RANK TWO ACM BUNDLES ON THE DEL PEZZO FOURFOLD OF DEGREE 6 AND ITS GENERAL HYPERPLANE SECTION

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Abstract. In the present paper we completely classify locally free sheaves of rank 2 with vanishing intermediate cohomology modules on the image of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \subseteq \mathbb{P}^8$ and its general hyperplane sections.

Such a classification extends similar already known results regarding del Pezzo varieties with Picard numbers 1 and 3 and dimension at least 3.

1. Introduction

Let $\mathbb{P}^N$ be the N-dimensional projective space over an algebraically closed field k of characteristic 0. If $X \subseteq \mathbb{P}^N$ is a smooth closed submanifold of dimension $n \ge 1$ we set $\mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_X$.

Among vector bundles $\mathcal{F}$ on such an X, the simplest ones from the cohomological point of view satisfy the vanishing

$$H^i\big(X, \mathcal{F}(t)\big) = 0, \qquad \forall\ i = 1, \dots, n-1,\ t \in \mathbb{Z}.$$

Such vector bundles are called arithmetically Cohen-Macaulay (aCM for short).

It is possible to classify aCM bundles completely for varieties belonging to a short precise list, consisting (for $n \ge 2$) of projective spaces, smooth quadrics, the Veronese surface in $\mathbb{P}^5$ and rational normal surface scrolls of degree up to 4 (see [21] for quartic scrolls and [26], [28], [5], [32] for the remaining cases: see also also [17]). However, for any other variety such a classification is hopeless, due to the "wild" behavior of the category of maximal Cohen–Macaulay modules (which are the algebraic counterpart of aCM bundles), as described in [22].

In spite of this, a detailed study of aCM bundles of low rank, in particular of rank 2, is feasible for certain types of varieties, most notably Fano manifolds. These are smooth projective varieties X of dimension n, such that ω_X^{-1} is ample (see [27] for a review about Fano varieties). Let r be the greatest positive integer such that $\omega_X \cong \mathcal{L}^{-r}$ for some ample $\mathcal{L} \in \mathrm{Pic}(X)$. It is known that $1 \le r \le n+1$ and $r = n+1$ (resp. $r = n$) if and only if $X = \mathbb{P}^n$ (resp. X is a quadric hypersurface).

We are thus interested in the case $r \le n-1$. Fano manifolds satisfying the equality $r = n-1$ are called del Pezzo. They are completely classified and fall into a very short list (see [23], [27]).

We focus our attention to the case $n \ge 3$: for simplicity we assume $\mathcal{L}$ to be very ample. Several results about aCM bundles on Fano and del Pezzo threefolds can be found in [2], [4], [7], [10], [30], [19], [3], when the Picard number $\varrho(X) = \mathrm{rk}\,\mathrm{Pic}(X)$ of X equals 1. Let us point out that aCM bundles of rank 2 on our variety X are intimately related to several important features of X, as for instance for a cubic hypersurface X these bundles are in bijection with inequivalent Pfaffian representations of X. In a different direction, stable aCM bundles $\mathcal{F}$ of rank 2 with $c_1(\mathcal{F}) = 0$ on a del Pezzo threefold X correspond to instantons of minimal charge (see [20], [29]) which appear in connection with the intermediate Jacobian of X, and with periods of other classes of Fano manifolds (see [31]). Anyway these bundles have been studied mainly under the assumption $\varrho(X) = 1$, while

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technical difficulties arise for higher values of $\varrho(X)$, mainly because the correspondence between a rank 2 bundle $\mathcal{F}$ and the zero locus of a global section of $\mathcal{F}$ is unclear a priori: this is indeed the starting point of our analysis.

The first complete classification of aCM bundles of rank 2 on del Pezzo manifolds X with $\varrho(X) \geq 2$ is presented in [11] for the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^7$. The moduli spaces of these bundles are studied in [12]. In this case $\deg(X) = 6$ and $\varrho(X)$ is as high as possible.

The aim of the present paper is to give a complete description of indecomposable aCM bundles of rank 2 on the other del Pezzo manifolds of degree 6, namely the Segre embedding $\Phi := \mathbb{P}^2 \times \mathbb{P}^2 \subseteq \mathbb{P}^8$ and its general hyperplane section F. As we explain below, both F and Φ have Picard number 2.

In order to formulate our main results, we introduce some notation. Let $A(X)$ be the Chow ring of X, so that $A^r(X)$ denotes the set of cycles of codimension r. The two projections $\pi_i\colon \Phi \to \mathbb{P}^2$, $i = 1, 2$, are isomorphic to the canonical map $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}) \to \mathbb{P}^2$. We denote by η_i, $i = 1, 2$, the classes of $\pi_i^*\mathcal{O}_{\mathbb{P}^2}(1)$ in $A^1(\Phi)$, so that $\operatorname{Pic}(\Phi) \cong \mathbb{Z}^{\oplus 2}$ is generated by η_1 and η_2. Clearly, η_2 and η_1 are respectively identified with the tautological divisors of π_1 and π_2, and the class of the hyperplane divisor on Φ is $\eta = \eta_1 + \eta_2$. Then:

$$A(\Phi) \cong A(\mathbb{P}^2) \otimes A(\mathbb{P}^2) \cong \mathbb{Z}[\eta_1, \eta_2]/(\eta_1^3, \eta_2^3).$$

The morphisms π_i induce maps $p_i\colon F \to \mathbb{P}^2$ by restriction, $i = 1, 2$. Such maps are isomorphic to the canonical map $\mathbb{P}(\Omega^1_{\mathbb{P}^2}(2)) \to \mathbb{P}^2$. Thinking of the second copy of $\mathbb{P}^2$ as the dual of the first one, then F can also be viewed naturally as the flag variety of pairs point–line in $\mathbb{P}^2$. Let h_i, $i = 1, 2$, be the classes of $p_i^*\mathcal{O}_{\mathbb{P}^2}(1)$ in $A^1(F)$ respectively. As for Φ we have that h_2 and h_1 are respectively identified with the tautological divisors of p_1 and p_2, so that the class of the hyperplane divisor on F is $h = h_1 + h_2$. The above discussion proves the isomorphisms

$$A(F) \cong A(\mathbb{P}^2)[h_1]/(h_1^2 - h_1h_2 + h_2^2) \cong \mathbb{Z}[h_1, h_2]/(h_1^2 - h_1h_2 + h_2^2, h_1^3, h_2^3).$$

In particular, $\operatorname{Pic}(F) \cong \mathbb{Z}^{\oplus 2}$ with generators h_1 and h_2.

In Section 2 we briefly discuss the necessary background, which includes the definition of an initialized vector bundle and the Hartshorne–Serre correspondence on a variety $X \subseteq \mathbb{P}^N$, a correspondence between rank 2 vector bundles on X and locally complete intersection subschemes $Y \subseteq X$ of codimension 2 and subcanonical, i.e. such that the dualizing sheaf satisfies $\omega_Y \cong \mathcal{O}_{\mathbb{P}^N}(\alpha) \otimes \mathcal{O}_Y$ for some $\alpha \in \mathbb{Z}$.

In particular we give therein a characterization of aCM line bundles on F and we prove that initialized, indecomposable, aCM bundles of rank 2 on Φ restrict on F to bundles with the same properties in the final part of the section. Thus the descriptions of bundles on F and Φ are strictly interlaced. We will use such a relationship in the whole paper: indeed, usually, we will first deal with one of the del Pezzo sextics, then using the obtained results for dealing with the other one.

Our main results are the following (see Lemma 4.3 and Theorems 5.2, 5.7, 6.3, 5.1, 5.8, 6.5 for more detailed statements).

Main Theorem for F. *If $\mathcal{E}$ is an indecomposable, initialized, aCM bundle of rank 2 on F, then the zero locus E of a general section $s \in H^0\big(F, \mathcal{E}\big)$ has pure codimension 2.*

Moreover one of the following cases occur up to permutations of the h_i's.

1. $c_1(\mathcal{E}) = 0$ *and* $c_2(\mathcal{E}) = h_2^2$: *E is a line.*
2. $c_1(\mathcal{E}) = h_2$ *and* $c_2(\mathcal{E}) = h_2^2$: $\mathcal{E} \cong p_2^*\Omega^1_{\mathbb{P}^2}(2h_2)$ *and E is a line.*
3. $c_1(\mathcal{E}) = 2h_1 + h_2$ *and* $c_2(\mathcal{E}) = 2h_1^2 + 2h_2^2$: $\mathcal{E} \cong p_2^*\Omega^1_{\mathbb{P}^2}(h_1 + 2h_2)$ *and E is an integral, smooth, rational curve of degree 4.*
4. $c_1(\mathcal{E}) = 2h$ *and* $c_2(\mathcal{E}) = 3h_2^2 + 5h_1^2$: *$E$ is an elliptic normal curve in $\mathbb{P}^8$.*
5. $c_1(\mathcal{E}) = 2h$ *and* $c_2(\mathcal{E}) = 4h_2^2 + 4h_1^2$: *$E$ is an elliptic normal curve in $\mathbb{P}^8$.*

Main Theorem for Φ. *If $\mathcal{G}$ is an indecomposable, initialized, aCM bundle of rank 2 on Φ, then the zero locus Σ of a general section $\sigma \in H^0\big(\Phi, \mathcal{G}\big)$ has pure codimension 2.*

Moreover one of the following cases occur up to permutations of the η_i's.

1. $c_1(\mathcal{G}) = 0$ *and* $c_2(\mathcal{G}) = \eta_2^2$: *$\Sigma$ is a plane.*

(2) *$c_1(\mathcal{G})=\eta_2$ and $c_2(\mathcal{G})=\eta_2^2$: $\mathcal{G}\cong\pi_2^*\Omega^1_{\mathbb{P}^2}(2\eta_2)$ and Σ is a plane.*
(3) *$c_1(\mathcal{G})=2\eta_1+\eta_2$ and $c_2(\mathcal{G})=2\eta_1^2+2\eta_2^2$: $\mathcal{G}\cong\pi_2^*\Omega^1_{\mathbb{P}^2}(\eta_1+2\eta_2)$ and Σ is an integral, smooth surface of degree 4 and sectional genus 0.*
(4) *$c_1(\mathcal{G})=2\eta$ and $c_2(\mathcal{G})=\eta_2^2+3\eta_1^2+2\eta_1\eta_2$: Σ is a del Pezzo surface of degree 8 isomorphic to the blow up $\mathbb{F}_1$ of $\mathbb{P}^2$ at a point.*

In Sections 3 and 4 we give general bounds on c_1 and γ_1, proving that the general sections of $\mathcal{E}$ and $\mathcal{G}$ vanish exactly in codimension 2. In Section 5 we provide a complete classification of bundles on F and Φ with first Chern class $c_1=0,2h$ and $\gamma_1=0,2\eta$ respectively. Finally, in Section 6, we will determine which intermediate cases are actually admissible.

The study of (semi)stability of aCM bundles of rank 2 on F and Φ and of the moduli spaces of such (semi)stable bundles will be the object of a future paper.

Throughout the whole paper we refer to [27] and [25] for all the unmentioned definitions, notations and results.

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2. Some facts on aCM locally free sheaves

Throughout the whole paper k will denote an algebraically closed field of characteristic 0. Let $X\subseteq\mathbb{P}^N$ be a subvariety, i.e. an integral closed subscheme defined over k, of dimension $n\ge1$. We set $\mathcal{O}_X(1):=\mathcal{O}_{\mathbb{P}^N}(1)\otimes\mathcal{O}_X$. We start this section by recalling two important definitions.

The variety $X\subseteq\mathbb{P}^N$ is called aCM if the restriction maps $H^0\big(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(t)\big)\to H^0\big(X,\mathcal{O}_X(t)\big)$ are surjective and $h^i\big(X,\mathcal{O}_X(t)\big)=0$, $1\le i\le n-1$.

In what follows X will be an aCM, integral and smooth subvariety of $\mathbb{P}^N$ of dimension $n\ge1$.

Definition 2.1. Let $\mathcal{F}$ be a vector bundle on X. We say that $\mathcal{F}$ is *arithmetically Cohen–Macaulay* (aCM for short) if the modules $H^i_*\big(X,\mathcal{F}\big)=\bigoplus_{t\in\mathbb{Z}}H^i\big(X,\mathcal{F}(t)\big)$ vanish for all $i=1,\dots,n-1$.

If $\mathcal{F}$ is an aCM bundle, then the minimal number of generators $m(\mathcal{F})$ of $H^0_*\big(X,\mathcal{F}\big)$ as a module over the graded coordinate ring of X is $\operatorname{rk}(\mathcal{F})\deg(X)$ at most (e.g. see [8]).

Definition 2.2. Let $\mathcal{F}$ be a vector bundle on X. We say that $\mathcal{F}$ is *initialized* if

$$\min\{\ t\in\mathbb{Z}\mid h^0\big(X,\mathcal{F}(t)\big)\ne0\ \}=0.$$

We say that $\mathcal{F}$ is *Ulrich* if it is initialized, aCM and $h^0\big(X,\mathcal{F}\big)=\operatorname{rk}(\mathcal{F})\deg(X)$.

Let $\mathcal{F}$ be an Ulrich bundle. On the one hand we know that $m(\mathcal{F})\le\operatorname{rk}(\mathcal{F})\deg(X)$. On the other hand the generators of $H^0\big(X,\mathcal{F}\big)$ are minimal generators of $H^0_*\big(X,\mathcal{F}\big)$ due to the vanishing of $H^0\big(X,\mathcal{F}(-1)\big)$. We conclude that $\mathcal{F}$ is necessarily globally generated. Several other results are known for Ulrich bundles (e.g. see [18], [8], [9], [14]).

Let f be a global section of a rank 2 vector bundle $\mathcal{F}$. Its zero–locus $(f)_0\subseteq X$ is either empty or its codimension is at most 2. Thus we can always write $(f)_0=C\cup\Delta$ where C has pure codimension 2 (or it is empty) and Δ has pure codimension 1 (or it is empty). In particular $\mathcal{F}(-\Delta)$ has a section f_Δ vanishing on C, which is thus locally complete intersection inside X. Moreover, the Koszul complex of f_Δ twisted by $\mathcal{O}_X(\Delta)$ is

$$0\longrightarrow\mathcal{O}_X(\Delta)\longrightarrow\mathcal{F}\longrightarrow\mathcal{I}_{C|X}(c_1(\mathcal{F})-\Delta)\longrightarrow0. \tag{1}$$

Moreover we also have the following exact sequence

$$0\longrightarrow\mathcal{I}_{C|X}\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{O}_C\longrightarrow0. \tag{2}$$

The above construction can be reversed, because the Hartshorne–Serre correspondence holds (for further details see [33], [24], [1]).

Theorem 2.3. *Let $C\subseteq X$ be a local complete intersection subscheme of codimension 2 and $\mathcal{L}$ an invertible sheaf on X such that $H^2\big(X,\mathcal{L}^\vee\big)=0$. If $\det(\mathcal{N}_{C|X})\cong\mathcal{L}\otimes\mathcal{O}_C$, then there exists a vector bundle $\mathcal{F}$ of rank 2 on X such that:*

(1) *$\det(\mathcal{F}) \cong \mathcal{L}$;*
(2) *$\mathcal{F}$ has a section f such that C coincides with the zero locus $(f)_0$ of f.*

Moreover, if $H^1\big(X, \mathcal{L}^\vee\big) = 0$, the two conditions above determine $\mathcal{F}$ up to isomorphism.

From now on we will focus our attention on the del Pezzo fourfold $\Phi \subseteq \mathbb{P}^8$ of degree 6 and its general hyperplane section $F \subseteq \mathbb{P}^7$ which are both aCM varieties. The rest of this section is devoted to inspecting the relations between aCM bundles of ranks 1 and 2 on F and Φ.

As pointed out in the Introduction

$$A(\Phi) \cong \mathbb{Z}[\eta_1, \eta_2]/(\eta_1^3, \eta_2^3), \qquad A(F) \cong \mathbb{Z}[h_1, h_2]/(h_1^2 - h_1h_2 + h_2^2, h_1^3, h_2^3).$$

We recall the following representation of F. Let $B \subseteq \mathrm{GL}_3$ be the subgroup of upper triangular matrices. The group B is mapped by the natural morphism $\mathrm{GL}_3 \to \mathrm{PGL}_3$ onto the subgroup fixing both the point $[1, 0, 0]$ and the line $\{\ x_2 = 0\ \}$. It follows the existence of a natural isomorphism $F \cong \mathrm{GL}_3 / B$. Such a representation of F is particularly helpful, because it allows us to state the following consequence of Borel–Weil–Bott Theorem (see [16],Theorem 2) in order to compute the cohomology of the line bundles on F.

Theorem 2.4. *If $\mathcal{L} \in \mathrm{Pic}(F)$, then $h^i\big(F, \mathcal{L}\big) \neq 0$ for at most one $i = 0, \dots, 3$.*

We have the following standard restriction exact sequence

$$0 \longrightarrow \mathcal{O}_\Phi(\alpha_1\eta_1 + \alpha_2\eta_2 - \eta) \longrightarrow \mathcal{O}_\Phi(\alpha_1\eta_1 + \alpha_2\eta_2) \longrightarrow \mathcal{O}_F(\alpha_1h_1 + \alpha_2h_2) \longrightarrow 0$$

for each $\mathcal{O}_F(\alpha_1h_1 + \alpha_2h_2) \in \mathrm{Pic}(F)$. Since the cohomology of $\mathcal{O}_\Phi(\alpha_1\eta_1 + \alpha_2\eta_2)$ vanishes in odd dimension thanks to Künneth formulas, it follows that

$$\begin{aligned} h^1\big(F, \mathcal{O}_F(\alpha_1h_1 + \alpha_2h_2)\big) - h^2\big(F, \mathcal{O}_F(\alpha_1h_1 + \alpha_2h_2)\big) = \\ = h^2\big(\Phi, \mathcal{O}_\Phi(\alpha_1\eta_1 + \alpha_2\eta_2 - \eta)\big) - h^2\big(\Phi, \mathcal{O}_\Phi(\alpha_1\eta_1 + \alpha_2\eta_2)\big). \end{aligned}$$

Thanks to Theorem 2.4, only one of $h^1\big(F, \mathcal{O}_F(\alpha_1h_1 + \alpha_2h_2)\big)$ and $h^2\big(F, \mathcal{O}_F(\alpha_1h_1 + \alpha_2h_2)\big)$ is non–zero, according to the sign of the second member of the equality above.

More precisely we have the following proposition.

Proposition 2.5. *For each $\alpha_1, \alpha_2 \in \mathbb{Z}$ with $\alpha_1 \leq \alpha_2$, we have*

$$h^i\big(F, \mathcal{O}_F(\alpha_1h_1 + \alpha_2h_2)\big) \neq 0$$

if and only if

- *$i = 0$ and $\alpha_1 \geq 0$;*
- *$i = 1$ and $\alpha_1 \leq -2$, $\alpha_1 + \alpha_2 + 1 \geq 0$;*
- *$i = 2$ and $\alpha_2 \geq 0$, $\alpha_1 + \alpha_2 + 3 \leq 0$;*
- *$i = 3$ and $\alpha_2 \leq -2$.*

In all these cases

$$h^i\big(F, \mathcal{O}_F(\alpha_1h_1 + \alpha_2h_2)\big) = (-1)^i \frac{(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_1 + \alpha_2 + 2)}{2}.$$

An helpful tool for summarizing the first part of the above proposition is the following picture dealing with the subsets of the (α_1, α_2)–plane whose points correspond to some non–zero cohomology group of the sheaf $\mathcal{O}_F(\alpha_1h_1 + \alpha_2h_2)$ (without restriction on α_1 and α_2).

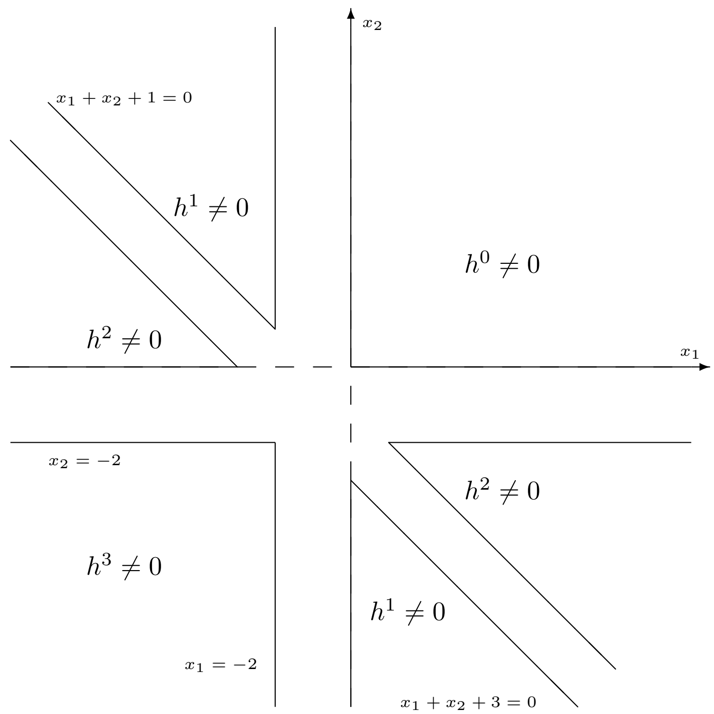

Figure 1

The points corresponding to the direct summands of $H^i_*\big(F, \mathcal{O}_F(\alpha_1 h_1 + \alpha_2 h_2)\big)$ lie on the line through (α_1, α_2) and parallel to $x_1 - x_2 = 0$, hence the following two corollaries are immediate.

Corollary 2.6. *If $\alpha_1, \alpha_2 \in \mathbb{Z}$, then*

$$H^1_*\big(F, \mathcal{O}_F(\alpha_1 h_1 + \alpha_2 h_2)\big) \neq 0 \quad \Leftrightarrow \quad H^2_*\big(F, \mathcal{O}_F(\alpha_1 h_1 + \alpha_2 h_2)\big) \neq 0.$$

Proof. Looking at Figure 1, one immediately checks that the line $x_2 = x_1 + c$ intersects the subset $h^1 \neq 0$ if and only if it intersects the subset $h^2 \neq 0$. □

The following second important corollary shows that initialized aCM (resp. Ulrich) line bundles on F are exactly the restrictions of initialized aCM (resp. Ulrich) line bundles on Φ.

Corollary 2.7. *If $\alpha_1, \alpha_2 \in \mathbb{Z}$, then $\mathcal{O}_F(\alpha_1 h_1 + \alpha_2 h_2)$ is initialized and aCM if and only if (α_1, α_2) is one of the following:*

$$(0,0), \quad (0,1), \quad (1,0), \quad (0,2), \quad (2,0).$$

Moreover, $\mathcal{O}_F(\alpha_1 h_1 + \alpha_2 h_2)$ is Ulrich if and only if (α_1, α_2) is one of the following:

$$(0,2), \quad (2,0).$$

Proof. On the one hand the sheaf $\mathcal{O}_F(\alpha_1 h_1 + \alpha_2 h_2)$ is initialized if and only if α_1 and α_2 are non–negative and $\alpha_1\alpha_2 = 0$. On the other hand, looking at Figure 1, one checks that the line $x_2 = x_1 + c$ does not intersect the subsets $h^1 \neq 0$ and $h^2 \neq 0$ if and only if $-2 \leq c \leq 2$. Thus the first part of the statement follows.

The second part of the statement is an easy direct computation. □

The following remark will be used in the paper.

Remark 2.8. Let $\mathcal{G}$ be an aCM bundle of rank r on Φ and let $\mathcal{E} := \mathcal{G}\otimes\mathcal{O}_F$ be its restriction to F. By computing the cohomology of sequence

$$0 \longrightarrow \mathcal{G}(-\eta) \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow 0 \tag{3}$$

twisted by $\mathcal{O}_\Phi(t\eta)$ one checks that $\mathcal{E}$ is aCM too.

If $\mathcal{G}$ is initialized it is immediate to check that the same is true for $\mathcal{E}$. Moreover the restriction morphism induced by the inclusion $F \subseteq \Phi$ yields an isomorphism $H^0\big(\Phi,\mathcal{G}\big) \cong H^0\big(F,\mathcal{E}\big)$. In this isomorphism the zero loci of the corresponding sections $\sigma \in H^0\big(\Phi,\mathcal{G}\big)$ and $s \in H^0\big(F,\mathcal{E}\big)$ satisfy $(s)_0 = F \cap (\sigma)_0$.

Conversely if $\mathcal{E}$ is initialized, the cohomology of Sequence (3) twisted by $\mathcal{O}_\Phi(t\eta)$ yields that $h^0\big(\Phi,\mathcal{G}(t\eta)\big) = h^0\big(\Phi,\mathcal{G}(-\eta)\big)$ when $t \le -1$. We conclude that $h^0\big(\Phi,\mathcal{G}(-\eta)\big) = 0$. Moreover $h^0\big(\Phi,\mathcal{G}\big) = h^0\big(F,\mathcal{E}\big) \ne 0$, because $\mathcal{E}$ is initialized. It follows that also $\mathcal{G}$ is initialized. Moreover $\mathcal{G}$ is Ulrich if and only if the same holds for $\mathcal{E}$, because $\deg(\Phi) = \deg(F)$ and $\mathrm{rk}(\mathcal{G}) = \mathrm{rk}(E)$.

Let $\mathcal{G}$ be an initialized, aCM bundle of rank r on Φ. Clearly if $\mathcal{G}$ is decomposable the same is true for $\mathcal{E}$. We will prove in the last part of this section that if $\mathcal{E} \cong \mathcal{G}\otimes\mathcal{O}_F$ is decomposable and $r = 2$, then $\mathcal{G}$ is decomposable too. To this purpose assume that $\mathcal{E}$ splits as a sum of line bundles. Such line bundles are necessarily aCM and they are either both initialized or one of them is initialized and the other one has no sections. In particular we can assume that

$$\mathcal{E} \cong \mathcal{O}_F(a_2h_2) \oplus \mathcal{O}_F(b_1h_1 + b_2h_2)$$

where $0 \le a_2 \le 2$, $|b_1 - b_2| \le 2$ and at least one of the b_i's is not positive.

Twisting Sequence (3) by $\mathcal{O}_\Phi(-a_2\eta_2)$ we obtain the sequence

$$0 \longrightarrow \mathcal{G}(-\eta - a_2\eta_2) \longrightarrow \mathcal{G}(-a_2\eta_2) \longrightarrow \mathcal{O}_F \oplus \mathcal{O}_F(b_1h_1 + (b_2 - a_2)h_2) \longrightarrow 0 \tag{4}$$

We first assume $h^1\big(F,\mathcal{O}_F((t+b_1)h_1 + (t+b_2-a_2)h_2)\big) = 0$ for each $t \le -1$. Hence the cohomology of the above sequence twisted by $\mathcal{O}_\Phi(t\eta)$ implies the existence of surjective maps

$$H^1\big(\Phi,\mathcal{G}(t\eta - a_2\eta_2)\big) \twoheadrightarrow H^1\big(\Phi,\mathcal{G}((t+1)\eta - a_2\eta_2)\big)$$

for each $t \le -1$. The vanishing $h^1\big(\Phi,\mathcal{G}(t\eta - a_2\eta_2)\big) = 0$ for $t \ll -1$, thus forces $h^1\big(\Phi,\mathcal{G}(t\eta - a_2\eta_2)\big) = 0$ for each $t \le -1$. It follows that $h^0\big(\Phi,\mathcal{G}(-a_2\eta_2)\big) = 1 + h^0\big(F,\mathcal{O}_F(b_1h_1 + (b_2-a_2)h_2)\big) \ne 0$.

If $\sigma \in H^0\big(\Phi,\mathcal{G}(-a_2\eta_2)\big)$ is a general section then we can write $(\sigma)_0 = \Theta \cup \Xi \subseteq \Phi$, where Θ has pure codimension 2 (or it is empty) and Ξ is an effective divisor (or it is empty).

Consider the restriction morphism

$$\psi\colon H^0\big(\Phi,\mathcal{G}(-a_2\eta_2)\big) \to H^0\big(F,\mathcal{O}_F \oplus \mathcal{O}_F(b_1h_1 + (b_2-a_2)h_2)\big).$$

The cohomology of Sequence (4) and the vanishing $h^1\big(\Phi,\mathcal{G}(-\eta - a_2\eta_2)\big) = 0$ imply that ψ is an isomorphism because $\mathcal{G}$ is assumed to be initialized. Thus $s := \psi(\sigma) \in H^0\big(F,\mathcal{O}_F \oplus \mathcal{O}_F(b_1h_1 + (b_2-a_2)h_2)\big)$ is general. We conclude that $(\sigma)_0 \cap F = (s)_0 = \emptyset$, hence $(\sigma)_0 = \emptyset$ too. In particular Sequence (1) for σ is

$$0 \longrightarrow \mathcal{O}_\Phi \longrightarrow \mathcal{G}(-a_2\eta_2) \longrightarrow \mathcal{O}_\Phi(c_1(\mathcal{G}) - 2a_2\eta_2) \longrightarrow 0.$$

Such a sequence corresponds to an element of

$$\mathrm{Ext}^1_\Phi\big(\mathcal{O}_\Phi(c_1(\mathcal{G}) - 2a_2\eta_2),\mathcal{O}_\Phi\big) \cong H^1\big(\Phi,\mathcal{O}_\Phi(2a_2\eta_2 - c_1(\mathcal{G}))\big).$$

The last space is zero, due to Künneth formula, thus the sequence above splits, whence $\mathcal{G} \cong \mathcal{O}_\Phi(a_2\eta_2) \oplus \mathcal{O}_\Phi(c_1(\mathcal{G}) - a_2\eta_2)$.

Now assume that there is at least one $t \le -1$ such that $h^1\big(F,\mathcal{O}_F((t+b_1)h_1 + (t+b_2-a_2)h_2)\big) \ne 0$. Only one case is actually admissible as the following lemma shows.

Lemma 2.9. *Let $a_2, b_1, b_2 \in \mathbb{Z}$ be such that $0 \le a_2 \le 2$, $|b_1 - b_2| \le 2$ and at least one of the b_i's is not positive. There is $t \le -1$ such that $h^1\big(F,\mathcal{O}_F((t+b_1)h_1 + (t+b_2-a_2)h_2)\big) \ne 0$, if and only if $a_2 = 1$, $b_1 = 2$, $b_2 = 0$ and $t = -1$.*

Proof. Notice that $b_1, b_2 \le 2$ thanks to the hypothesis. One implication is trivial. Indeed if $a_2 = 1$, $b_1 = 2$, $b_2 = 0$ and $t = -1$, then $h^1\big(F, \mathcal{O}_F((t+b_1)h_1 + (t+b_2-a_2)h_2)\big) = 1$ thanks to Proposition 2.5.

Let us now prove the opposite implication. Assume that $h^1\big(F, \mathcal{O}_F((t+b_1)h_1+(t+b_2-a_2)h_2)\big) \ne 0$ for some $t \le -1$.

Let $b_1 \le b_2 - a_2$. By Proposition 2.5, we have

$$t + b_1 \le -2, \qquad 2t + b_1 + b_2 - a_2 + 1 \ge 0.$$

Hence

$$0 \le 2t + b_1 + b_2 - a_2 + 1 = (t + b_1) + t + b_2 - a_2 + 1 \le b_2 - a_2 - 2 \le -a_2 \le 0,$$

for $t \le -1$. Indeed we know that $|b_1 - b_2| \le 2$, thus $b_2 - 2 \le b_1$. If $b_1 \le 0$, then $b_2 - 2 \le 0$. If $b_1 > 0$, then $b_2 \le 0$ thanks to the hypothesis. We conclude that all the inequalities above must be actually equalities, whence $t = -1$, $t + b_1 = -2$, $b_2 = 2$, $a_2 = 0$, thus $b_1 = -1$. Since $3 = |b_1 - b_2| > 2$, it follows that such a case cannot occur.

Let $b_1 > b_2 - a_2$. As in the previous case we deduce that

$$t + b_2 - a_2 \le -2, \qquad 2t + b_1 + b_2 - a_2 + 1 \ge 0,$$
$$0 \le 2t + b_1 + b_2 - a_2 + 1 = (t + b_2 - a_2) + t + b_1 + 1 \le b_1 - 2 \le 0$$

whence $t = -1$, $b_1 = 2$, $t + b_2 - a_2 = -2$. Taking into account of the restrictions on a_2 and b_2 we have either $b_2 = -1$ and $a_2 = 0$, or $b_2 = 0$ and $a_2 = 1$. Again the case $b_2 = -1$ and $a_2 = 0$ cannot occur because $3 = |b_1 - b_2| > 2$. □

It follows from the above lemma that if $\mathcal{E}$ splits, then we may assume that $a_2 = 1$, $b_1 = 2$, $b_2 = 0$ i.e. $\mathcal{E} = \mathcal{O}_F(h_2) \oplus \mathcal{O}_F(2h_1)$. Notice that $\mathcal{G}^\vee(\eta)$ is aCM by the Serre duality, because $\omega_\Phi \cong \mathcal{O}_\Phi(-3\eta)$. Moreover, in this case

$$\mathcal{G}^\vee(\eta) \otimes \mathcal{O}_F \cong \mathcal{E}^\vee(h) \cong \mathcal{O}_F(h_1) \oplus \mathcal{O}_F(-h_1 + h_2)$$

is initialized, thus $\mathcal{G}^\vee(\eta)$ is also initialized. Now we can substitute $\mathcal{G}$ with $\mathcal{G}^\vee(\eta)$: since $h^1\big(F, \mathcal{O}_F(th - 2h_1 + h_2)\big) = 0$ for each $t \le -1$, it follows that we can repeat almost verbatim (we only have to permute the roles of η_1 and η_2) the first part of the above discussion, proving that $\mathcal{G}^\vee(\eta)$, hence $\mathcal{G}$, must be decomposable.

We summarize the above discussion in the following statement.

Theorem 2.10. *If $\mathcal{G}$ is an initialized, indecomposable, aCM bundle of rank 2 on Φ, then $\mathcal{G} \otimes \mathcal{O}_F$ is an initialized, indecomposable, aCM bundle of rank 2 on F.*

Taking into account the above result, in what follows we will first deal with initialized, indecomposable, aCM bundles of rank 2 on F, giving their complete classification. Then we will use the results obtained on F in order to classify also initialized, indecomposable, aCM bundles of rank 2 on Φ and viceversa.

3. The lower bound

The aim of this section is to establish a lower bound for the first Chern class $c_1 := \alpha_1 h_1 + \alpha_2 h_2$ of an indecomposable, initialized, aCM bundle $\mathcal{E}$ of rank 2 on F. Due to Theorem 2.10, such a bound also holds for the first Chern class of each initialized indecomposable aCM bundle of rank 2 on Φ. As usual, in what follows, $h = h_1 + h_2$ will denote the hyperplane class on F.

Lemma 3.1. *Let $\mathcal{E}$ be an initialized aCM bundle of rank 2 on F. Then $\mathcal{E}^\vee(2h)$ is aCM and globally generated.*

Proof. See the proof of Lemma 3.1 in [11]. □

Remark 3.2. The above lemma implies that $4h - c_1 = c_1(\mathcal{E}^\vee(2h))$ is effective, whence $\alpha_i \le 4$. Moreover, if $\alpha_1, \alpha_2 \ge 3$, then there would be a monomorphism $H^0\big(F, \mathcal{E}(2h - c_1)\big) \to H^0\big(F, \mathcal{E}(-h)\big)$. On the one hand we know that the target space is zero. On the other hand $H^0\big(F, \mathcal{E}(2h - c_1)\big) \ne 0$, because $\mathcal{E}(2h - c_1) \cong \mathcal{E}^\vee(2h)$ is globally generated, a contradiction. Thus we can assume $\alpha_1 \le 2$.

We first check that the zero–locus of each section of an indecomposable, initialized, aCM bundle of rank 2 on F is non–empty.

Lemma 3.3. *Let $\mathcal{E}$ be an indecomposable, initialized, aCM bundle of rank 2 on F. Then the zero locus $(s)_0$ of a section of $\mathcal{E}$ is non–empty.*

Proof. We assume $\alpha_1 \le \alpha_2$. If $(s)_0 = \emptyset$, then sequence (1) becomes

$$0 \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_F(c_1) \longrightarrow 0.$$

The class of such an extension should be a non–zero element in $H^1\big(F, \mathcal{O}_F(-c_1)\big)$. Thanks to Proposition 2.5, we deduce that $\alpha_2 \ge 2$ and $\alpha_1 + \alpha_2 \le 1$, whence $\alpha_1 \le -1$. The cohomology of the above sequence twisted by $\mathcal{O}_F((-2 - \alpha_1)h)$ and Proposition 2.5 yields

$$h^1\big(F, \mathcal{E}((-2 - \alpha_1)h)\big) = h^1\big(F, \mathcal{O}(-2h_1 + (-2 + \alpha_2 - \alpha_1)h_2)\big) \ge 1.$$

We conclude that $\mathcal{E}$ would not be aCM, a contradiction. □

For each non–zero $s \in H^0\big(F, \mathcal{E}\big)$ we have $(s)_0 = E \cup D$ where E has pure codimension 2 (or it is empty) and $D \in |\delta_1 h_1 + \delta_2 h_2|$ has pure codimension 1 (or it is empty). At least one of E and D is non–empty, due to Lemma 3.3. Sequence (1) for s becomes

$$0 \longrightarrow \mathcal{O}_F(D) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|F}(c_1 - D) \longrightarrow 0.$$

On the one hand, twisting Sequence (1) for s by $\mathcal{O}_F(-h)$ and taking its cohomology, the vanishing of $h^0\big(F, \mathcal{E}(-h)\big)$ implies $h^0\big(F, \mathcal{O}_F(D - h)\big) = 0$. In particular we know that at least one of the δ_i is zero.

On the other hand, twisting the same sequence by $\mathcal{O}_F(2h - c_1)$, using the isomorphism $\mathcal{E}(-c_1) \cong \mathcal{E}^\vee$ and taking into account that $\mathcal{E}^\vee(2h)$ is globally generated (see Lemma 3.1), we obtain that $\mathcal{I}_{E|F}(2h - D)$ is globally generated too. Thus

$$0 \ne H^0\big(F, \mathcal{I}_{E|F}(2h - D)\big) \subseteq H^0\big(F, \mathcal{O}_F(2h - D)\big),$$

hence $\delta_i \le 2$, $i = 1, 2$. We conclude that $(\delta_1, \delta_2) \in \{\ (0,0), (0,1), (0,2), (1,0), (2,0)\ \}$: in particular $\mathcal{O}_F(D)$ is aCM (see Corollary 2.7).

In what follows we will prove that $c_1 - D$, hence c_1, is effective.

Lemma 3.4. *Let $\mathcal{E}$ be an indecomposable, initialized, aCM bundle of rank 2 on F. Assume that $s \in H^0\big(F, \mathcal{E}\big)$ is such that $(s)_0 = E \cup D$ where E has codimension 2 (or it is empty) and D has codimension 1. If $c_1 - D$ is not effective, then $E = \emptyset$.*

Proof. Assume $c_1 - D$ is non–effective: thanks to Sequence (2) for s,

$$h^0\big(F, \mathcal{I}_{E|F}(c_1 - D - th)\big) \le h^0\big(F, \mathcal{O}_F(c_1 - D - th)\big) = 0 \tag{5}$$

for each positive integer t. Taking into account that $\mathcal{E}$ and $\mathcal{O}_F(D)$ are aCM and $\delta_i \ge 0$, the cohomology of Sequence (1) for s yields also the vanishings

$$\begin{gathered} h^1(F, \mathcal{I}_{E|F}(c_1 - D - th)) = 0, \qquad t \in \mathbb{Z} \\ h^2(F, \mathcal{I}_{E|F}(c_1 - D)) = h^3\big(F, \mathcal{O}_F(D)\big) = 0 \\ h^2(F, \mathcal{I}_{E|F}(c_1 - D - h)) = h^3\big(F, \mathcal{O}_F(D - h)\big) = 0. \end{gathered} \tag{6}$$

Finally, we also have

$$h^2\big(F, \mathcal{I}_{E|F}(c_1 - D - 2h)\big) \le h^3\big(F, \mathcal{O}_F(D - 2h)\big) = h^0\big(F, \mathcal{O}_F(-D)\big) \le 1. \tag{7}$$

Assume $E \neq \emptyset$, whence $\deg(E) \geq 1$. Let H be a general hyperplane in $\mathbb{P}^7$. Define $S := F \cap H$ and $Z := E \cap H$, so that $\dim(Z) = 0$. We have the following exact sequence

$$0 \longrightarrow \mathcal{I}_{E|F}(c_1 - D - h) \longrightarrow \mathcal{I}_{E|F}(c_1 - D) \longrightarrow \mathcal{I}_{Z|S}(c_1 - D) \longrightarrow 0.$$

The cohomology of the above sequence, Equalities (5), (6) and Inequality (7) imply

$$h^0\big(S, \mathcal{I}_{Z|S}(c_1 - D)\big) = h^1\big(S, \mathcal{I}_{Z|S}(c_1 - D)\big) = 0,$$
$$h^0\big(S, \mathcal{I}_{Z|S}(c_1 - D - h)\big) = 0, \qquad h^1\big(S, \mathcal{I}_{Z|S}(c_1 - D - h)\big) \leq 1.$$

The above relations and the cohomology of Sequence

$$0 \longrightarrow \mathcal{I}_{Z|S}(c_1 - D) \longrightarrow \mathcal{O}_S(c_1 - D) \longrightarrow \mathcal{O}_Z \longrightarrow 0 \tag{8}$$

give

$$h^1\big(S, \mathcal{O}_S(c_1 - D)\big) = 0, \tag{9}$$

$$h^0\big(S, \mathcal{O}_S(c_1 - D)\big) = h^0\big(Z, \mathcal{O}_Z\big) = \deg(Z) = \deg(E). \tag{10}$$

The same relations as above and the cohomology of Sequence (8) twisted by $\mathcal{O}_S(-h)$ also yield

$$h^0\big(Z, \mathcal{O}_Z\big) \leq h^0\big(S, \mathcal{O}_S(c_1 - D - h)\big) - h^1\big(S, \mathcal{O}_S(c_1 - D - h)\big) + 1.$$

Looking at the formulas for $h^0\big(Z, \mathcal{O}_Z\big)$ obtained above, we deduce the inequality

$$1 \geq h^0\big(S, \mathcal{O}_S(c_1 - D)\big) - h^0\big(S, \mathcal{O}_S(c_1 - D - h)\big) + h^1\big(S, \mathcal{O}_S(c_1 - D - h)\big). \tag{11}$$

The remaining part of the proof is devoted to show that the aforementioned inequality leads to a contradiction.

Set $c_1 - D := \epsilon_1 h_1 + \epsilon_2 h_2$ and assume $\epsilon_1 \leq \epsilon_2$. Obviously $\epsilon_1 \leq -1$ because $c_1 - D$ is assumed to be not effective. Thus the cohomology of sequence

$$0 \longrightarrow \mathcal{O}_F(c_1 - D - h) \longrightarrow \mathcal{O}_F(c_1 - D) \longrightarrow \mathcal{O}_S(c_1 - D) \longrightarrow 0 \tag{12}$$

and Equality (10) yield

$$1 \leq \deg(E) = h^0\big(S, \mathcal{O}_S(c_1 - D)\big) \leq h^1\big(F, \mathcal{O}_F(c_1 - D - h)\big).$$

Hence Proposition 2.5 implies

$$h^1\big(F, \mathcal{O}_F(c_1 - D - h)\big) = -\frac{\epsilon_1 \epsilon_2 (\epsilon_1 + \epsilon_2)}{2},$$

$\epsilon_1 \leq -1$ and $\epsilon_1 + \epsilon_2 \geq 1$, whence $\epsilon_2 \geq 2$. Since $\mathcal{E}$, hence $\mathcal{E}^\vee$, is aCM and $h^0(F, \mathcal{I}_{E|F}(-D)) = 0$, it follows from the cohomology of sequence

$$0 \longrightarrow \mathcal{O}_F(D - c_1) \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{I}_{E|F}(-D) \longrightarrow 0$$

that $h^1\big(F, \mathcal{O}_F(-\epsilon_1 h_1 - \epsilon_2 h_2)\big) = h^1\big(F, \mathcal{O}_F(D - c_1)\big) = 0$. Taking into account that $-\epsilon_2 \leq -\epsilon_1$ and $\epsilon_2 \geq 2$ whence $-\epsilon_2 \leq -2$, again by Proposition 2.5 we obtain

$$\epsilon_1 + \epsilon_2 \geq 2. \tag{13}$$

Recall that $\epsilon_1 - 2 \leq -3$: if $\epsilon_1 + \epsilon_2 \geq 3$, then $(\epsilon_1 - 2) + (\epsilon_2 - 2) + 1 \geq 0$, hence again Proposition 2.5 yields

$$h^1\big(F, \mathcal{O}_F(c_1 - D - 2h)\big) = -\frac{(\epsilon_1 - 1)(\epsilon_2 - 1)(\epsilon_1 + \epsilon_2 - 2)}{2}.$$

The same equality still holds when $\epsilon_1 + \epsilon_2 = 2$, because both the members of the equality above are zero. Finally, Proposition 2.5 yields

$$h^1\big(F, \mathcal{O}_F(c_1 - D)\big) = -\frac{(\epsilon_1 + 1)(\epsilon_2 + 1)(\epsilon_1 + \epsilon_2 + 2)}{2}$$

if $\epsilon_1 \leq -2$. Again the same equality holds also for $\epsilon_1 = -1$, due to the vanishing of both the members.

Notice that if $\epsilon_1 \le -1$ and $\epsilon_1+\epsilon_2 \ge 2$, then $h^2\big(F, \mathcal{O}_F(c_1-D-2h)\big)=0$ (Proposition 2.5). Thus Equality (9) and the cohomology of sequence (12) yield

$$h^0\big(S, \mathcal{O}_S(c_1-D)\big) = h^1\big(F, \mathcal{O}_F(c_1-D-h)\big) - h^1\big(F, \mathcal{O}_F(c_1-D)\big),$$
$$h^0\big(S, \mathcal{O}_S(c_1-D-h)\big) - h^1\big(S, \mathcal{O}_S(c_1-D-h)\big) = $$
$$= h^1\big(F, \mathcal{O}_F(c_1-D-2h)\big) - h^1\big(F, \mathcal{O}_F(c_1-D-h)\big).$$

In particular, by substituting the above expressions in Inequality (11), we obtain

$$1 \ge 2h^1\big(F, \mathcal{O}_F(c_1-D-h)\big) - h^1\big(F, \mathcal{O}_F(c_1-D)\big) - h^1\big(F, \mathcal{O}_F(c_1-D-2h)\big).$$

Thus the expressions obtained above in terms of the ϵ_i's for $h^1\big(F, \mathcal{O}_F(c_1-D-th)\big)$, $t=0,1,2$, and the Inequality (13) finally yield $1 \ge 3(\epsilon_1+\epsilon_2) \ge 6$, a contradiction. We conclude that if c_1-D is non–effective, then $E=\emptyset$. □

We conclude this section with the proof of the effectiveness of c_1-D.

Proposition 3.5. *If $\mathcal{E}$ is an indecomposable, initialized, aCM bundle of rank 2 on F, $s \in H^0\big(F, \mathcal{E}\big)$ and D denotes the component of pure codimension 1 of $(s)_0$, if any, then $c_1 \ge c_1-D \ge 0$.*

In particular, if $c_1=0$, then the zero–locus of each section $s \in H^0\big(F, \mathcal{E}\big)$ has pure codimension 2.

Proof. Let $c_1-D := \epsilon_1 h_1+\epsilon_2 h_2$ be non–effective: we can assume $\epsilon_1 \le -1$. The previous Lemma implies $E=\emptyset$ whence $\mathcal{I}_{E|F} \cong \mathcal{O}_F$. Sequence (1) thus becomes

$$0 \longrightarrow \mathcal{O}_F(D) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_F(c_1-D) \longrightarrow 0.$$

Recall that $(\delta_1,\delta_2) \in \{\,(0,0),(0,1),(0,2),(1,0),(2,0)\,\}$. The cohomology of the above sequence twisted by $\mathcal{O}_F(th)$ and Corollary 2.7 imply

$$0 = h^1\big(F, \mathcal{E}(th)\big) = h^1\big(F, \mathcal{O}_F(c_1-D+th)\big), \qquad t \in \mathbb{Z}.$$

Moreover the above sequence does not split because $\mathcal{E}$ is assumed to be indecomposable, hence

$$H^1(F, \mathcal{O}_F(2D-c_1)) \cong Ext^1_F\big(\mathcal{O}_F(c_1-D), \mathcal{O}_F(D))\big) \ne 0.$$

We have $2D-c_1 \in |(\delta_1-\epsilon_1)h_1+(\delta_2-\epsilon_2)h_2|$. Since $\delta_i \ge 0$, $i=1,2$, and $\epsilon_1 \le -1$, it follows that $\delta_1-\epsilon_1 \ge 1$. Proposition 2.5 thus gives $\delta_2-\epsilon_2 \le -2$, whence $\epsilon_2 \ge 2$. We conclude that $\epsilon_2-\epsilon_1 \ge 3$.

If we take $t := -2-\epsilon_1$, then $c_1-D+th = -2h_1+(\epsilon_2-\epsilon_1-2)h_2$, hence again Proposition 2.5 implies $h^1\big(F, \mathcal{O}_F(c_1-D+th)\big) \ne 0$, a contradiction. □

4. The upper bound and zero–loci of general sections

The aim of this section is to establish an upper bound for the first Chern class $c_1 = \alpha_1 h_1+\alpha_2 h_2$ of an indecomposable, initialized, aCM bundle $\mathcal{E}$ of rank 2 on F. From now on we will assume that the second Chern class of $\mathcal{E}$ is $c_2 := \beta_1 h_2^2+\beta_2 h_1^2$.

If ω_2 is the second Chern class of the sheaf Ω^1_F, the Riemann–Roch theorem yields

$$\chi(\mathcal{E}) = 2+\frac{1}{6}(c_1^3-3c_1c_2)+\frac{1}{2}(c_1^2h-2c_2h)+\frac{1}{12}(4c_1h^2+\omega_2c_1). \tag{14}$$

We easily obtain the identities

$$c_1^3 = 3(\alpha_1^2\alpha_2+\alpha_1\alpha_2^2), \qquad c_1^2h = \alpha_1^2+4\alpha_1\alpha_2+\alpha_2^2, \qquad c_1h^2 = 3(\alpha_1+\alpha_2). \tag{15}$$

Recall that p_1 is isomorphic to the canonical map $\mathbb{P}(\Omega^1_{\mathbb{P}^2}(2)) \to \mathbb{P}^2$, hence it is smooth. In particular we have the two exact sequences

$$0 \longrightarrow \Omega^1_{F|\mathbb{P}^2} \longrightarrow p_1^*\Omega^1_{\mathbb{P}^2}(2) \otimes \mathcal{O}_F(-h_2) \longrightarrow \mathcal{O}_F \longrightarrow 0,$$
$$0 \longrightarrow p_1^*\Omega^1_{\mathbb{P}^2} \longrightarrow \Omega^1_F \longrightarrow \Omega^1_{F|\mathbb{P}^2} \longrightarrow 0$$

Another Chern class computation thus gives $\omega_2 = 6h_1h_2$, hence $\omega_2c_1 = 6(\alpha_1+\alpha_2)$.

We have $h^i\big(F,\mathcal{E}^\vee(-h)\big)=h^{3-i}\big(F,\mathcal{E}(-h)\big)=0$ for $i\ge 1$. Since

$$h^0\big(F,\mathcal{I}_{E|F}(-D-h)\big)\le h^0\big(F,\mathcal{O}_F(-D-h)\big)=0$$

(see Sequence (2)), it follows that

$$\chi\big(\mathcal{E}^\vee(-h)\big)=h^0\big(F,\mathcal{E}^\vee(-h)\big)=h^0\big(F,\mathcal{O}_F(D-c_1-h)\big)=0$$

(due to sequence (1) and to the effectiveness of c_1-D proved in Proposition 3.5). The above equality, Formula (14) for $\mathcal{E}^\vee(-h)$ and Equalities (15) yield

$$c_1c_2=\alpha_1\beta_1+\alpha_2\beta_2=\alpha_1^2\alpha_2+\alpha_1\alpha_2^2. \tag{16}$$

We also have $h^i\big(F,\mathcal{E}^\vee\big)=h^{3-i}\big(F,\mathcal{E}(-2h)\big)=0$, for $i\ge 1$. If $D=0$, then $E\ne\emptyset$, hence $h^0\big(F,\mathcal{I}_{E|F}\big)=h^0\big(F,\mathcal{I}_{E|F}(-D)\big)=0$. If $D\ne 0$, then

$$h^0\big(F,\mathcal{I}_{E|F}(-D)\big)\le h^0\big(F,\mathcal{O}_F(-D)\big)=0.$$

Let

$$e(c_1,D):=\begin{cases}0 & \text{if } D\ne c_1,\\ 1 & \text{if } D=c_1.\end{cases}$$

Since $(s)_0\ne\emptyset$, it follows that

$$\chi(\mathcal{E}^\vee)=h^0\big(F,\mathcal{E}^\vee\big)=h^0\big(F,\mathcal{O}_F(D-c_1)\big)=e(c_1,D)$$

Combining as above Sequences (2), (1) and Formulas (14) for $\mathcal{E}^\vee$ and (16), we finally obtain

$$hc_2=\beta_1+\beta_2=2+\frac{1}{2}\left(\alpha_1^2+4\alpha_1\alpha_2+\alpha_2^2-3\alpha_1-3\alpha_2\right)-e(c_1,D). \tag{17}$$

Lemma 4.1. *Let $\mathcal{E}$ be a vector bundle of rank 2 on F and $s\in H^0\big(F,\mathcal{E}\big)$ a section such that $E:=(s)_0$ has pure codimension 2. Then*

$$\deg(E)=hc_2,\qquad p_a(E)=\frac{1}{2}c_1c_2-hc_2+1.$$

Proof. The result was proved in [11] (see Lemma 2.1) when $F\cong\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$. The same proof holds verbatim also in this case, because it only depends on the isomorphism $\omega_F\cong\mathcal{O}_F(-2h)$. □

Assume that $E\ne\emptyset$. Its class in $A^2(F)$ is $c_2(\mathcal{E}(-D))=c_2-c_1D+D^2$. Since $|h_i|$, $i=1,2$ is base–point–free on F, $D\ge 0$ and c_1-D is effective (see Proposition 3.5), it follows that

$$\beta_i\ge\beta_i-h_i(c_1-D)D=h_ic_2-h_i(c_1-D)D=h_ic_2(\mathcal{E}(-D))\ge 0,\qquad i=1,2. \tag{18}$$

Proposition 4.2. *If $\mathcal{E}$ is an indecomposable, initialized, aCM bundle of rank 2 on F, then $2h-c_1\ge 0$.*

Moreover, if equality holds, then $\mathcal{E}$ is Ulrich.

Proof. We start by proving the effectiveness of $2h-c_1$. We distinguish two cases, according to whether $\mathcal{E}$ is regular in the sense of Castelnuovo–Mumford or not. Notice that, being aCM, $\mathcal{E}$ is regular if and only if $h^3\big(F,\mathcal{E}(-3h)\big)=0$.

If $\mathcal{E}$ is not regular, then the Serre duality yields $h^0\big(F,\mathcal{E}^\vee(h)\big)=h^3\big(F,\mathcal{E}(-3h)\big)\ne 0$. If $t\in\mathbb{Z}$ is such that $\mathcal{E}^\vee(th)$ is initialized, we thus infer that $t\le 1$. Since $\mathcal{E}^\vee(th)$ is aCM too, it follows that $2th-c_1=c_1(\mathcal{E}^\vee(th))$ is effective for some $t\le 1$, due to Proposition 3.5. We conclude that $2h-c_1$ is effective too.

If $\mathcal{E}$ is regular, then it is globally generated. Thus the zero–locus $E:=(s)_0$ of a general section $s\in H^0\big(F,\mathcal{E}\big)$ is a smooth curve: in particular its divisorial part D is zero. Let $\alpha_1\le\alpha_2$: we already know that $0\le\alpha_1\le 2$ and $\alpha_2\le 4$ (see Remark 3.2). Obviously we are interested in the cases $\alpha_2=3,4$, so that $c_1\ne 0$.

We have $h^i\big(F,\mathcal{E}^\vee(h)\big)=h^{3-i}\big(F,\mathcal{E}(-3h)\big)=0$, $i=0,1,2,3$, because $\mathcal{E}$ is aCM, regular and initialized. Combining the above remarks with equality (14), Formulas (16), (17) and the vanishing of $e(c_1,D)$ (recall that $c_1\ne 0$ and $D=0$), we finally obtain

$$0=\chi(\mathcal{E}^\vee(h))=12-3\alpha_1-3\alpha_2.$$

Thus $(\alpha_1,\alpha_2)\in\{\ (1,3),(0,4)\ \}$.

If $c_1=h_1+3h_2$, then Formulas (17) and (16) give

$$\beta_1+\beta_2=hc_2=7,\qquad c_1c_2=\beta_1+3\beta_2=12.$$

subtracting the two equation each other we obtain $2\beta_2=5$, a contradiction: thus such a case cannot occur.

We will now prove that if $c_1=4h_2$, then $\mathcal{E}$ splits as sum of invertible sheaves, thus such a case cannot occur too. Equalities (16), (17) and the Inequalities (18) force $c_2=4h_2^2$. Lemma (4.1) implies $\deg(E)=4$ and $p_a(E)=-3$.

Recall that for each pair of curves C' and C'' without common components, the arithmetic genera of C', C'' and $C'\cup C''$ satisfy the equality

$$p_a(C'\cup C'')=p_a(C')+p_a(C'')+c-1$$

where c is the degree of the intersection $C'\cap C''$. It follows that E, being smooth, is necessarily the union of 4 pairwise skew lines whose cycle in $A^2(F)$ is h_2^2 (indeed it is easy to check that the class of each line on F is either h_1^2, or h_2^2).

Consider the projection $p_2\colon F\cong\mathbb{P}(\Omega^1_{\mathbb{P}^2}(2))\to\mathbb{P}^2$ and let $\Gamma:=p_2(E)\subseteq\mathbb{P}^2$. The scheme Γ is a set of 4 pairwise distinct points and $p_2^*\mathcal{I}_{\Gamma|\mathbb{P}^2}\cong\mathcal{I}_{E|F}$. Since $F\cong\mathbb{P}(\Omega^1_{\mathbb{P}^2}(2))$, it follows that $p_{2*}\mathcal{O}_F(h_1)\cong\Omega^1_{\mathbb{P}^2}(2)$. Moreover $\mathcal{O}_F(h_2)=p_2^*\mathcal{O}_{\mathbb{P}^2}(1)$, hence

$$\begin{aligned}h^0\big(\mathbb{P}^2,\otimes\Omega^1_{\mathbb{P}^2}\otimes\mathcal{I}_{\Gamma|\mathbb{P}^2}(3)\big)&=h^0\big(\mathbb{P}^2,\mathcal{I}_{\Gamma|\mathbb{P}^2}(1)\otimes\Omega^1_{\mathbb{P}^2}(2)\big)=\\&=h^0\big(\mathbb{P}^2,\mathcal{I}_{\Gamma|\mathbb{P}^2}\otimes\Omega^1_{\mathbb{P}^2}(2)\otimes p_{2*}\mathcal{O}_F(h_2)\big)=\\&=h^0\big(F,p_2^*\mathcal{I}_{\Gamma|\mathbb{P}^2}\otimes\mathcal{O}_F(h_1)\otimes\mathcal{O}_F(h_2)\big)=h^0\big(F,\mathcal{I}_{E|F}(h)\big).\end{aligned}$$

In order to compute the last dimension we compute the cohomology of Sequence (1) for E twisted by $\mathcal{O}_F(h-c_1)$. Proposition 2.5 and the isomorphism $\mathcal{E}^\vee\cong\mathcal{E}(-c_1)$ yield $h^0\big(F,\mathcal{I}_{E|F}(h)\big)=h^0\big(F,\mathcal{E}^\vee(h)\big)$. Thus the Serre duality finally yields $h^0\big(F,\mathcal{I}_{E|F}(h)\big)=h^3\big(F,\mathcal{E}(-3h)\big)$, which is zero because we are assuming $\mathcal{E}$ regular. We conclude that

$$h^0\big(\mathbb{P}^2,\otimes\Omega^1_{\mathbb{P}^2}\otimes\mathcal{I}_{\Gamma|\mathbb{P}^2}(3)\big)=0. \tag{19}$$

The points of Γ are either in general position, or three of them lie on a line not containing the fourth one, or they are aligned.

In the first case, we have the Koszul resolution

$$0\longrightarrow\mathcal{O}_{\mathbb{P}^2}(-2)\longrightarrow\mathcal{O}_{\mathbb{P}^2}^{\oplus2}\longrightarrow\mathcal{I}_{\Gamma|\mathbb{P}^2}(2)\longrightarrow0,$$

We have both Sequence (1) and the pull back of the Koszul resolution above via p_2^*, i.e.

$$0\longrightarrow\mathcal{O}_F\longrightarrow\mathcal{E}\longrightarrow\mathcal{I}_{E|F}(4h_2)\longrightarrow0,$$
$$0\longrightarrow\mathcal{O}_F\longrightarrow\mathcal{O}_F(2h_2)^{\oplus2}\longrightarrow\mathcal{I}_{E|F}(4h_2)\longrightarrow0,$$

(we used the isomorphisms $p_2^*\mathcal{I}_{\Gamma|\mathbb{P}^2}\cong\mathcal{I}_{E|F}$ and $p_2^*\mathcal{O}_{\mathbb{P}^2}(1)\cong\mathcal{O}_F(h_2)$ in order to obtain the second sequence). We conclude by observing that Theorem 2.3 yields $\mathcal{E}\cong\mathcal{O}_F(2h_2)^{\oplus2}$, because $h^1\big(F,\mathcal{O}_F(-4h_2)\big)=0$ (again by Proposition 2.5).

In the second case, up to a proper linear change of coordinates, we can assume that the homogeneous ideal of Γ is $(x_0x_1,x_0x_2,x_1x_2(x_1-x_2))$: it follows the existence of a minimal free resolution of the form

$$0\longrightarrow\mathcal{O}_{\mathbb{P}^2}(-1)\oplus\mathcal{O}_{\mathbb{P}^2}(-2)\longrightarrow\mathcal{O}_{\mathbb{P}^2}^{\oplus2}\oplus\mathcal{O}_{\mathbb{P}^2}(-1)\longrightarrow\mathcal{I}_{\Gamma|\mathbb{P}^2}(2)\longrightarrow0.$$

Notice that the minimality of the above resolution yield that the induced map $\mathcal{O}_{\mathbb{P}^2}(-1)\to\mathcal{O}_{\mathbb{P}^2}(-1)$ must be necessarily zero.

Taking the cohomology of the above sequence twisted by $\Omega^1_{\mathbb{P}^2}(1)$ we obtain the exact sequence

$$0\longrightarrow H^0\big(\mathbb{P}^2,\Omega^1_{\mathbb{P}^2}\otimes\mathcal{I}_{\Gamma|\mathbb{P}^2}(3)\big)\longrightarrow H^1\big(\mathbb{P}^2,\Omega^1_{\mathbb{P}^2}\big)\longrightarrow H^1\big(\mathbb{P}^2,\Omega^1_{\mathbb{P}^2}\big).$$

The last map is necessarily zero (see above), thus we would have $h^0\big(\mathbb{P}^2,\Omega^1_{\mathbb{P}^2}\otimes\mathcal{I}_{\Gamma|\mathbb{P}^2}(3)\big)=1$, contradicting Equality (19).

Finally, in the third case we have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \longrightarrow \mathcal{I}_{\Gamma|\mathbb{P}^2}(2) \longrightarrow 0.$$

It is immediate to check that $h^0\big(\mathbb{P}^2, \otimes\Omega^1_{\mathbb{P}^2} \otimes \mathcal{I}_{\Gamma|\mathbb{P}^2}(3)\big) = 3$, again contradicting Equality (19).

We conclude the proof by examining the case of indecomposable, aCM bundles $\mathcal{E}$ is initialized, aCM with $c_1 = 2h$. We have $\mathcal{E}^\vee \cong \mathcal{E}(-2h)$, hence $h^3\big(F, \mathcal{E}(-3h)\big) = h^0\big(F, \mathcal{E}^\vee(h)\big) = h^0\big(F, \mathcal{E}(-h)\big) = 0$. It follows that $\mathcal{E}$ is regular, whence globally generated. We can thus again conclude that the zero locus of a general section of $\mathcal{E}$ is a smooth curve E: in particular the divisorial part D is zero so that $e(c_1, D) = 0$. Equalities (17) and (16) imply that $hc_2 = 8$, $c_1c_2 = 16$. A direct substitution in Formula (14) finally implies that $h^0\big(F, \mathcal{E}\big) = \chi(\mathcal{E}) = 12$, hence $\mathcal{E}$ is Ulrich. □

We now show that if $\mathcal{E}$ is an indecomposable, initialized, aCM bundle, then its general section vanishes exactly along a curve. To this purpose we assume $\alpha_1 \le \alpha_2$. Moreover we can also assume that $c_1 \ne 0, 2h$ thanks to Propositions 3.5 and 4.2. In particular only the cases $(0,1)$, $(0,2)$, $(1,1)$, $(2,1)$ must be examined.

Let $s \in H^0\big(F, \mathcal{E}\big)$ be a general section and assume that satisfies $(s)_0 = E \cup D$ where E has codimension 2 (or it is empty) and $D \in |\delta_1 h_1 + \delta_2 h_2|$ is non–zero. We already know that $(\delta_1, \delta_2) \in \{\ (0,1), (0,2)\ \}$ up to permutations. For each (α_1, α_2) and (δ_1, δ_2) such that $c_1 - D \ge 0$, we compute hc_2 and c_1c_2 using Equalities (17) and (16), hence also (β_1, β_2). Taking into account that the class of E is $c_2(\mathcal{E}(-D))$ we obtain following table.

(α_1, α_2)	(δ_1, δ_2)	$e(c_1 - D)$	hc_2	c_1c_2	(β_1, β_2)	E
$(0,1)$	$(0,1)$	1	0	0	$(0,0)$	$\emptyset$
$(0,2)$	$(0,1)$	0	1	0	$(1,0)$	$\emptyset$
$(0,2)$	$(0,2)$	1	0	0	$(0,0)$	$\emptyset$
$(1,1)$	$(0,1)$	0	2	2	$(1,1)$	$\emptyset$
$(1,1)$	$(0,1)$	0	2	2	$(2,0)$	$h_2^2 - h_1^2$
$(1,1)$	$(0,1)$	0	2	2	$(0,2)$	$-h_2^2 + h_1^2$
$(2,1)$	$(1,0)$	0	4	6	$(2,2)$	h_2^2
$(2,1)$	$(2,0)$	0	4	6	$(2,2)$	$\emptyset$
$(2,1)$	$(0,1)$	0	4	6	$(2,2)$	$\emptyset$

Both D and $c_1 - D$ are effective on F, thus both $\mathcal{O}_F(D)$ and $\mathcal{O}_F(c_1 - D)$ are globally generated. Moreover $h^1\big(F, \mathcal{O}_F(D)\big) = 0$ by Proposition 2.5. If $E = \emptyset$, then $\mathcal{I}_{E|F}(c_1 - D) \cong \mathcal{O}_F(c_1 - D)$. Thus Sequence (1) would imply that $\mathcal{E}$ should be globally generated. If this were the case, then the general section of $\mathcal{E}$ would vanish along a curve, contradicting the initial hypothesis.

In the fifth case, we would have $Eh_2 = -1$, which is absurd, because the linear system $|h_2|$ is base–point–free. Thus the fifth case cannot occur. A similar argument shows that also the sixth case cannot occur as well.

We conclude that only the seventh case is possible. We have the following exact sequence

$$0 \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{O}_F(h_2)^{\oplus 2} \longrightarrow \mathcal{I}_{E|F}(2h_2) \longrightarrow 0.$$

It follows that $\mathcal{I}_{E|F}(h)$ is globally generated. Sequence (1) would imply that $\mathcal{E}$ should be globally generated, because $c_1 - D = h$ and $h^1\big(F, \mathcal{O}_F(h_1)\big) = 0$. We already checked that this leads to a contradiction, hence such a case cannot occur.

We can summarize the above discussion as follows.

Lemma 4.3. *If $\mathcal{E}$ is an indecomposable, initialized, aCM bundle of rank 2 on F, then the zero–locus $(s)_0$ of a general section $s \in H^0\big(F, \mathcal{E}\big)$ has pure codimension 2.*

5. The extremal cases

In this section we will deal with bundles on F and Φ of rank 2 whose first Chern class is either 0, or $2h$, or 2η.

Let $\mathcal{G}$ be an indecomposable, initialized, aCM vector bundle of rank 2 on Φ and let γ_1 and γ_2 be its Chern classes. If $\gamma_2 = \mu_1\eta_2^2 + \mu_2\eta_1^2 + \mu_3\eta_1\eta_2$, then the restriction $\mathcal{E} := \mathcal{G} \otimes \mathcal{O}_F$ satisfies $c_2 = (\mu_1 + \mu_3)h_2^2 + (\mu_2 + \mu_3)h_1^2$. Let Σ and E be the zero loci of general sections of $\mathcal{G}$ and $\mathcal{E}$ respectively. Thanks to Lemma 4.3, E has codimension 2, hence the same is true for Σ, by Remark 2.8. The class of Σ in $A^2(\Phi)$ is γ_2, thus

$$\deg(\Sigma) = \gamma_2\eta^2 = \mu_1 + \mu_2 + 2\mu_3 \geq 0.$$

Moreover

$$\mu_1 = \gamma_2\eta_1^2 \geq 0, \qquad \mu_2 = \gamma_2\eta_2^2 \geq 0, \qquad \mu_3 = \gamma_2\eta_1\eta_2 \geq 0, \tag{20}$$

because $|\eta_i|$ is base–point–free for $i = 1, 2$.

Assume $\gamma_1 = 0$. The zero–locus $\Sigma := (\sigma)_0$ of a general section $\sigma \in H^0\big(\Phi, \mathcal{G}\big)$ has codimension 2 inside Φ. Formula (17) and Lemma 4.1 give that E, the hyperplane section of Σ, is a line, thus Σ is a plane. Looking at the second Chern class c_2 of $\mathcal{E}$, either μ_1 or μ_2 is 1, μ_3 being zero. Hence the cycle of Σ is either η_1^2, or η_2^2.

Conversely, if we take a plane $\Sigma \subseteq \Phi$ we can assume that its cycle in $A^2(\Phi)$ is η_2^2, because $\deg(\Sigma) = 1$, and we have $\omega_\Sigma \cong \mathcal{O}_\Sigma(-3\eta)$. The isomorphism $\omega_\Phi \cong \mathcal{O}_\Phi(-3\eta)$, implies $\det(\mathcal{N}_{\Sigma|\Phi}) \cong \mathcal{O}_\Sigma$ by adjunction on Φ. Theorem 2.3 with $\mathcal{L} := \mathcal{O}_\Phi$ proves the existence of a unique vector bundle $\mathcal{G}$ of rank 2 fitting into a sequence of the form

$$0 \longrightarrow \mathcal{O}_\Phi \longrightarrow \mathcal{G} \longrightarrow \mathcal{I}_{\Sigma|\Phi} \longrightarrow 0.$$

Hence $h^1\big(\Phi, \mathcal{G}(t\eta)\big) \leq h^1\big(\Phi, \mathcal{I}_{\Sigma|\Phi}(t\eta)\big)$, $t \in \mathbb{Z}$. The cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_{\Phi|\mathbb{P}^8} \longrightarrow \mathcal{I}_{\Sigma|\mathbb{P}^8} \longrightarrow \mathcal{I}_{\Sigma|\Phi} \longrightarrow 0$$

twisted by $\mathcal{O}_{\mathbb{P}^8}(t)$ and the vanishing $h^1\big(\Phi, \mathcal{I}_{\Sigma|\mathbb{P}^8}(t)\big) = h^2\big(\Phi, \mathcal{I}_{\Phi|\mathbb{P}^8}(t)\big) = 0$ (recall that both Σ and Φ are aCM) give $h^1\big(\Phi, \mathcal{I}_{\Sigma|\Phi}(t\eta)\big) = 0$, hence $h^1\big(\Phi, \mathcal{G}(t\eta)\big) = 0$. A similar computation shows that $h^2\big(\Phi, \mathcal{G}(t\eta)\big) = 0$ too.

Finally $\mathcal{G}^\vee \cong \mathcal{G}$, because $\gamma_1 = 0$, thus the Serre duality also yields $h^3\big(\Phi, \mathcal{G}(t\eta)\big) = 0$. We conclude that $\mathcal{G}$ is an aCM bundle. It is easy to check that $\mathcal{G}$ is also initialized.

If $\mathcal{G}$ were decomposable, then $\mathcal{G} \cong \mathcal{M} \oplus \mathcal{M}^{-1}$ because $\gamma_1 = 0$. Thus $\eta_2^2 = \gamma_2 = -c_1(\mathcal{M})^2$. But if $c_1(\mathcal{M}) = m_1\eta_1 + m_2\eta_2$, then $c_1(\mathcal{M})^2 = m_1^2\eta_1^2 + m_2^2\eta_2^2 + 2m_1m_2\eta_1\eta_2$ which cannot coincide with $-\eta_2^2$.

Theorem 5.1. *Let $\mathcal{G}$ be an indecomposable, initialized, aCM bundle of rank 2 on Φ with $\gamma_1 = 0$. Then $\gamma_2 = \eta_2^2$ up to permutation of the η_i's.*

Conversely, there exists an indecomposable, initialized, aCM bundle $\mathcal{G}$ of rank 2 on Φ with Chern classes $\gamma_1 = 0$ and $\gamma_2 = \eta_2^2$.

Moreover the zero–locus of a general section of $\mathcal{G}$ is a plane and each plane on Φ can be obtained in such a way.

An analogous statement holds for F.

Theorem 5.2. *Let $\mathcal{E}$ be an indecomposable, initialized, aCM bundle of rank 2 on F with $c_1 = 0$. Then $c_2 = h_2^2$ up to permutation of the h_i's.*

Conversely, there exists an indecomposable, initialized, aCM bundle $\mathcal{E}$ of rank 2 on F with Chern classes $c_1 = 0$ and $c_2 = h_2^2$.

Moreover the zero–locus of a general section of $\mathcal{E}$ is a line and each line on F can be obtained in such a way.

Proof. The only change is in the proof of the indecomposability of $\mathcal{E}$. Indeed in this case $h_1h_2 = h_1^2 + h_2^2$, thus we have to verify that $(m_1^2 + 2m_1m_2)h_1^2 + (2m_1m_2 + m_2^2)h_2^2 = -h_2^2$ cannot hold, which is a trivial computation. □

Besides the above general construction via codimension 2 subschemes of F, we can also obtain examples of initialized, indecomposable aCM bundles with $c_1 = 0$ (and $\gamma_1 = 0$) by means of an interesting alternative construction via suitable pull–backs.

Example 5.3. Let $\mathcal{E}$ be any bundle defined by the exact sequence

$$0 \longrightarrow \mathcal{O}_F(-h_2) \longrightarrow \mathcal{O}_F \oplus p_2^*\Omega^1_{\mathbb{P}^2}(1) \longrightarrow \mathcal{E} \longrightarrow 0$$

(e.g. $\mathcal{E}$ is the pullback on F via p_2 of the restriction to a linear $\mathbb{P}^2$ of a null correlation bundle on $\mathbb{P}^3$). Hence $\mathcal{E}$ is initialized, indecomposable and aCM on F with $c_1 = 0$ and $c_2 = h_2^2$.

Obviously a similar construction can be repeated on Φ obtaining an initialized, indecomposable, aCM bundle with $\gamma_1 = 0$ and $\gamma_2 = \eta_2^2$.

Let us now consider the case $c_1 = 2h$. We first characterize such bundles in terms of the zero–locus of their general sections.

Assume the existence on F of an indecomposable, initialized, aCM bundle $\mathcal{E}$ of rank 2 with $c_1 = 2h$. Proposition 4.2 guarantees that $\mathcal{E}$ is Ulrich and that the zero–locus $E := (s)_0$ of a general section $s \in H^0\big(F, \mathcal{E}\big)$ is a smooth curve inside F.

From Sequence (1) we obtain $h^1\big(F, \mathcal{I}_{E|F}\big) = h^1\big(F, \mathcal{I}_{E|F}(h)\big) = h^0\big(F, \mathcal{I}_{E|F}(h)\big) = 0$, whence both $h^0\big(E, \mathcal{O}_E\big) = 1$ and E is linearly normal and non–degenerate thanks to Sequence (2). Moreover, combining Lemma 4.1 and Formulas (17), (16), we obtain $\deg(E) = hc_2 = 8$, $p_a(E) = 1$. Thus we can assume that $E \subseteq \mathbb{P}^7$ is an elliptic normal curve, i.e. a non–degenerate, irreducible, smooth, elliptic curve of degree 8.

Conversely, let E be an elliptic normal curve on F. By adjunction $\mathcal{O}_E \cong \omega_E \cong \det(\mathcal{N}_{E|F}) \otimes \mathcal{O}_F(-2h)$. The sheaf $\mathcal{O}_F(2h)$ satisfies the vanishing of the Theorem 2.3, thus E is the zero locus of a section s of a vector bundle $\mathcal{E}$ of rank 2 on F with $c_1 = 2h$: the class of E is $c_2 \in A^2(F)$.

The cohomology of Sequence (1) twisted by $\mathcal{O}_F(-h)$ implies that $\mathcal{E}$ is initialized, being E non–degenerate.

Let us prove that $\mathcal{E}$ is aCM. The curve E is aCM, due to Proposition 1.1 and Corollary 2.2 of [13], then $h^1\big(F, \mathcal{I}_{E|\mathbb{P}^7}(t)\big) = 0$. Moreover $h^2\big(F, \mathcal{I}_{F|\mathbb{P}^7}(t)\big) = 0$ because F is aCM. Taking the cohomology of sequence

$$0 \longrightarrow \mathcal{I}_{F|\mathbb{P}^7} \longrightarrow \mathcal{I}_{E|\mathbb{P}^7} \longrightarrow \mathcal{I}_{E|F} \longrightarrow 0,$$

twisted by $\mathcal{O}_{\mathbb{P}^7}(t)$, we also deduce that $h^1\big(F, \mathcal{I}_{E|F}(th)\big) = 0$. The twisted cohomology of the Sequence (1) yields $h^1\big(F, \mathcal{E}(th)\big) = 0$. Such a vanishing also implies $h^2\big(F, \mathcal{E}(th)\big) = 0$ by the Serre duality. We conclude that $\mathcal{E}$ is aCM. Thanks to Proposition 4.2 we also know that $\mathcal{E}$ is Ulrich. Thus elliptic normal curves on F correspond to Ulrich bundles on F with $c_1 = 2h$.

We now compute the possible values of c_2. The linear system $|h_i|$ on F has dimension 2 (see Proposition 2.5). If $D_i \in |h_i|$ is general, then the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_F(h - h_i) \longrightarrow \mathcal{O}_F(h) \longrightarrow \mathcal{O}_{D_i}(h) \longrightarrow 0$$

yields $h^0\big(D_i, \mathcal{O}_{D_i}(h)\big) = 5$, thus D_i spans a space of dimension 4 in $\mathbb{P}^7$. Since E is non–degenerate, it follows that the restriction of $|h_i|$ to E has dimension at least 2. Since E is elliptic, it follows that $h_iE = \beta_i \ge 3$, thanks to the Riemann–Roch theorem on E. Thus the possible cases for (β_1, β_2) are $(3, 5)$, $(4, 4)$ up to permutations, i.e. c_2 is either $3h_2^2 + 5h_1^2$, or $4h_1^1 + 4h_1^2$.

In the two following examples we will prove that both such cases actually occur.

Example 5.4. Proposition 2.5 implies $\mathrm{Ext}^1_F\big(\mathcal{O}_F(2h_1), \mathcal{O}_F(2h_2)\big) \cong H^1\big(\mathcal{O}_F(-2h_1 + 2h_2)\big) \cong k^{\oplus 3}$. In particular we obtain a family of indecomposable rank two Ulrich bundles on F given by the non–trivial extensions

$$0 \longrightarrow \mathcal{O}_F(2h_2) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_F(2h_1) \longrightarrow 0.$$

We have $c_1(\mathcal{E}) = 2h$ and $c_2(\mathcal{E}) = 4h_1h_2 = 4h_2^2 + 4h_1^2$.

We notice that the above construction cannot be extended on Φ: indeed Künneth formulas imply that $h^1\big(\Phi, \mathcal{L}\big) = 0$ for each line bundle $\mathcal{L} \in \mathrm{Pic}(\Phi)$.

Example 5.5. Let $a \ge 1$ be an integer and let $\mathcal{H}$ be any vector bundle fitting into the exact sequence

$$0 \longrightarrow \mathcal{H} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4a} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(2)^{\oplus 2a} \longrightarrow 0.$$

If $\mathcal{H}$ and $\mathcal{H}'$ are two vector bundles obtained as above, each isomorphism $\mathcal{H} \cong \mathcal{H}'$ lifts to a isomorphism of the corresponding exact sequences. It follows that the bundle fitting in the above sequence form a family $\mathcal{M}_{\mathbb{P}^2}(a)$ of dimension

$$h^0\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)\big)^{\oplus 8a^2} - \dim(\mathrm{GL}_{4a}) - \dim(\mathrm{GL}_{2a}) + 1 = 4a^2 + 1$$

of pairwise non–isomorphic vector bundles of rank $2a$.

Let $\mathcal{G} := (\pi_1^*\mathcal{H})(\eta) \cong (\pi_1^*\mathcal{H}(1)) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^2}(1)$. As pointed out in [15], Theorem 3.6 we obtain a family $\mathcal{M}_\Phi(a)$ of dimension $4a^2 + 1$ of vector bundles of rank $r := 2a$ with first Chern class $2a\eta$ which are Ulrich and simple, i.e. $h^0\big(\Phi, \mathcal{G}^\vee \otimes \mathcal{G}\big) = 1$: in particular $\mathcal{G}$ is also indecomposable. The bundles in $\mathcal{M}_\Phi(a)$ are pairwise non–isomorphic because the same is true for the bundles in $\mathcal{M}_{\mathbb{P}^2}(a)$.

Thanks to Remark 2.8 the bundle $\mathcal{E} := \mathcal{G} \otimes \mathcal{O}_F$ is an initialized, Ulrich bundle on F of rank r: moreover $c_1(\mathcal{E}) = 2ah$. Notice that if $a = 1$ it is easy to check that $c_2(\mathcal{G}) = \eta_2^2 + 3\eta_1^2 + 2\eta_1\eta_2$ and $c_2(\mathcal{E}) = 3h_2^2 + 5h_1^2$.

It follows the existence for each integer $a \geq 1$ of a non–empty family $\mathcal{M}_F(a)$ of Ulrich bundles of rank r with first Chern class $2ah$.

Claim 5.6. *The bundles in $\mathcal{M}_F(a)$ are simple (hence indecomposable) and pairwise non–isomorphic.*

Indeed, let $\mathcal{G}' \cong (\pi_1^*\mathcal{H}')(\eta)$ be another bundle in $\mathcal{M}_\Phi(a)$, and consider the cohomology of the restriction sequence

$$0 \longrightarrow \mathcal{G}^\vee \otimes \mathcal{G}'(-\eta) \longrightarrow \mathcal{G}^\vee \otimes \mathcal{G}' \longrightarrow \mathcal{E}^\vee \otimes \mathcal{E}' \longrightarrow 0,$$

where $\mathcal{E}' := \mathcal{G}' \otimes \mathcal{O}_F$. Since

$$\mathcal{G}^\vee \otimes \mathcal{G}'(-\eta) \cong (\pi_1^*\mathcal{H}^\vee \otimes \mathcal{H}'(-1)) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^2}(-1),$$

it follows from Künneth formulas that $h^1\big(\Phi, \mathcal{G}^\vee \otimes \mathcal{G}'(-\eta)\big) = 0$. We conclude that the natural map $H^0\big(\Phi, \mathcal{G}^\vee \otimes \mathcal{G}'\big) \to H^0\big(F, \mathcal{E}^\vee \otimes \mathcal{E}'\big)$ is surjective, i.e. each morphism $u\colon \mathcal{E} \to \mathcal{E}'$ can be lifted to a morphism $v\colon \mathcal{G} \to \mathcal{G}'$ such that

$$\det(v) \in H^0\big(\Phi, \det(\mathcal{G}^\vee) \otimes \det(\mathcal{G}')\big) \cong H^0\big(\Phi, \mathcal{O}_\Phi\big) \cong k,$$

because $\mathcal{G}$ and $\mathcal{G}'$ have the same first Chern class.

On the one hand, if u is an isomorphism, then v is an isomorphism too, because the restriction to F of $\det(v) \in k$ is exactly $\det(u) \in k$ which is non–zero. We conclude that the bundles in $\mathcal{M}_F(a)$ are pairwise non–isomorphic.

On the other hand, taking $\mathcal{G} \cong \mathcal{G}'$, we immediately deduce that $\mathcal{E}$ is a bundle on F satisfying

$$1 \leq h^0\big(F, \mathcal{E}^\vee \otimes \mathcal{E}\big) \leq h^0\big(\Phi, \mathcal{G}^\vee \otimes \mathcal{G}\big) \leq 1$$

i.e. $\mathcal{E}$ is simple, thus indecomposable too.

An immediate consequence of the above analysis is that F is of Ulrich–wild representation type, i.e. it supports p–dimensional families of non–isomorphic, indecomposable Ulrich bundles for arbitrary large p.

Theorem 5.7. *Let $\mathcal{E}$ be an indecomposable, initialized, aCM bundle of rank 2 on F with $c_1 = 2h$. Then c_2 is either $3h_2^2 + 5h_2^2$, or $4h_2^2 + 4h_1^2$ up to permutation of the h_i's.*

Conversely, for each such a c_2, there exists an indecomposable, initialized, aCM bundle $\mathcal{E}$ of rank 2 on F with Chern classes $c_1 = 2h$ and c_2.

Moreover each such bundle $\mathcal{E}$ is Ulrich, the zero–locus of a general section of $\mathcal{E}$ is an elliptic normal curve and each elliptic normal curve on F can be obtained in this way unless it is the complete intersection of two divisors in $|2h_2|$ and $|2h_1|$ (such a case can occur only if $c_2 = 4h_2^2 + 4h_1^2 = 4h_1h_2$).

Proof. Examples 5.4 and 5.5 with $a = 1$ prove the existence of the bundles. The remaining part of the proof follows from the discussion after Example 5.3. □

We will now focus our attention on the analogous problem of classifying indecomposable, initialized, aCM bundles $\mathcal{G}$ of rank 2 on Φ with $\gamma_1 = 2\eta$ and $\gamma_2 = \mu_1\eta_2^2 + \mu_2\eta_1^2 + \mu_3\eta_1\eta_2$. As pointed out at the beginning of this section, $\mathcal{G}\otimes\mathcal{O}_F$ is an indecomposable, initialized, aCM bundle of rank 2 on F with $c_1 = 2h$ and $c_2 = (\mu_1+\mu_3)h_2^2 + (\mu_2+\mu_3)h_1^2$. Hence $(\mu_1+\mu_3, \mu_2+\mu_3)$ is either $(3,5)$ or $(4,4)$ by Theorem 5.7.

Theorem 5.8. *Let $\mathcal{G}$ be an indecomposable, initialized, aCM bundle of rank 2 on Φ with $\gamma_1 = 2\eta$. Then $\gamma_2 = \eta_2^2 + 3\eta_1^2 + 2\eta_1\eta_2$ up to permutation of the η_i's.*

Conversely, there exists an indecomposable, initialized, aCM bundle $\mathcal{G}$ of rank 2 on Φ with Chern classes $\gamma_1 = 2\eta$ and $\gamma_2 = \eta_2^2 + 3\eta_1^2 + 2\eta_1\eta_2$.

Moreover each such bundle $\mathcal{G}$ is Ulrich, the zero–locus of a general section of $\mathcal{G}$ is a del Pezzo surface of degree 8 isomorphic to the blow up $\mathbb{F}_1$ of $\mathbb{P}^2$ at a point and each del Pezzo surface of degree 8 on Φ can be obtained in this way unless it is the complete intersection of two divisors in $|2\eta_2|$ and $|2\eta_1|$ (such a case can occur only if $\gamma_2 = 4\eta_1\eta_2$).

Proof. The claimed existence follows from Example 5.5.

In order to complete the proof of the statement, we preliminary observe that if $\gamma_1 = 2\eta$, then $\mathcal{G}$ is Ulrich, hence its general section vanishes on a smooth surface Σ whose class in $A^2(\Phi)$ is $\gamma_2 = \mu_1\eta_2^2 + \mu_2\eta_1^2 + \mu_3\eta_1\eta_2$. Recall that $(\mu_1+\mu_3, \mu_2+\mu_3)$ can be assumed either $(3,5)$, or $(4,4)$.

The same argument used for elliptic curves on F shows that $\Sigma \subseteq \mathbb{P}^8$ is non–degenerate. Restricting Sequence (1) for Σ (which is a Koszul complex) to Σ itself we obtain that $\mathcal{G}\otimes\mathcal{O}_\Sigma \cong \mathcal{N}_{\Sigma|\Phi}$. Adjunction formula finally yields $\omega_\Sigma \cong \mathcal{O}_\Sigma(-\eta)$. We conclude that Σ is a del Pezzo surface embedded anticanonically in $\mathbb{P}^8$, thus its degree is 8. Looking at the classification of del Pezzo surfaces we know that Σ is isomorphic to either $Q := \mathbb{P}^1\times\mathbb{P}^1$, or $\mathbb{F}_1 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-1))$. Conversely, again imitating the argument used for elliptic curves on F, recalling that each del Pezzo surface and each del Pezzo sextic fourfold in $\mathbb{P}^8$ are aCM, it is also possible to prove that each del Pezzo surface $\Sigma \subseteq \Phi$ is the zero locus of an Ulrich bundle $\mathcal{G}$ of rank 2 on Φ with $\gamma_1 = 2\eta$. Thus, in order to prove the statement it suffices to deal with the possible embeddings $\Sigma \subseteq \Phi$.

We start with the case $\Sigma \cong \mathbb{F}_1$. Recall that $\mathrm{Pic}(\mathbb{F}_1)$ is freely generated by divisors ℓ and m such that $\ell^2 = -1$, $m^2 = 0$ and $\ell m = 1$. Let $\Sigma \subseteq \Phi$ be any smooth del Pezzo octic surface isomorphic to $\mathbb{F}_1$ in the class γ_2.

Each morphism $\varphi\colon \mathbb{F}_1 \to \Phi$ induces, by compositions with the two natural projections $\Phi \to \mathbb{P}^2$, maps $\varphi_i\colon \mathbb{F}_1 \to \mathbb{P}^2$. In particular we know that $\varphi^*\mathcal{O}_\Phi(\eta_i) \cong \varphi_i^*\mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_{\mathbb{F}_1}(a_i\ell + b_i m)$ for suitable integers a_i, b_i. We thus have $\varphi^*\mathcal{O}_\Phi(\eta) \cong \mathcal{O}_{\mathbb{F}_1}((a_1+a_2)\ell + (b_1+b_2)m)$.

Since $\mathcal{O}_{\mathbb{F}_1}(a_i\ell + b_i m)$ is effective, it follows that both a_i and b_i are non–negative. Moreover the embedding $\Sigma \subseteq \mathbb{P}^8$ is anticanonical, thus $a_1 + a_2 = 2$ and $b_1 + b_2 = 3$. We have $\mu_i = \eta_i^2\Sigma = (a_i\ell + b_i m)^2 = a_i(2b_i - a_i)$, $i = 1,2$.

Let us first examine the case $\mu_1 + \mu_3 = \mu_2 + \mu_3 = 4$. If $a_1 = a_2 = 1$, then $2b_i - 1 = \mu_i$ must be odd. Moreover

$$2\mu_3 = 8 - \mu_1 - \mu_2 = 10 - 2b_1 - 2b_2 = 10 - 6 = 4.$$

We conclude that $\mu_1 = \mu_2 = \mu_3 = 2$, a contradiction. If $a_1 = 0$ and $a_2 = 2$, then $\mu_1 = 0$, whence $\mu_2 = 0$ and $\mu_3 = 4$. The equality $2(2b_2 - 2) = \mu_2 = 0$ thus yields $b_2 = 1$. The morphism φ_2 should be thus defined by three sections of $\mathcal{O}_{\mathbb{F}_1}(2\ell + m)$ without common zeros, which is absurd, because all such sections vanish identically on ℓ. A similar argument holds in the case $a_1 = 2$ and $a_2 = 0$.

Now we examine the second case $\mu_1 + \mu_3 = 3$, $\mu_2 + \mu_3 = 5$. Taking into account of the listed constraints, arguing as above it is not difficult to check that $a_1 = a_2 = b_1 = 1$ and $b_2 = 2$. It follows that $\mu_1 = 1$, $\mu_2 = 3$, $\mu_3 = 2$, i.e. $\gamma_2 = \eta_2^2 + 3\eta_1^2 + 2\eta_1\eta_2$.

Notice that each bundle $\mathcal{G}$ with such a γ_2 is, necessarily, indecomposable. Indeed, if $\mathcal{G} \cong \mathcal{L}_1\oplus\mathcal{L}_2$ for some $\mathcal{L}_i \in \mathrm{Pic}(\Phi)$, then $\mathcal{L}_i$ is aCM. Either both the $\mathcal{L}_i$ are initialized or one of them is initialized and the other one has no sections. Thanks to the equalities

$$h^0\big(\Phi, \mathcal{L}_1\big) + h^0\big(\Phi, \mathcal{L}_2\big) = h^0\big(\Phi, \mathcal{G}\big) = 12 \qquad c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2) = 2\eta$$

and to Proposition 2.5, it is easy to check that $\mathcal{G} \cong \mathcal{O}_\Phi(2\eta_2) \oplus \mathcal{O}_\Phi(2\eta_1)$ necessarily. In particular we should have $\gamma_2 = 4\eta_1\eta_2$, a contradiction.

We finally examine the case of the smooth quadric Q. Recall that $\mathrm{Pic}(Q)$ is freely generated by divisors ℓ and m such that $\ell^2 = m^2 = 0$ and $\ell m = 1$.

As in the previous case we have two morphisms $\varphi_i \colon Q \to \mathbb{P}^2$ with $\varphi^*\mathcal{O}_\Phi(\eta_i) \cong \varphi_i^*\mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_Q(a_i\ell + b_i m)$ for suitable integers a_i, b_i. Thus again $\varphi^*\mathcal{O}_\Phi(\eta) \cong \mathcal{O}_Q((a_1+a_2)\ell + (b_1+b_2)m)$.

In this case the list of constraints is as follows: $a_i, b_i \ge 0$, $a_1 + a_2 = b_1 + b_2 = 2$, $\mu_i = \eta_i^2\Sigma = (a_i\ell + b_i m)^2 = 2a_ib_i$, $i = 1, 2$, and we have the equality

$$2\mu_3 = 8 - \mu_1 - \mu_2 = 8 - 2b_1 - 2b_2 = 8 - 4 = 4,$$

hence $\mu_3 = 2$. If $\mu_1 + \mu_3 = 3$, $\mu_2 + \mu_3 = 5$, then $\mu_i = 2a_ib_i$ would be odd, a contradiction. Finally, let $\mu_1 + \mu_3 = \mu_2 + \mu_3 = 4$, hence $2 = \mu_i = 2a_ib_i$. We deduce that $a_1 = b_1 = a_2 = b_2 = 1$, thus the maps φ_i are the restriction of projections $\psi_i \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ from a point. Let $\psi \colon \mathbb{P}^3 \dashrightarrow \Phi$ be their product rational map. By construction $\varphi = \psi_{|Q}$.

By definition, the equations defining the composition of ψ with the inclusion $\Phi \subseteq \mathbb{P}^8$ are

$$(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (y_0y_3, y_0y_4, y_0y_5, y_1y_3, y_1y_4, y_1y_5, y_2y_3, y_2y_4, y_2y_5)$$

where y_0, y_1, y_2 and y_3, y_4, y_5 are two triples of independent sections of $H^0\big(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)\big)$. Let $V \subseteq H^0\big(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)\big)$ the subspace generated by all these sections: we distinguish the two cases $\dim(V) = 3, 4$.

In the last case we can assume $y_4 = \sum_{i=0}^3 u_iy_i$ and $y_5 = \sum_{j=0}^3 v_jy_j$. Since y_3, y_4, y_5 are independent, it follows that at least one of the u_i's, $i = 0, 1, 2$ is non–zero. It is easy to check that $\mathrm{im}(\psi)$ is contained in the hyperplane with equation

$$\begin{aligned}(u_0v_3 - u_3v_0)x_0 + v_0x_1 + (u_1v_3 - u_3v_1)x_3 + v_1x_4 + \\ + (u_2v_3 - u_3v_2)x_6 + v_2x_7 = u_0x_2 + u_1x_5 + u_2x_8.\end{aligned}$$

In the first case we can write $y_3 = \sum_{i=0}^2 u_iy_i$, $y_4 = \sum_{j=0}^2 v_jy_j$, $y_5 = \sum_{j=0}^2 w_jy_j$. Since y_3, y_4, y_5 are independent, it follows that at least one of the u_i's is non–zero. Again one checks easily that the hyperplane with equation

$$w_0x_0 + w_1x_3 + w_2x_6 = u_0x_2 + u_1x_5 + u_2x_8$$

contains $\mathrm{im}(\psi)$.

Thus $\Sigma \subseteq \mathrm{im}(\psi) \subseteq \mathbb{P}^8$ would be contained in a hyperplane too. In particular, Σ is not a del Pezzo octic, because it is degenerate. We conclude that both these cases cannot occur. □

6. The intermediate cases

In this section we will classify the indecomposable, initialized, aCM bundles of rank 2 on F and Φ whose first Chern class is neither 0, nor $2h$, nor 2η.

Inequality (18) with $D = 0$ and Formulas (16), (17) yield the following list (up to permutations) for the invariants associated with $\mathcal{E}$ and E.

Case	(α_1, α_2)	(β_1, β_2)	$\deg(E)$	$p_a(E)$
L	$(0,1)$	$(1,0)$	1	0
M	$(0,2)$	$(1,0)$	1	0
N	$(1,1)$	$(2,0)$	2	0
P	$(1,1)$	$(1,1)$	2	0
Q	$(2,1)$	$(2,2)$	4	0

Assume that $\mathcal{E}$ is an indecomposable, initialized, aCM bundle with invariants in the table above. The bundle $\mathcal{E}^\vee(th)$ is also an indecomposable aCM bundle. If t is such that $\mathcal{E}^\vee(th)$ is initialized, then Propositions 3.5 and 4.2 give $0 \le 2t - \alpha_i \le 2$. We conclude that $t = 1$. We have

$$c_1(\mathcal{E}^\vee(h)) = (2-\alpha_1)h_1 + (2-\alpha_2)h_2,$$
$$c_2(\mathcal{E}^\vee(h)) = (\beta_2 - 2\alpha_1 - \alpha_2 + 3)h_1^2 + (\beta_1 - \alpha_1 - 2\alpha_2 + 3)h_2^2.$$

Thus case L occurs if and only if case Q also occurs.

We will show that there there are only two possible bundles (both on F and on Φ) up to permuting the generators of the Picard group. Such bundles on F are described in the following example.

Example 6.1. We have $\mathrm{Ext}^1_F(\mathcal{O}_F(h_1), \mathcal{O}_F(-h_1+h_2)) \cong H^1(F, \mathcal{O}_F(-2h_1+h_2)) \cong k$ (see Proposition 2.5). It follows the existence of a unique indecomposable extension

$$0 \longrightarrow \mathcal{O}_F(-h_1+h_2) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_F(h_1) \longrightarrow 0. \tag{21}$$

It is immediate to check that $\mathcal{E}$ is initialized, indecomposable and aCM. Its Chern classes are $c_1 = h_2$ and $c_2 = h_2^2$.

Claim 6.2. $\mathcal{E} \cong p_2^*\Omega^1_{\mathbb{P}^2}(2h_2)$.

It suffices to prove the existence of another extension like Sequence (21) with $p_2^*\Omega^1_{\mathbb{P}^2}(2h_2)$ in the middle. In fact, let $\mathcal{G} := \pi_2^*\Omega^1_{\mathbb{P}^2}(2\eta_2)$ and consider the standard restriction sequence

$$0 \longrightarrow \mathcal{G}(-2\eta_2) \longrightarrow \mathcal{G}(\eta_1 - \eta_2) \longrightarrow \mathcal{G} \otimes \mathcal{O}_F(h_1 - h_2) \longrightarrow 0.$$

Künneth formula yields $h^1\big(\Phi, \mathcal{G}(-2\eta_2)\big) = 1$ and $h^0\big(\Phi, \mathcal{G}(\eta_1-\eta_2)\big) = h^1\big(\Phi, \mathcal{G}(\eta_1-\eta_2)\big) = 0$, whence $h^0\big(F, \mathcal{G} \otimes \mathcal{O}_F(h_1-h_2)\big) = 1$. Let $s \in H^0\big(F, \mathcal{G} \otimes \mathcal{O}_F(h_1-h_2)\big)$ be a general section and consider the corresponding Sequence (1)

$$0 \longrightarrow \mathcal{O}_F(D) \longrightarrow \mathcal{G} \otimes \mathcal{O}_F(h_1-h_2) \longrightarrow \mathcal{I}_E(2h_1 - h_2 - D) \longrightarrow 0,$$

where D is an effective divisor and E has pure codimension 2 in F or it is empty.

On the one hand $h^0\big(F, \mathcal{G} \otimes \mathcal{O}_F(h_1-h_2)\big) = 1$, thus $h^0\big(F, \mathcal{O}_F(D)\big) \le 1$. This last inequality forces the vanishing $D = 0$, due to Proposition 2.5. On the other hand an easy computation shows that $c_2(\mathcal{G} \otimes \mathcal{O}_F(h_1-h_2)) = 0$. It follows that $\mathcal{G} \otimes \mathcal{O}_F(h_1-h_2)$ fits into the exact sequence

$$0 \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{G} \otimes \mathcal{O}_F(h_1-h_2) \longrightarrow \mathcal{O}_F(2h_1-h_2) \longrightarrow 0$$

which is the claimed extension twisted by $\mathcal{O}_F(-h_1+h_2)$.

Notice that $\mathcal{E}^\vee(h) \cong p_2^*\Omega^1_{\mathbb{P}^2}(h_1+2h_2)$. Pulling back the suitably twisted Euler sequence via p_2 we thus immediately deduce that $\mathcal{E}^\vee(h)$ is globally generated. Thus the zero locus E of its general section is a smooth curve. From Proposition 2.5 and the cohomology of Sequence (2) applied for $\mathcal{E}^\vee(h)$ and twisted by $\mathcal{O}_F(-h_1-2h_2)$, one can deduce that $h^0\big(F, \mathcal{I}_{E|F}\big) = h^1\big(F, \mathcal{I}_{E|F}\big) = 0$, because $\mathcal{E}^\vee(h)$ is initialized and aCM. The cohomology of Sequence (1) yields that E is connected, hence integral. Thus E is also rational.

We are now ready to complete the description of initialized, indecomposable, aCM bundles of rank 2 on F.

6.1. The cases L and Q.

In case L the cycle of E is $h_2^2 \in A^2(F)$.

In particular E is a line, hence $\omega_E \cong \mathcal{O}_E(-2h)$. Thus $\det(\mathcal{N}_{E|F}) \cong \mathcal{O}_E \cong \mathcal{O}_F(h_2) \otimes \mathcal{O}_E$. Since $h^2\big(F, \mathcal{O}_F(-h_2)\big) = 0$ thanks to Proposition 2.5, it follows the existence of an exact sequence

$$0 \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|F}(h_2) \longrightarrow 0. \tag{22}$$

where $\mathcal{E}$ is a vector bundle of rank 2 on F (see Theorem 2.3). Trivially $\mathcal{E}$ is initialized.

The cohomology of Sequence (2) twisted by $\mathcal{O}_F(h_1)$ returns the exact sequence

$$H^0\big(F, \mathcal{O}_F(h_1)\big) \longrightarrow H^0\big(E, \mathcal{O}_E(h_1)\big) \longrightarrow H^1\big(F, \mathcal{I}_{E|F}(h_1)\big) \longrightarrow 0.$$

We have $h_1E = h_1h_2^2$, hence the first map is surjective. It follows $h^1\big(F, \mathcal{I}_{E|F}(h_1)\big) = 0$ and $h^0\big(F, \mathcal{I}_{E|F}(h_1)\big) = 1$. The cohomology of Sequence (22) twisted by $\mathcal{O}_F(h_1-h_2)$ thus yields $h^0\big(F, \mathcal{E}(h_1-h_2)\big) = 1$.

Let Z be the zero locus of a general section of $\mathcal{E}(h_1-h_2)$. As usual we decompose $Z = C \cup \Delta$ where C has pure codimension 2 (or it is empty) and Δ is an effective divisor. In this case Sequence (1) is

$$0 \longrightarrow \mathcal{O}_F(\Delta) \longrightarrow \mathcal{E}(h_1-h_2) \longrightarrow \mathcal{I}_{C|F}(2h_1-h_2-\Delta) \longrightarrow 0,$$

thus $h^0\big(F,\mathcal{O}_F(\Delta)\big) \le 1$. Proposition 2.5 and the effectiveness of Δ finally yield $\Delta = 0$. In particular the class of C is $c_2(\mathcal{E}(h_1-h_2)) = 0$, hence $C = \emptyset$. The above sequence twisted by $\mathcal{O}_F(-h_1+h_2)$ becomes

$$0 \longrightarrow \mathcal{O}_F(-h_1+h_2) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_F(h_1) \longrightarrow 0.$$

which is the extension of Example 6.1 because $\operatorname{Ext}^1_F\big(\mathcal{O}_F(2h_1-h_2),\mathcal{O}_F\big) \cong k$. We conclude that $\mathcal{E}$ is indecomposable, initialized and aCM, hence both the cases L and Q are admissible.

6.2. **The case M.** In case M again $E \subseteq F$ is a line, whose class is $h_2^2 \in A^2(F)$: we know that $\det(\mathcal{N}_{E|F}) \cong \mathcal{O}_F(2h_2)\otimes\mathcal{O}_E$. On the one hand Theorem 2.3 guarantees the existence of an exact sequence

$$0 \longrightarrow \mathcal{O}_F(-2h_2) \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{I}_{E|F} \longrightarrow 0,$$

where $\mathcal{E}$ is a vector bundle of rank 2 on F, because $h^2\big(F,\mathcal{O}_F(-2h_2)\big) = 0$. On the other hand, there is also the obvious exact sequence

$$0 \longrightarrow \mathcal{O}_F(-2h_2) \longrightarrow \mathcal{O}_F(-h_2)^{\oplus 2} \longrightarrow \mathcal{I}_{E|F} \longrightarrow 0.$$

Since $h^1\big(F,\mathcal{O}_F(-2h_2)\big) = 0$ by Proposition 2.5, it follows that Theorem 2.3 implies $\mathcal{E} \cong \mathcal{O}_F(h_2)^{\oplus 2}$.

Thus the case M is not admissible: indeed $\mathcal{E}$ exists, but it is decomposable.

6.3. **The cases N and P.** In both these cases E is a curve of degree 2 of genus 0. It is either integral, or the union of two concurrent lines, because $p_a(E) = 0$, or a double line.

In case P only the first two cases are possible because the class of E is $h_2^2+h_1^2$, thus E is reduced. We have $\omega_E \cong \mathcal{O}_E(-h)$, whence $\det(\mathcal{N}_{E|F}) \cong \mathcal{O}_E(h)$. As ususal, the vanishing $h^2\big(F,\mathcal{O}_F(h)\big) = 0$ implies the existence of an exact sequence

$$0 \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|F}(h) \longrightarrow 0.$$

where $\mathcal{E}$ is a vector bundle of rank 2 on F.

Since E is either a conic or the union of two distinct concurrent lines and $h_2c_2 = 1$, it follows that $h^0\big(E,\mathcal{O}_E(h_2)\big) = 2$. Consider the cohomology of Sequence (2) twisted by $\mathcal{O}_F(h_2)$. On the one hand $h^0\big(F,\mathcal{I}_{E|F}(h_2)\big) \ge 1$. On the other hand $h^0\big(F,\mathcal{I}_{E|F}(h_2)\big) < h^0\big(F,\mathcal{O}_F(h_2)\big)$, because $\mathcal{O}_F(h_2)$ is base point free, whence $h^0\big(F,\mathcal{I}_{E|F}(h_2)\big) \le 2$.

Thus the cohomology of the above sequence twisted by $\mathcal{O}_F(-h_1)$ and Corollary 2.7 give $1 \le h^0\big(F,\mathcal{E}(-h_1)\big) \le 2$.

Let Z be the zero locus of a general section of $\mathcal{E}(-h_1)$ and, as usual, write $Z = C\cup\Delta$. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_F(\Delta) \longrightarrow \mathcal{E}(-h_1) \longrightarrow \mathcal{I}_{C|F}(-h_1+h_2-\Delta) \longrightarrow 0,$$

thus $h^0\big(F,\mathcal{O}_F(\Delta)\big) \le 2$. Proposition 2.5 and the effectiveness of Δ yield $\Delta = 0$. The class of C is $c_2(\mathcal{E}(-h_1)) = 0$, hence $\mathcal{E}(-h_1)$ corresponds to an element of $\operatorname{Ext}^1_F\big(\mathcal{O}_F(-h_1+h_2),\mathcal{O}_F\big) = H^1\big(F,\mathcal{O}_F(h_2-h_1)\big) = 0$. It follows $\mathcal{E} = \mathcal{O}_F(h_1)\oplus\mathcal{O}_F(h_2)$.

In the case N we know that E is either irreducible or it is a double line because its class is $2h_2^2 \in A^2(F)$. Indeed two concurrent lines on F have necessarily classes h_1^2 and h_2^2 in $A^2(F)$.

In the first case there are distinct $U, V \in |h_2|$, $i = 1,2$ such that $E\cap U\cap V \ne \emptyset$. Since $EU = EV = 0$, it would follow that $E \subseteq U\cap V$, whence $2 = 2h_2^2h = \deg(E) \le \deg(U\cap V) = h_2^2h = 1$, a contradiction.

It follows that E is a double structure on a line E_{red} whose cycle in $A^2(F)$ is h_2^2. The general theory of double structures (see e.g. [6]) gives us an exact sequence of the form

$$0 \longrightarrow \mathcal{C}_{E_{red}|E} \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{O}_{E_{red}} \longrightarrow 0.$$

The conormal sheaf $\mathcal{C}_{E_{red}|E}$ is invertible on $E_{red} \cong \mathbb{P}^1$, hence $\mathcal{C}_{E_{red}|E} \cong \mathcal{O}_{\mathbb{P}^1}(-a)$. Moreover $a = 1$, because $p_a(E) = 0$ (see Section 2 of [6]).

The Hilbert scheme Γ of lines in F has exactly two components Γ_1 and Γ_2, each of them isomorphic to $\mathbb{P}^2$ (see Proposition 3.5.6 of [27]).

The lines with cycle h_2^2 correspond exactly to the points of one of the two component, say Γ_1, and the universal family $S_1 \to \Gamma_1$ is the natural projection $p_2\colon \mathbb{P}(\Omega^1_{\mathbb{P}^2}(2)) \to \mathbb{P}^2$. The variety F does not contain planes, hence the universal family $S_1 \to \Gamma_1$ maps surjectively on F (see Proposition 3.3.5 of [27]). Lemma 3.3.4 of [27] implies that the line U corresponding to the general point in Γ_1 satisfies $\mathcal{C}_{U|F} \cong \mathcal{N}^\vee_{U|F} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. The induced action of the automorphism group of F on Γ_1 is transitive, hence there is an automorphism of F fixing the class h_2^2 and sending E_{red} to U. We conclude that $\mathcal{C}_{E_{red}|F} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$.

From the inclusion $E_{red} \subseteq E \subseteq F$ we obtain an epimorphism $\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \cong \mathcal{C}_{E_{red}|F} \twoheadrightarrow \mathcal{C}_{E_{red}|E} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$, an absurd.

Thus the cases P and N are not admissible: $\mathcal{E}$ exists but it is decomposable in case P, it does not exist at all in case N.

We summarize the results obtained above as follows.

Theorem 6.3. *Let $\mathcal{E}$ be an indecomposable, initialized, aCM bundle of rank 2 on F and let $c_1 = \alpha_1 h_1 + \alpha_2 h_2$. Assume that c_1 is neither 0, nor $2h$. Then (c_1, c_2) is either (h_2, h_2^2), or $(h_1 + 2h_2, 2h_1^2 + 2h_2^2)$, up to permutations of the h_i's.*

Conversely, for each such a pair, there exists a unique indecomposable, initialized, aCM bundle $\mathcal{E}$ of rank 2 on F with Chern classes c_1 and c_2. If $c_1 = h_2$, then $\mathcal{E} \cong p_2^*\Omega^1_{\mathbb{P}^2}(2h_2)$. If $c_1 = 2h_1 + h_2$, then $\mathcal{E} \cong p_2^*\Omega^1_{\mathbb{P}^2}(h_1 + 2h_2)$.

Moreover the zero–locus of a general section of $\mathcal{E}$ is respectively a line, or an integral, smooth, rational quartic curve. Each line can be obtained in this way.

We conclude the section by dealing with initialized, indecomposable, aCM bundles $\mathcal{G}$ of rank 2 on Φ. As usual let $\gamma_1 = \alpha_1\eta_1 + \alpha_2\eta_2$ and $\gamma_2 = \mu_1\eta_2^2 + \mu_2\eta_1^2 + \mu_3\eta_1\eta_2$. As in the previous section we know $\mathcal{E} = \mathcal{G} \otimes \mathcal{O}_F$ has second Chern class $c_2 = (\mu_1 + \mu_3)h_2^2 + (\mu_2 + \mu_3)h_1^2$. Thus Remark 2.8, Theorem 6.3 and Inequalities (20) imply that we have to deal with the existence of the following cases

Case	(α_1, α_2)	(μ_1, μ_2, μ_3)
L	$(0, 1)$	$(1, 0, 0)$
Q	$(2, 1)$	$(1, 1, 1)$
Q′	$(2, 1)$	$(2, 2, 0)$
Q″	$(2, 1)$	$(0, 0, 2)$

As for F, if $\mathcal{G}$ is initialized, indecomposable aCM bundle, then the same holds true for $\mathcal{G}^\vee(\eta)$. The computation of $c_2(\mathcal{G}^\vee(\eta))$ shows that cases Q′ and Q″ cannot occur, hence only the cases L and Q are admissible.

Example 6.4. Consider the indecomposable bundle $\mathcal{G} := \pi_2^*\Omega^1_{\mathbb{P}^2}(2\eta_2)$. Using Künneth formulas it is easy to check that $\mathcal{G}$ is initialized and aCM. Moreover its Chern classes are $\gamma_1 = \eta_2$ and $\gamma_2 = \eta_2^2$. Thus case L on Φ occurs.

Notice that $\mathcal{G}^\vee(\eta) \cong \pi_2^*\Omega^1_{\mathbb{P}^2}(\eta_1 + 2\eta_2)$. Argueing as in Example 6.1 one easily checks that the zero locus Σ of its general section is an integral, smooth surface. Thanks to Remark 2.8 one notices that its general hyperplane section is the zero–locus of a general section of the bundle $p_2^*\Omega^1_{\mathbb{P}^2}(h_1 + 2h_2)$ on F, which is a rational curve (see Example 6.1). We conclude that the sectional genus of Σ is 0

The discussion above and Theorem 2.10 prove the first part of the statement below. The claimed existence of the bundles and the fact that planes can be all obtained via such a construction can be proved with the same argument used for the threefold F.

Theorem 6.5. *Let $\mathcal{G}$ be an indecomposable, initialized, aCM bundle of rank 2 on Φ and let $c_1 = \alpha_1\eta_1 + \alpha_2\eta_2$. Assume that γ_1 is neither 0, nor 2η. Then (γ_1, γ_2) is either (η_2, η_2^2), or $(2\eta_1 + \eta_2, \eta_1^2 + \eta_2^2 + \eta_1\eta_2)$, up to permutations of the η_i's.*

Conversely, for each such a pair, there exists a unique indecomposable, initialized, aCM bundle $\mathcal{G}$ of rank 2 on Φ with Chern classes γ_1 and γ_2. If $\gamma_1 = \eta_2$, then $\mathcal{G} \cong \pi_2^*\Omega^1_{\mathbb{P}^2}(2\eta_2)$. If $\gamma_1 = \eta_1 + 2\eta_2$, then $\mathcal{G} \cong \pi_2^*\Omega^1_{\mathbb{P}^2}(\eta_1 + 2\eta_2)$.

Moreover the zero–locus of a general section of $\mathcal{G}$ is respectively a plane, or an integral, smooth quartic surface with sectional genus 0. Each plane on F can be obtained in this way.

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