D) \longrightarrow 0. \tag{5}$$

The vanishing of the cohomologies of $\mathcal{I}_{E|F}(c_1 - D)$ and $\mathcal{I}_{E|F}(c_1 - D - h)$ implies

$$h^0\big(S, \mathcal{I}_{Z|S}(c_1 - D)\big) = h^1\big(S, \mathcal{I}_{Z|S}(c_1 - D)\big) = 0.$$

The above equalities and the cohomology of

$$0 \longrightarrow \mathcal{I}_{Z|S}(c_1 - D) \longrightarrow \mathcal{O}_S(c_1 - D) \longrightarrow \mathcal{O}_Z \longrightarrow 0, \tag{6}$$

suitably twisted give $h^1\big(S, \mathcal{O}_S(c_1 - D)\big) = 0$, $h^1\big(S, \mathcal{O}_S(c_1 - D - h)\big) \leq h^2(D)$ and

$$h^0\big(S, \mathcal{O}_S(c_1 - D)\big) = h^0\big(Z, \mathcal{O}_Z\big) = \deg(E) \geq 1.$$

Twisting Sequence (5) by $\mathcal{O}_F(-h)$, a similar argument also gives

$$h^0\big(S,\mathcal{I}_{Z|S}(c_1-D-h)\big)=h^1\big(F,\mathcal{I}_{E|F}(c_1-D-2h)\big)=h^1(D),$$
$$h^1\big(S,\mathcal{I}_{Z|S}(c_1-D-h)\big)=h^2\big(F,\mathcal{I}_{E|F}(c_1-D-2h)\big)=h^2(D).$$

The above equalities and the cohomology of Sequence (6) twisted by $\mathcal{O}_S(-h)$ yield

$$h^0\big(Z,\mathcal{O}_Z\big)=h^0\big(S,\mathcal{O}_S(c_1-D-h)\big)-h^1\big(S,\mathcal{O}_S(c_1-D-h)\big)-h^1(D)+h^2(D),$$

whence

$$h^0\big(S,\mathcal{O}_S(c_1-D-h)\big)-1\le h^0\big(Z,\mathcal{O}_Z\big)\le h^0\big(S,\mathcal{O}_S(c_1-D-h)\big)+1.$$

Set $c_1-D:=\epsilon_1h_1+\epsilon_2h_2+\epsilon_3h_3$ and assume $\epsilon_1\le\epsilon_2\le\epsilon_3$: $\epsilon_1\le-1$ because c_1-D is assumed to be not effective. Thus the cohomology of

$$0\longrightarrow\mathcal{O}_F(c_1-D-h)\longrightarrow\mathcal{O}_F(c_1-D)\longrightarrow\mathcal{O}_S(c_1-D)\longrightarrow 0$$

and the vanishing $h^1\big(S,\mathcal{O}_S(c_1-D)\big)=0$ proved above, yield

$$h^0\big(S,\mathcal{O}_S(c_1-D)\big)=h^1\big(F,\mathcal{O}_F(c_1-D-h)\big)-h^1\big(F,\mathcal{O}_F(c_1-D)\big).$$

Since $h^0\big(S,\mathcal{O}_S(c_1-D)\big)\ge 1$, it follows that $h^1\big(F,\mathcal{O}_F(c_1-D-h)\big)\ge 1$. In particular $1\le\epsilon_2\le\epsilon_3$, whence

$$h^0\big(F,\mathcal{O}_F(c_1-D)\big)=\epsilon_1\epsilon_2+\epsilon_1\epsilon_3+\epsilon_2\epsilon_3+\epsilon_1+\epsilon_2+\epsilon_3+1.$$

Taking again the cohomology of the above sequence twisted by $\mathcal{O}_F(c_1-D-h)$ we obtain the exact sequence (recall that c_1-D is not effective)

$$0\longrightarrow H^0(S,\mathcal{O}_S(c_1-D-h))\longrightarrow H^1(F,\mathcal{O}_F(c_1-D-2h))\xrightarrow{\varphi}$$
$$\longrightarrow H^1(F,\mathcal{O}_F(c_1-D-h))\longrightarrow H^1(S,\mathcal{O}_S(c_1-D-h)).$$

If φ is not surjective then necessarily $1\le h^1(S,\mathcal{O}_S(c_1-D-h))\le h^2(D)\le 1$, thus equality must hold. In particular $D=0$, $h^0\big(S,\mathcal{O}_S(c_1-D-h)\big)=h^0\big(Z,\mathcal{O}_Z\big)\ge 1$ and

$$h^0\big(S,\mathcal{O}_S(c_1-D-h)\big)=h^1\big(F,\mathcal{O}_F(c_1-D-2h)\big)-h^1\big(F,\mathcal{O}_F(c_1-D-h)\big)+1.$$

If $\epsilon_2=1$, then $h^1\big(F,\mathcal{O}_F(c_1-D-2h)\big)=0$ and $h^1\big(F,\mathcal{O}_F(c_1-D-h)\big)\ge 1$, whence $h^0\big(S,\mathcal{O}_S(c_1-D-h)\big)\le 0$, a contradiction. Thus $2\le\epsilon_2\le\epsilon_3$, whence

$$h^0\big(S,\mathcal{O}_S(c_1-D-h)\big)=\epsilon_1\epsilon_2+\epsilon_1\epsilon_3+\epsilon_2\epsilon_3-\epsilon_1-\epsilon_2-\epsilon_3+2.$$

The equality $h^0\big(S,\mathcal{O}_S(c_1-D)\big)=h^0\big(Z,\mathcal{O}_Z\big)=h^0\big(S,\mathcal{O}_S(c_1-D-h)\big)$ implies $2\epsilon_1+2\epsilon_2+2\epsilon_3+1=0$, which is not possible, because the ϵ_i's are integers.

If φ is surjective, then

$$h^0\big(S,\mathcal{O}_S(c_1-D-h)\big)=h^1\big(F,\mathcal{O}_F(c_1-D-2h)\big)-h^1\big(F,\mathcal{O}_F(c_1-D-h)\big),$$

and we can easily again infer $2\le\epsilon_2\le\epsilon_3$, whence

$$h^0\big(S,\mathcal{O}_S(c_1-D-h)\big)=\epsilon_1\epsilon_2+\epsilon_1\epsilon_3+\epsilon_2\epsilon_3-\epsilon_1-\epsilon_2-\epsilon_3+1.$$

If $h^0\big(Z,\mathcal{O}_Z\big)=h^0\big(S,\mathcal{O}_S(c_1-D-h)\big)\pm 1$ we obtain $2\epsilon_1+2\epsilon_2+2\epsilon_3\pm 1=0$ which is not possible.

If $h^0\big(Z,\mathcal{O}_Z\big)=h^0\big(S,\mathcal{O}_S(c_1-D-h)\big)$, then $\epsilon_1+\epsilon_2+\epsilon_3=0$, thus $\epsilon_1=-\epsilon_2-\epsilon_3$. Substituting in the expression of $h^0\big(S,\mathcal{O}_S(c_1-D)\big)$ we obtain

$$h^0\big(S,\mathcal{O}_S(c_1-D)\big)=-\epsilon_2^2-\epsilon_2\epsilon_3-\epsilon_3^2+1\le -11$$

an absurd.

We conclude that the assumption $E \neq \emptyset$ yields a contradiction. This completes the proof of the statement. □

We are now ready to prove the claimed effectiveness of $c_1 - D$.

Proposition 3.4. *If $\mathcal{E}$ is an indecomposable initialized aCM bundle of rank 2 on F, $s \in H^0\big(F, \mathcal{E}\big)$ and D denotes the component of codimension 1 of $(s)_0$, if any, then $c_1 \geq c_1 - D \geq 0$.*

In particular if $c_1 = 0$, then the zero–locus of each section $s \in H^0\big(F, \mathcal{E}\big)$ has codimension 2.

Proof. The equality $E = \emptyset$ implies $\mathcal{I}_{E|F} \cong \mathcal{O}_F$. Sequence (2) thus becomes

$$0 \longrightarrow \mathcal{O}_F(D) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_F(c_1 - D) \longrightarrow 0. \tag{7}$$

If $D \in |2h_2 + 2h_3|$, then the cohomology of Sequence (7) twisted by $\mathcal{O}_F(-2h)$ would give

$$0 = h^0\big(F, \mathcal{O}_F(c_1 - D)\big) \geq h^0\big(F, \mathcal{O}_F(c_1 - D - 2h)\big) = h^1\big(F, \mathcal{O}_F(D - 2h)\big) = 1,$$

a contradiction. If $D \in |2h_3|$, then $h^1\big(F, \mathcal{O}_F(c_1 - D - 2h)\big) = h^2\big(F, \mathcal{O}_F(D - 2h)\big) = 1$ contradicting the condition $\epsilon_1 \leq -1$.

In the remaining cases we know that $h^i\big(F, \mathcal{O}_F(D - h)\big) = h^i\big(F, \mathcal{O}_F(D - 2h)\big) = 0$, $i = 1, 2$. Taking the cohomology of Sequence (7) suitably twisted, we obtain $h^i\big(F, \mathcal{O}_F(c_1 - D - h)\big) = h^i\big(F, \mathcal{O}_F(c_1 - D - 2h)\big) = 0$ in the same range, because both $\mathcal{E}$ and $\mathcal{O}_F(D)$ are aCM.

If Sequence (7) does not split, then

$$H^1\big(F, \mathcal{O}_F(2D - c_1)\big) = \mathrm{Ext}^1_F\big(\mathcal{O}_F(c_1 - D), \mathcal{O}_F(D)\big) \neq 0.$$

Thus $\delta_j - \epsilon_j \leq -2$ for exactly one $j = 1, 2, 3$. Since $\delta_1 \geq 0$ and $\epsilon_1 \leq -1$ (as we assumed above) such a j is not 1. We can assume $\delta_3 - \epsilon_3 \leq -2$, whence we obtain $\epsilon_3 \geq 2$. If $\epsilon_2 \geq 1$, then $h^1\big(F, \mathcal{O}_F(c_1 - D - h)\big) \neq 0$. If $\epsilon_2 \leq 0$, then $h^2\big(F, \mathcal{O}_F(c_1 - D - 2h)\big) \neq 0$. In both cases we have a contradiction. □

4. An upper bound on the first Chern class

In this section we will find a bound from above for the first Chern class c_1 of an indecomposable initialized aCM bundle $\mathcal{E}$ of rank 2 on F. Thus $h^i\big(F, \check{\mathcal{E}}(-h)\big) = h^{3-i}\big(F, \mathcal{E}(-h)\big) = 0$ for $i \geq 1$. Since

$$h^0\big(F, \mathcal{I}_{E|F}(-D - h)\big) \leq h^0\big(F, \mathcal{O}_F(-D - h)\big) = 0$$

(see Sequence (3)), it follows that

$$\chi\big(\check{\mathcal{E}}(-h)\big) = h^0\big(F, \check{\mathcal{E}}(-h)\big) = h^0\big(F, \mathcal{O}_F(D - c_1 - h)\big) = 0$$

(due to sequence (2) and to the effectiveness of $c_1 - D$ proved in Proposition 3.4). Such an equality, Formula (1) and the equalities

$$c_1^3 = 6\alpha_1\alpha_2\alpha_3,$$
$$c_1^2 h = 2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3),$$
$$c_1 h^2 = 2(\alpha_1 + \alpha_2 + \alpha_3),$$
$$\omega_2 c_1 = 4(\alpha_1 + \alpha_2 + \alpha_3),$$

finally imply

$$c_1 c_2 = 2\alpha_1\alpha_2\alpha_3. \tag{8}$$

Similarly assume $h^i\big(F,\mathcal{E}(-2h)\big)=0$ for $i\le 2$. Thus $h^i\big(F,\check{\mathcal{E}}\big)=h^{3-i}\big(F,\mathcal{E}(-2h)\big)=0$, for $i\ge 1$. Notice that $H^0\big(F,\mathcal{I}_{E|F}\big)$ is contained in the kernel of the natural injective map $H^0\big(F,\mathcal{O}_F\big)\to H^0\big(E,\mathcal{O}_E\big)$, hence $h^0\big(F,\mathcal{I}_{E|F}\big)=0$. Moreover

$$h^0\big(F,\mathcal{I}_{E|F}(-D)\big)\le h^0\big(F,\mathcal{O}_F(-D)\big)=0$$

if $D\neq 0$. We conclude that $h^0\big(F,\mathcal{I}_{E|F}(-D)\big)=0$ whence

$$h^0\big(F,\check{\mathcal{E}}\big)=h^0\big(F,\mathcal{O}_F(D-c_1)\big)=e(c_1,D):=\begin{cases}0 & \text{if } D\neq c_1,\\ 1 & \text{if } D=c_1.\end{cases}$$

Combining as above Sequences (3), (2) and Formulas (1), (8) we finally obtain

$$hc_2=\alpha_1\alpha_2\alpha_3+(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)+1-e(c_1,D). \tag{9}$$

In what follows we will often combine the above equalities with the trivial ones

$$\begin{cases}\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3=c_1c_2\\ \beta_1+\beta_2+\beta_3=hc_2.\end{cases} \tag{10}$$

Finally, assume that $E\neq\emptyset$. Its class in $A^2(F)$ is $c_2(\mathcal{E}(-D))=c_2-c_1D+D^2$. Since $|h_i|$, $i=1,2,3$ is base–point–free on F, $D\ge 0$ and $c_1-D\ge 0$ (see Proposition 3.4), it follows that

$$\beta_i\ge\beta_i-h_i(c_1-D)D=h_ic_2-h_i(c_1-D)D=h_ic_2(\mathcal{E}(-D))\ge 0,\qquad i=1,2,3. \tag{11}$$

The numbers c_1c_2 and hc_2 will be very important in what follows.

Lemma 4.1. *Let $\mathcal{E}$ be a vector bundle of rank 2 on F and $s\in H^0\big(F,\mathcal{E}\big)$ a section such that $E:=(s)_0$ has codimension 2. Then*

$$\deg(E)=hc_2,\qquad p_a(E)=\frac{1}{2}c_1c_2-hc_2+1.$$

Proof. The assertion on $\deg(E)$ is immediate, because the class in $A^2(F)$ is c_2. Let us compute $p_a(E)$. Adjunction formula on F yields $\omega_E\cong\omega_F\otimes\det(\mathcal{N}_{E|F})$. We have $\mathcal{N}_{E|F}\cong\mathcal{E}\otimes\mathcal{O}_E$, thus $\det(\mathcal{N}_{E|F})\cong\mathcal{O}_E(c_1)$. We also have that $\omega_F\cong\mathcal{O}_F(-2h)$. Equating the degrees of the two members of the above formula we obtain also the second equality. □

Now we turn our attention on the upper bound on c_1.

Proposition 4.2. *If $\mathcal{E}$ is an indecomposable initialized aCM bundle of rank 2 on F, then either $2h-c_1\ge 0$ or $c_1=h_1+2h_2+3h_3$ up to permutations of the h_i's.*

Moreover, if either $c_1=2h$ or $c_1=h_1+2h_2+3h_3$ up to permutations of the h_i's, then $\mathcal{E}$ is Ulrich, hence the zero–locus $E:=(s)_0$ of a general section $s\in H^0\big(F,\mathcal{E}\big)$ is a smooth curve.

Proof. We distinguish two cases according to whether $\mathcal{E}$ is regular in the sense of Castelnuovo–Mumford or not.

In the latter case we infer $h^0\big(F,\check{\mathcal{E}}(h)\big)=h^3\big(F,\mathcal{E}(-3h)\big)\neq 0$ because $\mathcal{E}$ is aCM. Thus, if $t\in\mathbb{Z}$ is such that $\check{\mathcal{E}}(th)$ is initialized, we know that $t\le 1$. Since $\check{\mathcal{E}}(th)$ is obviously also aCM, we know that $2th-c_1=c_1(\check{\mathcal{E}}(th))$ is effective, due to Proposition 3.4. We conclude that the same is true for $2h-c_1$, because $t\le 1$.

Now we assume that $\mathcal{E}$ is regular, whence it is globally generated. Thus the zero–locus $E:=(s)_0$ of a general section $s\in H^0\big(F,\mathcal{E}\big)$ is a smooth curve. Let $\alpha_1\le\alpha_2\le\alpha_3$: we already know that $0\le\alpha_1\le 2$ and $\alpha_2,\alpha_3\le 4$. We want to examine

the cases $\alpha_3 = 3, 4$. In any case we have that $h^i\big(F, \check{\mathcal{E}}(h)\big) = h^{3-i}\big(F, \mathcal{E}(-3h)\big) = 0$, $i = 0, 1, 2, 3$, because $\mathcal{E}$ is assumed aCM, regular and initialized. Combining the above remarks with equality (1), Formulas (8), (9) and the vanishing of $e(c_1, D)$, we finally obtain

$$0 = \chi(\check{\mathcal{E}}(h)) = 12 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3.$$

Thus we have the following possibilities for $(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_3 \geq 3$: $(0, 2, 4)$, $(1, 1, 4)$, $(0, 3, 3)$, $(1, 2, 3)$.

If $c_1 = h_1 + 2h_2 + 3h_3$ up to permutations, then Formula (9) give $hc_2 = 7$ and $c_1c_2 = 12$. Substituting in Equality (1) we obtain $\chi(\mathcal{E}) = 12$. Since $\mathcal{E}$ is aCM, it follows that $h^1\big(F, \mathcal{E}\big) = h^2\big(F, \mathcal{E}\big) = 0$. Since $\mathcal{E}$ is initialized, it follows that $h^3\big(F, \mathcal{E}\big) = h^0\big(F, \mathcal{E}(-2h - c_1)\big) = 0$. We conclude that $h^0\big(F, \mathcal{E}\big) = 12$ and $\mathcal{E}$ turns out to be Ulrich. Hence $\mathcal{E}$ is globally generated, thus the zero–locus of a general section of $\mathcal{E}$ again trivially vanishes on a curve. Similarly, one can also prove that $\mathcal{E}$ is Ulrich in all the other cases.

We now go to prove that if c_1 is either $h_1 + h_2 + 4h_3$ or $2h_2 + 4h_3$ or $3h_2 + 3h_3$ up to permutations, then $\mathcal{E}$ splits as sum of invertible sheaves. To this purpose, let us assume we have proved the statement in the last case, i.e. if $c_1(\mathcal{F}) = 3h_2 + 3h_3$, then $\mathcal{F} \cong \mathcal{O}_F(h_2 + 2h_3) \oplus \mathcal{O}_F(2h_2 + h_3)$.

Let $c_1 = h_1 + h_2 + 4h_3$. Then we know that $\check{\mathcal{E}}(2h)$ is Ulrich (see Lemma 2.4 of [11]): in particular it is aCM and initialized. We have $c_1(\check{\mathcal{E}}(2h)) = 3h_1 + 3h_2$, thus the claim above implies $\check{\mathcal{E}}(2h) \cong \mathcal{O}_F(h_1 + 2h_2) \oplus \mathcal{O}_F(2h_1 + h_2)$, whence $\mathcal{E} \cong \mathcal{O}_F(h_2 + 2h_3) \oplus \mathcal{O}_F(2h_2 + h_3)$. Hence it suffices to examine only the two last cases. We will examine only the last case $c_1 = 3h_2 + 3h_3$, the other one being similar.

Recall that $\mathcal{E}$ is Ulrich, hence globally generated. In particular the scheme $E := (s)_0$ can be assumed to be a smooth curve. Thus the System (10) and the Inequalities (11) with $D = 0$ force $c_2 = 5h_2h_3$. Lemma (4.1) implies $\deg(E) = 5$ and $p_a(E) = -4$. It follows that E is necessarily non–connected (because it is smooth). Since Riemann–Roch on E yields $\chi(\mathcal{O}_E) = 5$, it follows that E is the union of 5 pairwise skew lines whose class in $A^2(F)$ is h_2h_3. In particular

$$\mathcal{O}_E(a_1h_1 + a_2h_2 + a_3h_3) \cong \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus 5}. \tag{12}$$

Let $p\colon F \to \mathbb{P}^1 \times \mathbb{P}^1$ be the projection on the two last factors of $F \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Each $Q \in |h_1|$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, thus we have a natural section $\sigma\colon \mathbb{P}^1 \times \mathbb{P}^1 \to F$ of p whose image is Q. The restrictions $h_{2|Q}$ and $h_{3|Q}$ in $A^1(Q)$ of $h_2, h_3 \in A^1(F)$ are the two rulings on Q. If Q is general, then $\Gamma := Q \cap E$ consists of five distinct points on Q. Pulling back Sequence (2) via σ we obtain a Koszul complex of the form

$$0 \longrightarrow \mathcal{O}_Q \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_{\Gamma|Q} \otimes \sigma^*\mathcal{O}_F(3h_2 + 3h_3) \longrightarrow 0$$

where $\mathcal{F} := \sigma^*\mathcal{E}$. Since Γ is locally complete intersection inside Q, it follows that the Koszul complex above is everywhere exact. Pulling back the above complex via p, we still obtain an exact sequence because $\mathcal{O}_F$ is a vector bundle on Q. It is easy to check that $p^*\mathcal{I}_{\Gamma|Q} \cong \mathcal{I}_{E|F}$ and $p^*\sigma^*\mathcal{O}_F(3h_2 + 3h_3) \cong \mathcal{O}_F(3h_2 + 3h_3)$. Thus we have the two extensions

$$0 \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|F}(3h_2 + 3h_3) \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{O}_F \longrightarrow p^*\mathcal{F} \longrightarrow \mathcal{I}_{E|F}(3h_2 + 3h_3) \longrightarrow 0.$$

Theorem 2.3, yields $\mathcal{E} \cong p^*\mathcal{F}$, because $h^1\big(F, \mathcal{O}_F(-3h_2 - 3h_3)\big) = 0$.

With this in mind, Künneth's formulas for $\mathcal{E}(th) \cong \mathcal{O}_F(th_1) \otimes p^*\mathcal{F}(th_2 + th_3)$ and the projection isomorphism $\sigma_* p^*\mathcal{F} \cong \mathcal{F}$ give

$$\begin{aligned} h^0\big(F, \mathcal{O}_F(th_1)\big) h^1\big(Q, \mathcal{F}(th_{2|Q} + th_{3|Q})\big) + \\ h^1\big(F, \mathcal{O}_F(th_1)\big) h^0\big(Q, \mathcal{F}(th_{2|Q} + th_{3|Q})\big) = h^1\big(F, \mathcal{E}(th)\big) = 0, \\ h^1\big(F, \mathcal{O}_F(th_1)\big) h^1\big(Q, \mathcal{F}(th_{2|Q} + th_{3|Q})\big) = h^2\big(F, \mathcal{E}(th)\big) = 0, \end{aligned}$$

which yield $h^1\big(Q, \mathcal{F}(th_{2|Q} + th_{3|Q})\big) = 0$, $t \neq -1$. We claim that $\mathcal{F}(-h_{2|Q} - h_{3|Q})$ is regular in the sense of Castenuovo–Mumford (see [31]). We have to check that

$$h^1\big(Q, \mathcal{F}(-2h_{2|Q} - 2h_{3|Q})\big) = h^2\big(Q, \mathcal{F}(-3h_{2|Q} - 3h_{3|Q})\big) = 0.$$

Due to the discussion above it thus suffices to check the second vanishing. On the one hand we have $\det(\mathcal{F}) \cong \sigma^* \det(\mathcal{E}) \cong \mathcal{O}_Q(-3h_{2|Q} - 3h_{3|Q})$, thus the isomorphism $\check{\mathcal{F}} \cong \mathcal{F}(-3h_{2|Q} - 3h_{3|Q})$ and Serre's duality on Q give $h^2\big(Q, \mathcal{F}(-3h_{2|Q} - 3h_{3|Q})\big) = h^0\big(Q, \mathcal{F}(-2h_{2|Q} - 2h_{3|Q})\big)$.

On the other hand, taking the cohomology of sequence $0 \to \mathcal{O}_F(-h_1) \to \mathcal{O}_F \to \mathcal{O}_Q \to 0$ twisted by $\mathcal{E}(-2h)$, we obtain

$$\begin{aligned} H^0\big(F, \mathcal{E}(-2h_1 - 2h_2 - 2h_3)\big) \longrightarrow H^0\big(Q, \mathcal{F}(-2h_{2|Q} - 2h_{3|Q})\big) \longrightarrow \\ \longrightarrow H^1\big(F, \mathcal{E}(-3h_1 - 2h_2 - 2h_3)\big). \end{aligned}$$

Since $\mathcal{E}$ is initialized, it follows that $h^0\big(F, \mathcal{E}(-2h_1 - 2h_2 - 2h_3)\big) = 0$. Thanks to Sequence (2) it is easy to check that $H^1\big(F, \mathcal{E}(-3h_1 - 2h_2 - 2h_3)\big) \cong H^1\big(F, \mathcal{I}_{E|F}(-3h_1 + h_2 + h_3)\big)$, which is a quotient of $H^0\big(E, \mathcal{O}_E(-3h_1 + h_2 + h_3)\big)$ (use Sequence (3)). But this last cohomology group is zero, thanks to Formula (12).

The regularity of $\mathcal{F}(-h_{2|Q} - h_{3|Q})$ implies that it is globally generated and that $\mathcal{F}$ is regular too, thus $h^1\big(Q, \mathcal{F}(-h_{2|Q} - h_{3|Q})\big) = 0$. Due to a Theorem of Knörrer (see [1], Corollary 6.8 and the references therein), we know that $\mathcal{F}$ splits as the sum of two invertible sheaves on Q of one of the following forms

$$\mathcal{O}_Q(ah_{2|Q} + ah_{3|Q}), \quad \mathcal{O}_Q(ah_{2|Q} + (a+1)h_{3|Q}), \quad \mathcal{O}_Q((a+1)h_{2|Q} + ah_{3|Q}),$$

with $a \geq 0$. Since we know that $c_2(\mathcal{F}) = c_2(\mathcal{E})h_1 = 5h_1h_2h_3$, it follows that necessarily that $\mathcal{F} \cong \mathcal{O}_Q(h_{2|Q} + 2h_{3|Q}) \oplus \mathcal{O}_Q(2h_{2|Q} + h_{3|Q})$, whence $\mathcal{E} \cong p^*\mathcal{F} \cong \mathcal{O}_F(h_2 + 2h_3) \oplus \mathcal{O}_F(2h_2 + h_3)$. □

5. The proof of the theorem A

In this section we will complete the proof of Theorem A stated in the introduction. It only remains to show that if $\mathcal{E}$ is an indecomposable, initialized, aCM bundle, then its general section vanishes exactly along a curve (see Proposition 3.4).

We already checked in the previous sections (see Propositions 3.4 and 4.2) that either $c_1 \geq 0$ and $2h - c_1 \geq 0$ or $c_1 = h_1 + 2h_2 + 3h_3$ up to permutations of the h_i's for such kind of bundles and that the general section vanishes exactly along a curve when either $c_1 = 0$, or $2h - c_1 = 0$, or $c_1 = h_1 + 2h_2 + 3h_3$ up to permutations of the h_i's.

It follows that we can restrict our attention to the remaining cases. If we assume $\alpha_1 \leq \alpha_2 \leq \alpha_3$, such cases satisfy $\alpha_1 \leq 1 \leq \alpha_3$.

Lemma 5.1. *Let $\mathcal{E}$ be an indecomposable initialized aCM bundle of rank 2 on F whose general section $s \in H^0\big(F, \mathcal{E}\big)$ satisfies $(s)_0 = E \cup D$ where E has codimension 2 (or it is empty) and $D \in |\delta_1 h_1 + \delta_2 h_2 + \delta_3 h_3|$ is non–zero. Then:*

(1) *up to permutations*

$$(\delta_1, \delta_2, \delta_3) \in \{\ (0,0,1), (0,1,1), (0,0,2), (0,1,2)\ \};$$

(2) *$\mathcal{E}$ is not globally generated;*
(3) *$E \neq \emptyset$.*

Proof. We already know that

$$(\delta_1, \delta_2, \delta_3) \in \{\ (0,0,0), (0,0,1), (0,1,1), (0,0,2), (0,1,2), (0,2,2)\ \}$$

(see Relation (4)). Since $D \neq 0$, it follows that $(\delta_1, \delta_2, \delta_3) \neq (0,0,0)$. Let $D \in |2h_2 + 2h_3|$ and look at the cohomologies of Sequences (2) and (3) respectively twisted by $\mathcal{O}_F(-2h)$ and $\mathcal{O}_F(c_1 - D - 2h)$. On the one hand, taking into account that $h^0\big(F, \mathcal{E}(-2h)\big) = h^1\big(F, \mathcal{E}(-2h)\big) = 0$ by hypothesis we obtain

$$1 = h^1\big(F, \mathcal{O}_F(D - 2h)\big) = h^0\big(F, \mathcal{I}_{E|F}(c_1 - D - 2h)\big) \leq h^0\big(F, \mathcal{O}_F(c_1 - D - 2h)\big).$$

On the other hand we also know that the last dimension is zero, due to the restrictions $D \geq 0$, $2h - c_1 \geq 0$ and $c_1 \neq 2h$. It follows that equality $(\delta_1, \delta_2, \delta_3) = (0,2,2)$ cannot occur.

If $\mathcal{E}$ would be globally generated, then the general section $s \in H^0\big(F, \mathcal{E}\big)$ should vanish on a curve, contradicting the hypothesis.

Finally, assume that $E = \emptyset$. Thus $\mathcal{I}_{E|F}(c_1 - D) \cong \mathcal{O}_F(c_1 - D)$ in Sequence (2). We obviously have that $\mathcal{O}_F(D)$ is globally generated and $h^1\big(F, \mathcal{O}_F(D)\big) = 0$. Moreover $\mathcal{O}_F(c_1 - D)$ is globally generated too by Proposition 3.4. We conclude that $\mathcal{E}$ is globally generated too thanks to Sequence (2) contradicting what we just proved above. □

From now on in this section we will assume that $\mathcal{E}$ satisfies the hypothesis of the above lemma, hence we can assume $E \neq \emptyset$. Recall that E is in the class $c_2(\mathcal{E}(-D)) = c_2 - c_1 D + D^2 \in A^2(F)$. The class c_2 can be computed thanks to System (10) and Formulas (8), (9). Moreover we have

$$c_1 D = (\delta_1\alpha_2 + \delta_2\alpha_1)h_1h_2 + (\delta_1\alpha_3 + \delta_3\alpha_1)h_1h_3 + (\delta_2\alpha_3 + \delta_3\alpha_2)h_2h_3,$$
$$D^2 = 2(\delta_1\delta_2 h_1h_2 + \delta_1\delta_3 h_1h_3 + \delta_2\delta_3 h_2h_3).$$

In particular, if $D \neq 0$, then Inequalities (11) give further restrictions on the β_i's.

Careful but tedious computations show that, up to permutations of the h_i's, only the following few cases are admissible.

$(\alpha_1, \alpha_2, \alpha_3)$	$(\beta_1, \beta_2, \beta_3)$	$(\delta_1, \delta_2, \delta_3)$	E
$(1,1,2)$	$(1,1,1)$	$(0,0,1)$	h_1h_2
$(1,2,2)$	$(2,2,1)$	$(0,0,1)$	$h_1h_3 + h_1h_2$
$(1,2,2)$	$(2,3,0)$	$(0,0,1)$	$2h_1h_3$
$(1,2,2)$	$(2,2,1)$	$(0,1,1)$	h_1h_3

When the class of E is h_1h_2, then E is a line and we have an exact sequence of the form

$$0 \longrightarrow \mathcal{O}_F(-h_1 - h_2) \longrightarrow \mathcal{O}_F(-h_1) \oplus \mathcal{O}_F(-h_2) \longrightarrow \mathcal{I}_{E|F} \longrightarrow 0.$$

It follows that $\mathcal{I}_{E|F}(h) \cong \mathcal{I}_{E|F}(c_1 - D)$ is globally generated. Thus we conclude again that $\mathcal{E}$ should be globally generated, contradicting Lemma 5.1. A similar argument holds when the class of E is h_1h_3.

When the class of E is $h_1h_3+h_1h_2$, then we cannot repeat the above argument. In this case we have that $h^2\big(F,\mathcal{O}_F(D-h)\big)=h^3\big(F,\mathcal{O}_F(D-2h)\big)=0$. Moreover $\mathcal{E}$ is aCM, hence we obtain

$$h^1\big(F,\mathcal{I}_{E|F}(c_1-D-h)\big)=h^2\big(F,\mathcal{I}_{E|F}(c_1-D-2h)\big)=0.$$

Moreover the cohomology of Sequence (3) twisted by $\mathcal{O}_F(c_1-D-3h)$ implies that

$$h^3\big(F,\mathcal{I}_{E|F}(c_1-D-3h)\big)\leq h^3\big(F,\mathcal{O}_F(c_1-D-3h)\big).$$

Since $c_1-D-3h=-2h_1-h_2-2h_3$ we conclude that the last dimension is zero. It follows that $\mathcal{I}_{E|F}(c_1-D)$ is regular in the sense of Castelnuovo–Mumford, hence it is globally generated. As in the previous cases we also obtain that $\mathcal{E}$ is globally generated, contradicting again Lemma 5.1. A similar argument also proves that the class of E cannot be $2h_1h_3$.

By combining Propositions 3.4 and 4.2 and the above discussion we have the proof of Theorem A.

The next sections are devoted to the classification of all indecomposable initialized aCM bundle of rank 2 on F, or, in other words, to the proof of Theorem B stated in the introduction.

6. The standard bound

In this section we deal with the extremal cases $c_1=0$ and $c_1=2h$. In the first case Proposition 3.4 implies that the zero–locus $E:=(s)_0$ of a general section $s\in H^0\big(F,\mathcal{E}\big)$ has codimension 2 inside F. Formula (9) and Lemma 4.1 (notice that we know that $D=0$) give $\deg(E)=hc_2=1$, thus E is a line. In particular, if its class in $A^2(F)$ is $\beta_1h_2h_3+\beta_2h_1h_3+\beta_3h_1h_2$, then necessarily all the β_i's are zero but one which is 1, i.e. the class of E in $A^2(F)$ is either h_2h_3 or h_1h_3 or h_1h_2.

Conversely we show that each line on F arises a zero locus of a section of an indecomposable initialized aCM bundle of rank 2 on F with $c_1=0$. To this purpose, assume that the class of E is h_2h_3. We know that $\omega_E\cong\mathcal{O}_E(-2h)$, thus adjunction formula on F implies $\det(\mathcal{N}_{E|F})\cong\mathcal{O}_E$. Theorem 2.3 with $\mathcal{L}:=\mathcal{O}_F$ guarantees the existence of a vector bundle $\mathcal{E}$ of rank 2 fitting into a sequence of the form

$$0\longrightarrow\mathcal{O}_F\longrightarrow\mathcal{E}\longrightarrow\mathcal{I}_{E|F}\longrightarrow 0.$$

Hence $h^1\big(F,\mathcal{E}(th)\big)\leq h^1\big(F,\mathcal{I}_{E|F}(th)\big)$, $t\in\mathbb{Z}$. The vanishing $h^1\big(F,\mathcal{I}_{E|\mathbb{P}^7}(th)\big)=h^2\big(F,\mathcal{I}_{F|\mathbb{P}^7}(th)\big)=0$ (recall that both E and F are aG), imply $h^1\big(F,\mathcal{I}_{E|F}(th)\big)=0$, i.e. $h^1\big(F,\mathcal{E}(th)\big)=0$. Since $c_1=0$, it follows that $\check{\mathcal{E}}\cong\mathcal{E}$, thus Serre's duality also yields $h^2\big(F,\mathcal{E}(th)\big)=0$. We conclude that $\mathcal{E}$ is an aCM bundle. It is trivial to check that $\mathcal{E}$ is initialized. Finally, if $\mathcal{E}$ were decomposable, then $\mathcal{E}\cong\mathcal{M}\oplus\mathcal{M}^{-1}$ because $c_1=0$. Thus $h_2h_3=c_2=-c_1(\mathcal{M})^2$. But if $c_1(\mathcal{M})=\mu_1h_1+\mu_2h_2+\mu_3h_3$, then $c_1(\mathcal{M})^2=2(\mu_2\mu_3h_2h_3+\mu_1\mu_3h_1h_3+\mu_1\mu_2h_1h_2)$ which cannot coincide with h_2h_3.

Theorem 6.1. *There exist indecomposable initialized aCM bundles $\mathcal{E}$ of rank 2 with $c_1=0$.*

Moreover the zero–locus of a general section of $\mathcal{E}$ is a line and each line on F can be obtained in this way.

Proof. The statement follows from the correspondence described above. □

Now we turn our attention to the case $c_1 = 2h$. Proposition 4.2 guarantees that $\mathcal{E}$ is Ulrich (see Proposition 4.2), hence globally generated. Thus also in this case the zero–locus $E := (s)_0$ of a general section $s \in H^0\big(F, \mathcal{E}\big)$ is a smooth curve inside F.

Using Sequence (2) we obtain that $h^1\big(F, \mathcal{I}_{E|F}\big) = h^1\big(F, \mathcal{I}_{E|F}(h)\big) = 0$, whence both $h^0\big(E, \mathcal{O}_E\big) = 1$ and E is linearly normal (use Sequence (3)). We conclude that we can assume E to be a non–degenerate, smooth, elliptic curve. Moreover, combining Lemma 4.1 and Formula (9), we obtain $\deg(E) = hc_2 = 8$.

Conversely, let E be a smooth, non–degenerate elliptic curve of degree 8 on F. We know by adjunction that $\mathcal{O}_E \cong \omega_E \cong \det(\mathcal{N}_{E|F}) \otimes \mathcal{O}_F(-2h)$. The invertible sheaf $\mathcal{O}_F(2h)$ satisfies the vanishing of the Theorem 2.3, thus E is the zero locus of a section s of a vector bundle $\mathcal{E}$ of rank 2 on F with $c_1 = 2h$ and the class of E in $A^2(F)$ is c_2.

Due to Proposition 1.1 and Corollary 2.2 of [15], we know that E is aCM, then $h^1\big(F, \mathcal{I}_{E|\mathbb{P}^7}(th)\big) = 0$. Since F is aG, whence $h^2\big(F, \mathcal{I}_{F|\mathbb{P}^7}(th)\big) = 0$, taking the cohomology of sequence

$$0 \longrightarrow \mathcal{I}_{F|\mathbb{P}^7} \longrightarrow \mathcal{I}_{E|\mathbb{P}^7} \longrightarrow \mathcal{I}_{E|F} \longrightarrow 0,$$

it also follows that $h^1\big(F, \mathcal{I}_{E|F}(th)\big) = 0$. The cohomology of the Sequence (2) yields $h^1\big(F, \mathcal{E}(th)\big) = 0$. Such a vanishing also implies $h^2\big(F, \mathcal{E}(th)\big) = 0$ by Serre's duality. We conclude that $\mathcal{E}$ is aCM. Thanks to Proposition 4.2 we also know that $\mathcal{E}$ is Ulrich. Thus non–degenerate elliptic curves on a del Pezzo threefold F correspond to Ulrich bundles on F with $c_1 = 2h$.

First we deal with the possible values of c_2. Notice that the linear system $|h_i|$ on F has dimension 1. Let $D_i \in |h_i|$ be general. The cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_F(h - h_i) \longrightarrow \mathcal{O}_F(h) \longrightarrow \mathcal{O}_{D_i}(h) \longrightarrow 0$$

yields $h^0\big(D_i, \mathcal{O}_{D_i}(h)\big) = 4$, thus D_i spans a space of dimension 3 in $\mathbb{P}^7$. Since E is non–degenerate, we know that the restriction to E of $|h_i|$ has dimension at least 1. Since E is elliptic, it follows that its degree, which is β_i, is greater than 2, thanks to Riemann–Roch theorem on the curve E. We conclude that the possible cases for $(\beta_1, \beta_2, \beta_3)$ are the following up to permutations: $(2, 2, 4)$, $(2, 3, 3)$.

We now proceed to construct explicitly such curves and bundles. To this purpose we will make us of degenerate elliptic curves. Since the proofs of the results below depend only on the fact that F is a del Pezzo threefold, we will state and prove such results in general. Thus, in what follows, F will be an arbitrary del Pezzo threefold of degree $d \geq 3$, embedded in $\mathbb{P}^{d+1}$. Let H be a general hyperplane in $\mathbb{P}^{d+1}$, and define $S := F \cap H$. The surface S is smooth and connected of degree d. Moreover, adjunction on F implies that S is a del Pezzo surface. In particular we know that S can be represented as the blow up of $\mathbb{P}^2$ in $9 - d$ points in general position or $d = 2$ and $S \cong \mathbb{P}^1 \times \mathbb{P}^1$.

We focus our attention on the first case, thus the Picard group of S is freely generated by the class ℓ of the pull–back of a general line in $\mathbb{P}^2$ and by the classes of the $9 - d$ exceptional divisors $e_1, \ldots, e_{9-d}$ of the blow up.

Clearly we have a natural map $A^1(S) \to A^2(F)$ and we want to inspect which elliptic curves on F come from curves on S. We will make such an analysis later on in the section. We now make some comments on the deformation theory of degenerate elliptic curves on a del Pezzo threefolds.

Proposition 6.2. *Let F be a del Pezzo threefold of degree $d \geq 3$. If $H \subseteq \mathbb{P}^{d+1}$ is a general hyperplane and $C \subseteq S := F \cap H$ is a smooth, connected elliptic curve of degree δ, then*

$$h^0\big(S, \mathcal{O}_S(C)\big) = \delta+1, \qquad h^0\big(C, \mathcal{N}_{C|F}\big) = 2\delta, \qquad h^1\big(S, \mathcal{O}_S(C)\big) = h^1\big(C, \mathcal{N}_{C|F}\big) = 0.$$

Proof. We have the isomorphisms $\mathcal{N}_{C|S} \cong \mathcal{O}_C(C)$ and $\mathcal{N}_{S|F} \cong \mathcal{O}_S(h)$. Thus we have the exact sequences

$$0 \longrightarrow \mathcal{O}_C(C) \longrightarrow \mathcal{N}_{C|F} \longrightarrow \mathcal{O}_C(h) \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0,$$

Riemann–Roch on C implies $h^0\big(C, \mathcal{O}_C(h)\big) = \delta$ and $h^1\big(C, \mathcal{O}_C(h)\big) = 0$. Let $a\ell - \sum_{i=1}^d b_i e_i$ be the class of C in $A^1(S)$. The conditions $\deg(C) = \delta$ and $p_a(C) = 1$ yield $C^2 = a^2 - \sum_{i=1}^d b_i^2 = \delta > 0$, thus $h^0\big(C, \mathcal{O}_C(C)\big) = \delta$ and $h^1\big(C, \mathcal{O}_C(C)\big) = 0$. Thus the first of the exact sequences above gives $h^0\big(C, \mathcal{N}_{C|F}\big) = 2\delta$, $h^1\big(C, \mathcal{N}_{C|F}\big) = 0$.

Since $h^1\big(S, \mathcal{O}_S\big) = 0$, it follows from the second exact sequence above that $h^0\big(S, \mathcal{O}_S(C)\big) = \delta + 1$. □

Now we come back to analyse the case $F \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, identifying the elliptic curves on F contained in a smooth hyperplane section. We use the same notation introduced at the end of the previous section: thus H is a general hyperplane in $\mathbb{P}^7$, and $S := F \cap H$.

Lemma 6.3. *C is a smooth, connected elliptic curve of degree 8 on S if and only if, up to automorphisms of S, C is a smooth connected element in the class either $3\ell - e_1$, or $4\ell - 2e_1 - 2e_2$.*

Proof. Let $a\ell - \sum_{i=1}^3 b_i e_i$ be the class of C in $A^1(S)$. Since C is effective, it follows that $a, b_i \geq 0$. Moreover, one easily checks that the conditions $\deg(C) = 8$ and $p_a(C) = 1$ yield

$$3a - \sum_{i=1}^3 b_i = 8, \qquad a^2 - \sum_{i=1}^3 b_i^2 = 8. \tag{13}$$

Schwarz's inequality implies $(\sum_{i=1}^3 b_i)^2 \leq 3 \sum_{i=1}^3 b_i^2$. Combining such an inequality with equalities (13), we finally obtain $6a^2 - 48a + 88 \leq 0$, i.e. $a = 3, 4, 5$.

Now, again by equalities (13), we obtain the cases $3\ell - e_1$, $4\ell - 2e_1 - 2e_2$, $5\ell - 3e_1 - 2e_2 - 2e_3$. The automorphism of S induced by the quadratic transformation of $\mathbb{P}^2$ centered at the blown up points maps (ℓ, e_1, e_2, e_3) to $(2\ell - e_1 - e_2 - e_3, \ell - e_2 - e_3, \ell - e_1 - e_3, \ell - e_1 - e_2)$, thus it transforms $5\ell - 3e_1 - 2e_2 - 2e_3$ in $3\ell - e_1$. It follows that we can restrict our attention to $3\ell - e_1$ and $4\ell - 2e_1 - 2e_2$ only. □

We recall that if E is a non–degenerate, smooth elliptic curve on F, its class in $A^2(F)$ is either $2h_2h_3 + 3h_1h_3 + 3h_1h_2$ or $2h_2h_3 + 2h_1h_3 + 4h_1h_2$. We will show in the following the existence of non–degenerate, smooth, connected elliptic curves on F of both types. To this purpose we will first show the existence of degenerate elliptic curves in both the classes and we will postpone the proof of the existence of non–degenerate curves at the end of the present section.

Lemma 6.4. *Let $H \subseteq \mathbb{P}^7$ be a general hyperplane and let C be a smooth, connected elliptic curve of degree 8 on $S := F \cap H$. The class of C in $A^2(F)$ is either $2h_2h_3 + 3h_1h_3 + 3h_1h_2$ or $2h_2h_3 + 2h_1h_3 + 4h_1h_2$.*

Proof. First we go to deal with the classes of ℓ and e_i in $A^2(F)$. Recall that the class of S inside $A^1(F)$ is the class of the hyperplane section, i.e. $h = h_1 + h_2 + h_3$: thus, e.g. $h_1S = h_1h = h_1h_2 + h_1h_3$. Moreover the class of the hyperplane section of S is $3\ell - e_1 - e_2 - e_3$ in $A^1(S)$.

We have $h_ih^2 = 2h_1h_2h_3$, thus $h_iS = h_ih$ is the class of a conic on S. Arguing as above one easily checks that the classes of conics on S are $\ell - e_i$. Thus we can assume that $h_iS = \ell - e_i$ in $A^1(S)$.

Now, we turn our attention to ℓ. Let $\gamma_1h_2h_3+\gamma_2h_1h_3+\gamma_3h_1h_2$ its class in $A^2(F)$. We have $\gamma_i = \ell h_i \geq 0$ and

$$\gamma_1 + \gamma_2 + \gamma_3 = \ell h = \deg(\ell) = \ell(3\ell - e_1 - e_2 - e_3) = 3$$

If $\gamma_1 = 0$, then $e_1h_3 = (\ell - h_1S)h_3 = -h_1h_2h_3$. Since e_1 and h_3 are both effective, we obtain an absurd. Thus $\gamma_1 \geq 1$ and, similarly, $\gamma_2, \gamma_3 \geq 1$. We conclude that the classes of ℓ, e_1, e_2, e_3 in $A^2(F)$ are $h_2h_3 + h_1h_3 + h_1h_2$, h_2h_3, h_1h_3, h_1h_2 respectively.

Thanks to Lemma 6.3, the assertion about the class of C in $A^2(F)$ now follows from direct substitution. □

The curves C described in Lemma 6.4 are degenerate, thus they do not correspond to Ulrich bundles (more precisely to aCM bundles) at all. Thus we have finally to show the existence of non–degenerate curves in the same classes. We will check this with a well–known deformation argument (see [20]).

Proposition 6.5. *If $E \subseteq F$ is a general element in the class either $2h_2h_3+3h_1h_3+3h_1h_2$ or $2h_2h_3 + 2h_1h_3 + 4h_1h_2$. Then E is a non–degenerate smooth, connected elliptic curve.*

Proof. Let $\mathcal{H}$ be the union of the components of $\mathcal{H}ilb^{8t}(\mathbb{P}^7)$ containing non–degenerate smooth connected elliptic curves of degree 8. Take the incidence variety

$$\mathcal{X} := \{\ (C, H) \in \mathcal{H} \times \check{\mathbb{P}}^7 \mid C \subseteq H\ \}$$

Let $(C, H) \in \mathcal{X}$ be general. In particular $H \subseteq \mathbb{P}^7$ is general, then $S := F \cap H$ is a smooth del Pezzo surface of degree 6 as above and $C \subseteq S$ is a smooth connected elliptic curve whose class in $A^2(F)$ is $\gamma_1h_2h_3+\gamma_2h_1h_3+\gamma_3h_1h_2$. Anyhow $\deg(C) = 8$, hence Proposition 6.2 yields $h^0\big(C, \mathcal{N}_{C|S}\big) = 16$ and $h^1\big(C, \mathcal{N}_{C|S}\big) = 0$. It follows that $\mathcal{H}$ is smooth at the point corresponding to C (briefly we say that C is unobstructed) and it has dimension 16. Assume that each deformation inside F of C were degenerate.

On the one hand, the image of the projection $\mathcal{X} \to \mathcal{H}$ would contain an open neighbourhood of the point corresponding to C. Since the points in the fibre over C are parametrized by the hyperplanes containing C, it follows that such a projection would be generically injective, hence $\mathcal{X}$ would contain an irreducible component $\widehat{\mathcal{X}}$ of dimension 16 containing (C, H).

On the other hand the fibre of the projection $\widehat{\mathcal{X}} \to \check{\mathbb{P}}^7$ over H is isomorphic to the projectivization of $H^0\big(S, \mathcal{O}_S(C)\big)$ which is $\mathbb{P}^8$ (see Proposition 6.2). We conclude that $\dim(\widehat{\mathcal{X}}) \leq 15$, a contradiction. We conclude that each general deformation of C inside F is non–degenerate in $\mathbb{P}^7$.

Let $\mathcal{C} \subseteq F \times B \to B$ a flat family of curves in $\mathcal{H}$ with special fibre $\mathcal{C}_{b_0} \cong C$ over b_0. Since C is unobstructed, we can assume that B is integral. Since $\mathcal{C} \to B$ is flat and C is integral, it follows that $\mathcal{C}$ is integral too. Take a general element in the class h_i, say Q_i, and consider the family $\mathcal{Q}_i := \mathcal{C} \cap (Q_i \times B) \to B$. Since Q_i is general and h_i is globally generated, we can assume that $\mathcal{Q}_i$ is a family of 0–cycles of F. Up to a proper choice of homogeneous coordinates $t_0^{(i)}, t_1^{(i)}$ on the ith copy of $\mathbb{P}^1$ inside the product F, we can assume $\mathcal{O}_{\mathcal{Q}_i} \cong \mathcal{O}_{\mathcal{C}}/(t_1^{(i)})$. Thanks to the choice of Q_i, the element $t_1^{(i)}$ is regular element in $\mathcal{O}_{\mathcal{C}}$. Thus the Corollary of Theorem 22.5 of [30], implies that $\mathcal{Q}_i$ is flat over B. By semicontinuity the degree of the fibre of $\mathcal{Q}_i$ over b, which is $\mathcal{C}_b h_i$, is upper semicontinuous, thus there exists an open subset $B_i \subseteq B$ containing b_0 such that $\mathcal{C}_b h_i \leq \gamma_i$. Since

$$\gamma_1 + \gamma_2 + \gamma_3 = 8 = \deg(\mathcal{C}_b) = \mathcal{C}_b h = \sum_{i=1}^{3} \mathcal{C}_b h_i \leq \gamma_1 + \gamma_2 + \gamma_3,$$

at each $b \in B' := B_1 \cap B_2 \cap B_3 \subseteq B$, we finally conclude that $\mathcal{C}_b h_i = \gamma_i$, i.e. $\mathcal{C}_b$ is in the same class of C inside $A^2(F)$. Hence each general element $E \subseteq F$ in the classes $2h_2h_3 + 3h_1h_3 + 3h_1h_2$ and $2h_2h_3 + 2h_1h_3 + 4h_1h_2$ is non–degenerate. □

We finally state our existence result.

Theorem 6.6. *There exist indecomposable initialized aCM bundles $\mathcal{E}$ of rank 2 with $c_1 = 2h$ and c_2 either $2h_2h_3 + 3h_1h_3 + 3h_1h_2$ or $2h_2h_3 + 2h_1h_3 + 4h_1h_2$ up to permutations of the h_i's.*

Moreover the zero–locus of a general section of $\mathcal{E}$ is an elliptic normal curve and each elliptic normal curve on F can be obtained in this way.

Proof. The statement follows from the previous proposition and from the correspondence between non–degenerate smooth elliptic curve and bundle with $c_1 = 2h$. □

7. The sporadic extremal case

In this section we deal with the case $c_1 = h_1 + 2h_2 + 3h_3$. Again Proposition 4.2 guarantees that $\mathcal{E}$ is Ulrich, hence globally generated, hence the zero–locus $E := (s)_0$ of a general section $s \in H^0\big(F, \mathcal{E}\big)$ is a smooth curve inside F.

Notice that $\mathcal{O}_F(c_1)$ is aCM. This fact and the cohomology of Sequence (2) yield $h^1\big(F, \mathcal{I}_{E|F}\big) = h^1\big(F, \mathcal{I}_{E|F}(h)\big) = 0$. It follows that both $h^0\big(E, \mathcal{O}_E\big) = 1$ and E is linearly normal (use Sequence (3)). We conclude that we can assume E to be a non–degenerate, smooth, rational curve. Moreover again by combining Lemma 4.1 with $D = 0$ and Formula (9), one easily obtains $\deg(E) = hc_2 = 7$. It follows that E is a rational normal curve in $\mathbb{P}^7$.

Conversely, let E be a non–degenerate, smooth, rational curve of degree 7. We know by adjunction that $\mathcal{O}_{\mathbb{P}^1}(-2) \cong \omega_E \cong \det(\mathcal{N}_{E|F}) \otimes \mathcal{O}_F(-2h)$. The invertible sheaf $\mathcal{O}_F(h_1+2h_2+3h_3)$ satisfies the vanishing of the Theorem 2.3, thus E is the zero locus of a section s of a vector bundle $\mathcal{E}$ of rank 2 on F with $c_1 = h_1+2h_2+3h_3$ and the class of E in $A^2(F)$ is c_2. The curve E is aCM (Proposition 1.1 and Corollary 2.2 of [15]), thus the natural restriction maps

$$\varphi_{-(2+t)} \colon H^0\big(F, \mathcal{O}_F(-(2+t)h)\big) \longrightarrow H^0\big(E, \mathcal{O}_E(-(2+t)h)\big)$$

are surjective. The same argument used for the previous case of elliptic curves shows that $h^1\big(F, \mathcal{I}_{E|F}(-(2+t)h)\big)$. Thus $h^2(F, \mathcal{E}(th)\big) = h^1(F, \check{\mathcal{E}}(-(2+t)h)\big) = 0$.

Thanks to System (10) we obtain that the possible values of $(\beta_1, \beta_2, \beta_3)$ are the following: $(2,5,0)$, $(4,1,2)$, $(3,3,1)$. The first case cannot occur, otherwise E would be contained in a divisor of the system $|h_1|$, thus it would be degenerate.

We check that $h^1(F, \mathcal{E}(th)) = 0$. Looking at Sequence (2), it suffices to check that $h^1\big(F, \mathcal{I}_{E|F}(c_1+th)\big) = 0$, $t \in \mathbb{Z}$. To this purpose we notice that $\deg(\mathcal{O}_E(c_1+th)) = (c_1+th)c_2 = 12+7t$. Since $E \cong \mathbb{P}^1$, it follows that $h^1\big(F, \mathcal{I}_{E|F}(c_1+th)\big) \leq h^0\big(E, \mathcal{O}_E(c_1+th)\big) = 0$ when $t \leq -2$.

Let us examine the case $t = -1$ and let $M := c_1 - h = h_2 + 2h_3$. On the one hand $\deg(\mathcal{O}_E(M)) = 5$. On the other hand E is integral and non–degenerate thus there is no hypersurface in the class $c_1 - h = h_2 + 2h_3$ containing E. Since $h^0\big(F, \mathcal{O}_F(M)\big) = 6$, it follows that the natural restriction map

$$\vartheta_{-1} \colon H^0\big(F, \mathcal{O}_F(M)\big) \to H^0\big(E, \mathcal{O}_E(M)\big),$$

whence again $h^1\big(F, \mathcal{I}_{E|F}(c_1-h)\big) = 0$.

Let us examine the case $t \geq 0$. In this case we have a commutative diagram

$$\begin{array}{ccc} H^0\big(F, \mathcal{O}_F(th)\big) \otimes H^0\big(F, \mathcal{O}_F(M)\big) & \xrightarrow{\varphi_t \otimes \vartheta_{-1}} & H^0\big(E, \mathcal{O}_E(th)\big) \otimes H^0\big(E, \mathcal{O}_E(M)\big) \\ \big\downarrow & & \vartheta_t \big\downarrow \\ H^0\big(F, \mathcal{O}_F(c_1+th)\big) & \longrightarrow & H^0\big(E, \mathcal{O}_E(c_1+th)\big) \end{array}$$

where the top horizontal arrow is surjective. Since $E \cong \mathbb{P}^1$ the right vertical arrow is an epimorphism. We conclude that ϑ_t is surjective too, hence $h^1\big(F, \mathcal{I}_{E|F}(c_1+th)\big) = 0$, $t \geq 0$.

The vanishing of $h^i(F, \mathcal{E}(th)) = 0$, $i = 1,2$ and $t \in \mathbb{Z}$, proved above implies that $\mathcal{E}$ is aCM, hence $\mathcal{E}$ is again an Ulrich bundle thanks to Proposition 4.2.

Thus non–degenerate rational curves of degree 7 on F correspond to Ulrich bundles on F with $c_1 = h_1 + 2h_2 + 3h_3$. Moreover we already checked that c_2 is either $3h_2h_3 + 3h_1h_3 + h_1h_2$ or $4h_2h_3 + h_1h_3 + 2h_1h_2$.

We first construct examples in the latter case. We will construct such bundles as extension of suitable invertible sheaves.

We have that

$$\dim_k \left(\operatorname{Ext}^1_F\big(\mathcal{O}_F(2h_2+h_3), \mathcal{O}_F(h_1+2h_3)\big)\right) = h^1\big(F, \mathcal{O}_F(h_1-2h_2+h_3)\big) = 4,$$

thus there are non–trivial extensions of the form

$$0 \longrightarrow \mathcal{O}_F(h_1+2h_3) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_F(2h_2+h_3) \longrightarrow 0. \tag{14}$$

Being an extension of invertible sheaves, then $\mathcal{E}$ is a vector bundle of rank 2. Its Chern classes are $c_1 = h_1 + 2h_2 + 3h_3$ and $c_2 = 4h_2h_3 + h_1h_3 + 2h_1h_2$.

Taking the cohomology of Sequence (14) twisted by $\mathcal{O}_F(th)$, then the vanishing $h^i\big(F, \mathcal{O}_F(2h_2+h_3+th)\big) = h^i\big(F, \mathcal{O}_F(h_1+2h_3+th)\big) = 0$, $i = 1,2$ and $t \in \mathbb{Z}$, implies that $\mathcal{E}$ is aCM. Moreover, the same cohomology sequence and vanishing also yield that $\mathcal{E}$ is initialized, $h^0\big(F, \mathcal{E}\big) = 12$ and $\mathcal{E}$ is generated by its global sections.

We have to check that $\mathcal{E}$ is indecomposable, i.e. it is not the sum of invertible sheaves. Assume $\mathcal{E} \cong \mathcal{L}_1 \oplus \mathcal{L}_2$ for some $\mathcal{L}_i \in \operatorname{Pic}(F)$. Thus we know that $\mathcal{L}_i$ is aCM. Moreover either both the $\mathcal{L}_i$ are initialized or one of them is initialized and the other has no sections.

Thanks to Lemma 2.4 and to the equalities

$$h^0\big(F,\mathcal{L}_1\big)+h^0\big(F,\mathcal{L}_2\big)=h^0\big(F,\mathcal{E}\big)=12,$$
$$c_1(\mathcal{L}_1)+c_1(\mathcal{L}_1)=c_1=h_1+2h_2+3h_3$$

we know that $\mathcal{E}\cong\mathcal{O}_F(2h_2+h_3)\oplus\mathcal{O}_F(h_1+2h_3)$. If this is the case, since

$$\dim\big(\mathrm{Hom}_F\big(\mathcal{O}_F(h_1+2h_3),\mathcal{O}_F(2h_2+h_3)\big)\big)=h^0\big(F,\mathcal{E}(-h_1+2h_2-h_3)\big)=0,$$

it would follow that Sequence (14) splits, contradicting its non–triviality.

The above discussion partially proves the following result.

Theorem 7.1. *There exist indecomposable initialized aCM bundles $\mathcal{E}$ of rank 2 with $c_1=h_1+2h_2+3h_3$ and c_2 either $4h_2h_3+h_1h_3+2h_1h_2$ or $3h_2h_3+3h_1h_3+h_1h_2$.*

Moreover the zero–locus of a general section of $\mathcal{E}$ is an rational normal curve and each rational normal curve on F can be obtained in this way.

Proof. The above discussion proves the existence of bundles with $c_2=4h_2h_3+h_1h_3+2h_1h_2$.

Now we deal with the case $c_2=3h_2h_3+3h_1h_3+h_1h_2$. In this case we work by deforming suitable curves on a hyperplane section of F as we did in the previous section. Let $S:=H\cap F$ be a general hyperplane section of F. S is a del Pezzo surface of degree 6 in $\mathbb{P}^6$. We know by the proof of Lemma 6.4 that the classes of ℓ, e_1, e_2, e_3 in $A^2(F)$ are $h_2h_3+h_1h_3+h_1h_2$, h_2h_3, h_1h_3, h_1h_2 respectively. Using the same notations of Lemma 6.3 one easily checks that S contains rational curves of degree 7 and their classes are $3\ell-2e_3$, $4\ell-e_1-e_2-3e_3$ which are both in the class of $3h_2h_3+3h_1h_3+h_1h_2$.

Take a smooth rational curve $C\subseteq S$ of degree 7. We have $\mathcal{N}_{C|S}\cong\mathcal{O}_C(C)$ and $\mathcal{N}_{S|F}\cong\mathcal{O}_S(hS)$, thus the exact sequences

$$0\longrightarrow\mathcal{O}_C(C)\longrightarrow\mathcal{N}_{C|F}\longrightarrow\mathcal{O}_C(h)\longrightarrow 0,$$
$$0\longrightarrow\mathcal{O}_S\longrightarrow\mathcal{O}_S(C)\longrightarrow\mathcal{O}_C(C)\longrightarrow 0,$$

Since $C\cong\mathbb{P}^1$, it follows that $\mathcal{O}_C(C)\cong\mathcal{O}_{\mathbb{P}^1}(5)$, $\mathcal{O}_C(h)\cong\mathcal{O}_{\mathbb{P}^1}(7)$, thus $h^0\big(S,\mathcal{O}_S(C)\big)=7$, $h^0\big(C,\mathcal{N}_{E|F}\big)=14$, $h^1\big(C,\mathcal{N}_{C|F}\big)=0$.

Arguing as in the proof of Proposition 6.5 one checks the existence of a non–degenerate smooth rational curve E whose class is $3h_2h_3+3h_1h_3+h_1h_2$. □

8. The intermediate cases

In this section we will show which other indecomposable, initialized, aCM bundles $\mathcal{E}$ of rank 2 are actually admissible besides the ones occurring in the extremal cases. We know that for the general section $s\in H^0\big(F,\mathcal{E}\big)$ its zero–locus $E:=(s)_0$ has pure dimension 1.

System (10), Inequality (11) with $D = 0$ and Formulas (8), (9) yield the following list (up to permutations).

Case	$(\alpha_1, \alpha_2, \alpha_3)$	$(\beta_1, \beta_2, \beta_3)$	$\deg(E)$	$p_a(E)$
L	$(0, 0, 2)$	$(0, 0, 0)$	0	-1
M	$(0, 0, 1)$	$(1, 0, 0)$	1	0
N	$(0, 1, 1)$	$(1, 0, 0)$	1	0
P	$(0, 1, 2)$	$(1, 0, 0)$	1	0
Q	$(1, 1, 1)$	$(1, 1, 0)$	2	0
R	$(1, 1, 1)$	$(2, 0, 0)$	2	0
S	$(0, 2, 2)$	$(2, 0, 0)$	2	-1
T	$(1, 1, 2)$	$(1, 1, 1)$	3	0
U	$(1, 1, 2)$	$(0, 2, 1)$	3	0
V	$(1, 2, 2)$	$(2, 2, 1)$	5	0
W	$(1, 2, 2)$	$(2, 3, 0)$	5	0

It is immediate to exclude the case L because $E \neq \emptyset$ due to Lemma 3.2.

Let $\mathcal{E}$ be in the list above. Then $\check{\mathcal{E}}(th)$ is an indecomposable aCM bundle too. We take t in such a way that it is also initialized. Thus $0 \leq 2t - \alpha_1 \leq 2$, whence $t = 1$. We have

$$\begin{aligned} c_1(\check{\mathcal{E}}(h)) &= (2-\alpha_1)h_1 + (2-\alpha_2)h_2 + (2-\alpha_3)h_3, \\ c_2(\check{\mathcal{E}}(h)) &= (\beta_1 + 2 - (\alpha_2+\alpha_3))h_2h_3 + \\ &\quad + (\beta_2 + 2 - (\alpha_1+\alpha_3))h_1h_3 + (\beta_3 + 2 - (\alpha_1+\alpha_2))h_1h_2 \end{aligned}$$

In case S, then $c_2(\check{\mathcal{E}}(h)) = 0$, thus we can again exclude it. In case U (resp. W), then $c_2(\check{\mathcal{E}}(h)) = -h_2h_3 + h_1h_3 + h_1h_2$ (resp. $2h_1h_3 - h_1h_2$). Due to the positivity of the β_i's (see Formula (11)) also these two cases cannot occur. The same argument shows that cases T and V occur if and only if cases N and M occur respectively.

We now go to examine the remaining cases one by one, checking which of them actually occurs. We have to analyze cases M, N, P, Q, R.

8.1. **The case M (and V).** In case M we know by the table that E must be a line.

Let $E \subseteq F$ be a line in the class $h_2h_3 \in A^2(F)$. Since $\omega_E \cong \mathcal{O}_E(-2h)$, it follows that $\det(\mathcal{N}_{E|F}) \cong \mathcal{O}_E$ which thus coincides with $\mathcal{O}_F(h_3) \otimes \mathcal{O}_E$. Taking into account that $h^2\big(F, \mathcal{O}_F(-h_3)\big) = 0$, we thus know the existence of an exact sequence

$$0 \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|F}(h_3) \longrightarrow 0.$$

where $\mathcal{E}$ is a vector bundle of rank 2 on F (see Theorem 2.3). It is immediate to check that $\mathcal{E}$ is initialized. We now check that $\mathcal{E}$ is aCM. Trivially we have $h^1\big(F, \mathcal{E}(th)\big) = h^1\big(F, \mathcal{I}_{E|F}(th_1 + th_2 + (t+1)h_3)\big)$.

The cohomology of Sequence (3) twisted by $\mathcal{O}_F(th_1 + th_2 + (t+1)h_3)$ gives rise to the exact sequence

$$\begin{aligned} H^0\big(F, \mathcal{O}_F(th_1 + th_2 + (t+1)h_3)\big) &\longrightarrow H^0\big(E, \mathcal{O}_E(th_1 + th_2 + (t+1)h_3)\big) \longrightarrow \\ &\longrightarrow H^1\big(F, \mathcal{I}_{E|F}(th_1 + th_2 + (t+1)h_3)\big) \longrightarrow 0. \end{aligned}$$

The multiplication by h_3 gives rise to the commutative diagram

$$\begin{array}{ccc} H^0\big(F,\mathcal{O}_F(th)\big) & \xrightarrow{\varphi_t} & H^0\big(E,\mathcal{O}_E(th)\big) \\ \big\downarrow & & \big\downarrow{\scriptstyle \psi_t} \\ H^0\big(F,\mathcal{O}_F(th_1+th_2+(t+1)h_3)\big) & \longrightarrow & H^0\big(E,\mathcal{O}_E(th_1+th_2+(t+1)h_3)\big) \end{array}$$

Since both E and F are aG, it is easy to check that φ_t is surjective (see the proof of the analogous fact in Section 6). Moreover E is in the class of h_2h_3, thus ψ_t is an isomorphism. It follows that the map below is surjective, whence $h^1\big(F,\mathcal{I}_{E|F}(th_1+th_2+(t+1)h_3)\big)=0$.

Now we look at the vanishing of H^2. We know that we can also write the exact sequence

$$0\longrightarrow \mathcal{O}_F(-h_3)\longrightarrow \check{\mathcal{E}}\longrightarrow \mathcal{I}_{E|F}\longrightarrow 0.$$

We thus obtain

$$h^2\big(F,\mathcal{E}(th)\big)=h^1\big(F,\check{\mathcal{E}}(-(2+t)h)\big)=h^1\big(F,\mathcal{I}_{E|F}(-(2+t)h)\big).$$

But the last number is the dimension of the cokernel of the map $\varphi_{-(2+t)}$ which is zero. We conclude that $\mathcal{E}$ is aCM.

Assume that $\mathcal{E}$ is decomposable. Thus it is the sum of two initialized aCM invertible sheaves. Such sheaves are listed in Lemma 2.4. Looking at c_1 the unique possibility is then $\mathcal{E}\cong\mathcal{O}_F\oplus\mathcal{O}_F(h_3)$. But, in this case, $c_2=0$. We conclude that $\mathcal{E}$ is indecomposable.

In particular both the cases M and V are admissible.

8.2. **The case N (and T).** In case N we know by the table that $E\subseteq F$ is a line in the class $h_2h_3\in A^2(F)$. On the one hand we know that $\det(\mathcal{N}_{E|F})\cong\mathcal{O}_F(h_2+h_3)\otimes\mathcal{O}_E$. Since $h^2\big(F,\mathcal{O}_F(-h_2-h_3)\big)=0$, it follows the existence of an exact sequence

$$0\longrightarrow \mathcal{O}_F(-h_2-h_3)\longrightarrow \check{\mathcal{E}}\longrightarrow \mathcal{I}_{E|F}\longrightarrow 0.$$

where $\mathcal{E}$ is a vector bundle of rank 2 on F. On the other hand we also have the exact sequence

$$0\longrightarrow \mathcal{O}_F(-h_2-h_3)\longrightarrow \mathcal{O}_F(-h_2)\oplus\mathcal{O}_F(-h_3)\longrightarrow \mathcal{I}_{E|F}\longrightarrow 0.$$

Recall that we have an exact sequence of the form (e.g., see the proof of Theorem 1 of [2]) is

$$0\longrightarrow H^1\big(F,\mathcal{O}_F(-h_2-h_3)\big)\longrightarrow \mathrm{Ext}^1_F\big(\mathcal{I}_{E|F},\mathcal{O}_F(-h_2-h_3)\big)\longrightarrow H^0\big(E,\mathcal{O}_E\big)\longrightarrow 0.$$

Since $h^0\big(E,\mathcal{O}_E\big)=1$, it follows that there exists a unique non–trivial extension of $\mathcal{I}_{E|F}$ by $\mathcal{O}_F(-h_2-h_3)$, thus $\mathcal{E}\cong\mathcal{O}_F(h_2)\oplus\mathcal{O}_F(h_3)$.

In particular both the cases N and T are not admissible in the sense that the sheaf $\mathcal{E}$ exists, but it is decomposable.

8.3. **The case P.** Arguing as in the case N we check that $\mathcal{E}\cong\mathcal{O}_F(h_2+h_3)\oplus\mathcal{O}_F(h_3)$. Hence this case is not admissible in the sense that the sheaf $\mathcal{E}$ exists, but it is decomposable.

8.4. **The cases Q and R.** In these cases E is a curve of degree 2 of genus 0. E is either irreducible or the union of two concurrent lines, because $p_a(E) = 0$, or a double line.

In case Q only the first two cases are possible, thus E is reduced. In this case $\omega_E \cong \mathcal{O}_E(-h)$, whence $\det(\mathcal{N}_{E|F}) \cong \mathcal{O}_E(h)$. Since $h^2\big(F, \mathcal{O}_F(h)\big) = 0$, we thus know the existence of an exact sequence

$$0 \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|F}(h) \longrightarrow 0.$$

where $\mathcal{E}$ is a vector bundle of rank 2 on F. $\mathcal{E}$ is easily seen to be initialized and aCM. Looking at c_1 and taking into account Lemma 2.4 one also immediately verify that $\mathcal{E}$ is indecomposable too.

Let us examine case R. We claim that E is not reduced. E.g., assume E is the union of two distinct concurrent lines, say L, M, and let $p := L \cap M$ be their intersection point. Thus there are $Q_i \in |h_i|$, $i = 1, 2$ such that $p \in Q_1 \cap Q_2$. Since $LQ_i + MQ_i = EQ_i = 0$, it would follow that $E = L \cup M \subseteq Q_1 \cap Q_2$, whence its class inside $A^2(F)$ would be h_2h_3. But we know that its class is $c_2 = 2h_2h_3$, a contradiction. If E is irreducible we can argue similarly.

Thus E is a double structure on a line E_{red} whose class in $A^2(F)$ is h_2h_3 necessarily. The general theory of double structures (see e.g. [7]) gives us an exact sequence of the form

$$0 \longrightarrow \mathcal{C}_{E_{red}|E} \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{O}_{E_{red}} \longrightarrow 0.$$

The conormal sheaf $\mathcal{C}_{E_{red}|E}$ is an invertible sheaf on $E_{red} \cong \mathbb{P}^1$, thus $\mathcal{C}_{E_{red}|E} \cong \mathcal{O}_{\mathbb{P}^1}(-a)$. Moreover $a = 1$, because $p_a(E) = 0$ (see Section 2 of [7]).

Recall that the Hilbert scheme Γ of lines in F is the union of three components Γ_i, each of them isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (Proposition 3.5.6 of [24]).

On the one hand, all the lines on F in the class $h_2h_3 \in A^2(F)$ are parameterized by $\mathbb{P}^1 \times \mathbb{P}^1$, hence we can assume that Γ_1 parameterizes exactly the set of lines on F in the class h_2h_3. The total space of the universal family $S_1 \to \Gamma_1$ is $S_1 \cong \Gamma_1 \times \mathbb{P}^1 \cong F$, thus the natural projection $p_1 \colon S_1 \to F$ is an isomorphism. Thus the line L corresponding to the general point in Γ_i satisfies $\mathcal{C}_{L|F} \cong \tilde{\mathcal{N}}_{L|F} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$, by combining Lemma 3.3.4 and Proposition 3.3.5 of [24]. Moreover, the isomorphism $S \cong F$ also implies the existence of an automorphism of F fixing the class h_2h_3 and sending E_{red} to L. We conclude that $\mathcal{C}_{E_{red}|F} \cong \mathcal{C}_{L|F} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$.

On the other hand, the inclusion $E_{red} \subseteq E \subseteq \mathcal{I}_{F|\mathbb{P}^7}$ implies the existence of an epimorphism $\mathcal{C}_{E_{red}|F} \twoheadrightarrow \mathcal{C}_{E_{red}|E}$. We conclude the existence of an epimorphism $\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(-1)$, an absurd.

In particular case Q is admissible and case R is not admissible, because the bundle $\mathcal{E}$ does not exist at all.

We summarize the results of this section in the following result.

Theorem 8.1. *Let $\mathcal{E}$ be an indecomposable initialized aCM bundle of rank 2 on F and let $c_1 = \alpha_1h_1 + \alpha_2h_2 + \alpha_3h_3$. Assume that c_1 is neither 0, nor $2h$, nor $h_1 + 2h_2 + 3h_3$ up to permutations of the h_i's. Then (c_1, c_2) is either (h_3, h_2h_3), or $(h_1 + h_2 + h_3, h_2h_3 + h_1h_3)$, or $(h_1 + 2h_2 + 2h_3, 2h_2h_3 + 2h_1h_3 + h_1h_2)$, up to permutations of the h_i's.*

Conversely, for each such a pair, there exists and there exist an indecomposable initialized aCM bundle $\mathcal{E}$ of rank 2 on F with Chern classes c_1 and c_2.

Moreover the zero–locus of a general section of $\mathcal{E}$ is respectively a line, a, possibly reducible, reduced conic, a, possibly reducible, quintic with arithmetic genus 0 *and each such curve on F can be obtained in this way.*

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