A REFINED STABLE RESTRICTION THEOREM FOR VECTOR BUNDLES ON QUADRIC THREEFOLDS

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ABSTRACT. Let E be a stable rank 2 vector bundle on a smooth quadric threefold Q in the projective 4-space P. We show that the hyperplanes H in P for which the restriction of E to the hyperplane section of Q by H is not stable form, in general, a closed subset of codimension at least 2 of the dual projective 4-space, and we explicitly describe the bundles E which do not enjoy this property. This refines a restriction theorem of Ein and Sols [*Nagoya Math. J.* 96, 11–22 (1984)] in the same way the main result of Coandă [*J. reine angew. Math.* 428, 97–110 (1992)] refines the restriction theorem of Barth [*Math. Ann.* 226, 125–150 (1977)].

1. Introduction

Let E be a stable rank 2 vector bundle on the quadric threefold $Q = Q_3 \subset \mathbb{P}^4$, with $c_1(E) = 0$ or -1 and with $c_2(E) = c_2[L]$, where $[L]$ is the cohomology class of a line $L \subset Q$ and $c_2 \in \mathbb{Z}$. As it is well known, if $c_1 = 0$ then $c_2 \geq 2$ and it is even, and if $c_1 = -1$ then $c_2 \geq 1$. Moreover, if $c_1 = -1$ and $c_2 = 1$ then E is isomorphic to the *spinor bundle* $\mathcal{S}$. Examples of such bundles can be obtained, for $c_1 = 0$, by considering extensions:

$$0 \longrightarrow \mathcal{O}_Q(-1) \longrightarrow E \longrightarrow \mathcal{I}_Y(1) \longrightarrow 0 \tag{1.1}$$

where Y is a disjoint union of $c_2/2 + 1$ conics contained in Q_3, and for $c_1 = -1$ by considering extensions:

$$0 \longrightarrow \mathcal{O}_Q(-1) \longrightarrow E \longrightarrow \mathcal{I}_Y \longrightarrow 0 \tag{1.2}$$

where Y is a disjoint union of c_2 lines. For $c_1 = 0$ or -1 and $c_2 = 2$, any stable rank 2 vector bundle E on Q can be obtained in this way (see [SSW91, Cor. of Prop. 1] and [OS94, Prop. 4.4]). Alternatively, for $c_1 = 0$ and $c_2 = 2$, any such bundle E is the pull-back of a null-correlation bundle on $\mathbb{P}^3$ by a linear projection $Q \to \mathbb{P}^3$ with centre a point of $\mathbb{P}^4 \setminus Q$. Let $\mathbb{P}^{4\vee}$ be the dual projective space parametrizing the hyperplanes in $\mathbb{P}^4$ and let $Q^\vee \subset \mathbb{P}^{4\vee}$ be the dual quadric parametrizing the tangent hyperplanes to Q_3. For $h \in \mathbb{P}^{4\vee}$, let us denote by H the corresponding hyperplane of $\mathbb{P}^4$. Ein and Sols [ES84, Thm. 1.6] showed that, for a general $h \in \mathbb{P}^{4\vee} \setminus Q^\vee$, the restriction $E|_{H \cap Q}$ is stable on $H \cap Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$. However, *they missed an exception*, namely the spinor bundle $\mathcal{S}$. Actually, the two authors, guided (probably) by the case of $\mathbb{P}^3$ where the exceptions appear for $c_1 = 0$, worked out the details for the case $c_1 = 0$ of their result and left the (similar) details for the case $c_1 = -1$

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to the reader. We shall provide, in Remark 5.9, an extra argument showing that the spinor bundle is, in fact, the only exception.

In this paper we prove the following refinement of the result of Ein and Sols (analogous to the refinement of the restriction theorem of Barth [Ba77] from [Co92]):

Theorem 1.1. *Let E be as above and let $\Sigma \subset \mathbb{P}^{4\vee}$, $\Sigma \neq Q^\vee$, be an irreducible hypersurface. If, for a general point $h \in \Sigma \setminus Q^\vee$, $E|_{H \cap Q}$ is not stable then either:*

(i) *$c_1(E) = 0$, Σ consists of the hyperplanes passing through a point $x \in \mathbb{P}^4 \setminus Q$, and E is the pull-back of a nullcorrelation bundle on $\mathbb{P}^3$ by the linear projection of centre x restricted to $Q \to \mathbb{P}^3$, or:*

(ii) *$c_1(E) = -1$, E can be realized as an extension (1.2) with all the components of Y contained in a smooth hyperplane section $H_0 \cap Q$ of Q and $\Sigma = (H_0 \cap Q)^\vee$.*

The proof follows the strategy from [Co92]. It is based on a variant of the so called Standard Construction which is explained in Section 2. The sections 3 and 4 contain some auxiliary results, needed in the proof of Thm. 1.1 which is given in Section 5. We close Section 5 with a remark pointing out an important simplification in the proof of the main result of [Co92].

Finally, we study, in Section 6, the restrictions of E (or, more generally, of a stable rank 2 reflexive sheaf) to the singular hyperplane sections of Q and prove, using the same method, the following:

Theorem 1.2. *Let $\mathcal{E}$ be a stable rank 2 reflexive sheaf on Q, with $c_1(\mathcal{E}) = 0$ or -1. Then, for a general point $y \in Q$ such that, in particular, the tangent hyperplane $\mathrm{T}_y Q$ contains no singular point of $\mathcal{E}$, the restriction of $\mathcal{E}$ to $Y := Q \cap \mathrm{T}_y Q$ is stable, unless $\mathcal{E}$ is isomorphic to the spinor bundle $\mathcal{S}$.*

One derives immediately, from the above two theorems, the following:

Corollary 1.3. *Let E be a stable rank 2 vector bundle on Q, with $c_1(E) = 0$ or -1. Then, except for the case where E is one of the bundles appearing in the conclusion of Theorem 1.1, the set of the hyperplanes $H \subset \mathbb{P}^4$ for which $E|_{H \cap Q}$ is not stable has codimension ≥ 2 in $\mathbb{P}^{4\vee}$.*

The main result of [Co92] has been used in the study of the moduli spaces of mathematical instanton bundles on $\mathbb{P}^3$: see, for example, the paper of Katsylo and Ottaviani [KO03]. We hope that the results of the present paper might have applications in the study of the moduli spaces of *odd instanton bundles* on $Q_3 \subset \mathbb{P}^4$, introduced recently by Faenzi [Fa11].

Notation and conventions. (i) We work only with quasi-projective schemes over the field $\mathbb{C}$ of complex numbers. By a *point* we always mean a *closed point*.

(ii) If E is a rank n vector bundle (= locally free sheaf) on a scheme X, we denote by $\mathbb{G}_r(E)$ (resp., $\mathbb{G}^r(E)$) the relative Grassmannian of rank r subbundles (resp., quotient bundles) of E. Of course, $\mathbb{G}_r(E) \simeq \mathbb{G}^{n-r}(E)$ and $\mathbb{G}_r(E) \simeq \mathbb{G}^r(E^*)$. We use the classical convention (dual to Grothendieck's convention) for projective bundles, namely $\mathbb{P}(E) := \mathbb{G}_1(E)$. We shall also use the notation: $\mathbb{G}_a(\mathbb{P}^n) := \mathbb{G}_{a+1}(\mathbb{C}^{n+1})$.

(iii) If $X \subset \mathbb{P}^n$ is a non-singular, connected projective variety we denote by $X^\vee$ its dual variety, i.e., the set of points h of the dual projective space $\mathbb{P}^{n\vee}$ with the property that the corresponding hyperplane $H \subset \mathbb{P}^n$ contains the tangent linear subspace $\mathrm{T}_x X$ to X at some point $x \in X$. In particular, if L (resp., x) is a linear subspace (resp., point) of X, then $L^\vee$ (resp., $x^\vee$) consists of the points $h \in \mathbb{P}^{n\vee}$ such that $H \supset L$ (resp., $H \ni x$). $L^\vee$ is a linear subspace of $\mathbb{P}^{n\vee}$.

(iv) When we say that a sheaf is "(semi)stable" we mean that it is (semi)stable in the sense of Mumford and Takemoto (or μ-(semi)stable, or slope (semi)stable) with respect to a polarization which should be obvious in each of the cases under consideration. In particular, if $Q_2 \subset \mathbb{P}^3$ is a nonsingular quadric surface then we use the polarization $\mathcal{O}_{Q_2}(1,1)$.

(v) If Y is a closed subscheme of a scheme X, we shall denote by $\mathcal{I}_{Y,X}$ the kernel of the canonical epimorphism $\mathcal{O}_X \to \mathcal{O}_Y$, i.e., the ideal sheaf of $\mathcal{O}_X$ defining Y as a closed subscheme of X.

(vi) If $f : X \to Y$ is a morphism of schemes, we shall denote, occasionally, the sheaf $\Omega_{X/Y}$ of relative Kähler differentials by Ω_f.

2. The Standard Construction

Definition 2.1. Let $p : X \to Y$ be a morphism of schemes, let $\mathcal{F}$ be a coherent sheaf on Y and let:

$$0 \longrightarrow \mathcal{F}' \longrightarrow p^*\mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0 \tag{2.1}$$

be an exact sequence of coherent sheaves on X. Consider the fibre product $X \times_Y X$, the projections $p_1, p_2 : X \times_Y X \to X$ and let $\Delta = \Delta_{X/Y} \subset X \times_Y X$ be the image of the diagonal embedding $\delta = \delta_{X/Y} : X \to X \times_Y X$. The composite morphism:

$$p_1^*\mathcal{F}' \longrightarrow p_1^*p^*\mathcal{F} = p_2^*p^*\mathcal{F} \longrightarrow p_2^*\mathcal{F}'' \tag{2.2}$$

vanishes along Δ, hence the composite morphism:

$$p_1^*\mathcal{F}' \longrightarrow p_2^*\mathcal{F}'' \longrightarrow p_2^*\mathcal{F}''|_\Delta$$

is 0. But the sequence:

$$0 \longrightarrow \mathcal{I}_\Delta \otimes p_2^*\mathcal{F}'' \longrightarrow p_2^*\mathcal{F}'' \longrightarrow p_2^*\mathcal{F}''|_\Delta \longrightarrow 0$$

is exact (also to the left). [*Indeed*, we may assume that $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$ and $\mathcal{F} = \widetilde{M}$. The sequence:

$$0 \longrightarrow I \longrightarrow B \otimes_A B \longrightarrow B \longrightarrow 0$$

is a split exact sequence of right B-modules and if N is a $B \otimes_A B$-module then

$$N \otimes_{B \otimes_A B} (B \otimes_A M) \simeq N \otimes_B M,$$

hence the sequence:

$$0 \longrightarrow I \otimes_{B \otimes_A B} (B \otimes_A M) \longrightarrow B \otimes_A M \longrightarrow B \otimes_{B \otimes_A B} (B \otimes_A M) \longrightarrow 0$$

is exact.] One deduces that the morphism (2.2) induces a morphism $p_1^*\mathcal{F}' \to \mathcal{I}_\Delta \otimes p_2^*\mathcal{F}''$ which restricted to Δ gives us a morphism $\mathcal{F}' \to \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{F}''$. This morphism is called the *second fundamental form* of the exact sequence (2.1).

Lemma 2.1. *Let $p : X \to Y$ be a morphism of schemes, let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module and let $0 \to \mathcal{E}' \to p^*\mathcal{E} \to \mathcal{E}'' \to 0$ be an exact sequence of locally free sheaves on X. Let r be the rank of $\mathcal{E}''$, let $\pi : \mathbb{G} = \mathbb{G}^r(\mathcal{E}) \to Y$ be the relative Grassmannian of rank r quotients of $\mathcal{E}$ and let $f : X \to \mathbb{G}$ be the Y-morphism corresponding to the epimorphism $p^*\mathcal{E} \to \mathcal{E}''$. Then the composite morphism*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}'', \mathcal{E}') \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}'', \Omega_{X/Y} \otimes \mathcal{E}'') \xrightarrow{\mathrm{Tr}} \Omega_{X/Y}$$

deduced from the second fundamental form of the above exact sequence can be identified with the relative differential $\mathrm{d}f : f^*\Omega_{\mathbb{G}/Y} \to \Omega_{X/Y}$ of f.

Proof. We follow the argument from the proof of [HL97, Cor. 2.2.10]. Let $\mathcal{B}$ be the universal quotient of $\pi^*\mathcal{E}$ and let $0 \to \mathcal{A} \to \pi^*\mathcal{E} \to \mathcal{B} \to 0$ be the tautological exact sequence on $\mathbb{G}$. Let $q_1, q_2 : \mathbb{G} \times_Y \mathbb{G} \to \mathbb{G}$ be the canonical projections and let $\mathcal{J} \subset \mathcal{O}_{\mathbb{G} \times_Y \mathbb{G}}$ be the ideal sheaf of the diagonal $\Delta_{\mathbb{G}/Y} \subset \mathbb{G} \times_Y \mathbb{G}$. Consider the morphism $q_1^*\mathcal{A} \to q_2^*\mathcal{B}$ analogous to the morphism (2.2) from Def. 2.1. Using the universal property of the relative Grassmannian, one sees easily that, for every Y-scheme Z, an Y-morphism $\varphi : Z \to \mathbb{G} \times_Y \mathbb{G}$ factors through $\Delta_{\mathbb{G}/Y}$ if and only if the morphism $\varphi^* q_1^*\mathcal{A} \to \varphi^* q_2^*\mathcal{B}$ is 0. One deduces that $\Delta_{\mathbb{G}/Y}$ is the zero scheme of the morphism $q_1^*\mathcal{A} \to q_2^*\mathcal{B}$, which means that the image of the composite morphism:

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{G} \times_Y \mathbb{G}}}(q_2^*\mathcal{B}, q_1^*\mathcal{A}) \longrightarrow \mathcal{H}om_{\mathcal{O}_{\mathbb{G} \times_Y \mathbb{G}}}(q_2^*\mathcal{B}, q_2^*\mathcal{B}) \xrightarrow{\mathrm{Tr}} \mathcal{O}_{\mathbb{G} \times_Y \mathbb{G}}$$

is exactly $\mathcal{J}$. Restricting the epimorphism $\mathcal{H}om(q_2^*\mathcal{B}, q_1^*\mathcal{A}) \to \mathcal{J}$ to $\Delta_{\mathbb{G}/Y}$ one gets an epimorphism $\mathcal{H}om_{\mathcal{O}_\mathbb{G}}(\mathcal{B}, \mathcal{A}) \to \mathcal{J}/\mathcal{J}^2 = \Omega_{\mathbb{G}/Y}$, which must be an isomorphism because its source and its target are locally free sheaves of the same rank.

Now, one has only to recall that the relative differential $\mathrm{d}f : f^*\Omega_{\mathbb{G}/Y} \to \Omega_{X/Y}$ can be identified with the morphism $(f \times_Y f)^*(\mathcal{J}/\mathcal{J}^2) \to \mathcal{I}_\Delta/\mathcal{I}_\Delta^2$. □

Definition 2.2. Let $\mathcal{E}$ be a locally free sheaf on a nonsingular, connected variety X. A coherent subsheaf $\mathcal{E}'$ of $\mathcal{E}$ is called *saturated* if the quotient $\mathcal{E}/\mathcal{E}'$ is torsion-free. This is equivalent to the fact that $\mathcal{E}'$ is reflexive and $\mathcal{E}/\mathcal{E}'$ is locally free outside a closed subset of X of codimension ≥ 2. See, for example, [OSS80, II, Sect. 1.1] or [Ha80, Sect. 1].

Definition 2.3. We say that a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of coherent sheaves on a nonsingular, connected variety X is *generically zero* if there exists a non-empty open subset U of X such that $\varphi|_U = 0$.

Assume that such a morphism is not generically zero. There exists a non-empty open subset U of X such that $\mathcal{F}|_U$ and $\mathcal{G}|_U$ are locally free. Since $\varphi|_U \neq 0$, it follows that there exists a non-empty open subset U' of U such that $\varphi(x) : \mathcal{F}(x) \to \mathcal{G}(x)$ is non-zero, $\forall x \in U'$ (here $\mathcal{F}(x) := \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$). We say, in this case, that φ *has generically rank* ≥ 1.

Proposition 2.2. *Let $p : X \to Y$ be a dominant morphism of nonsingular, connected varieties, with all the non-empty fibres of dimension $\dim X - \dim Y$, and with irreducible general fibres. Let $\mathcal{E}$ be a locally free sheaf on Y, let $\mathcal{E}'$ be a saturated subsheaf of $p^*\mathcal{E}$ and let $\mathcal{E}'' := p^*\mathcal{E}/\mathcal{E}'$.*

If the second fundamental form $\mathcal{E}' \to \Omega_{X/Y} \otimes \mathcal{E}''$ associated to the short exact sequence $0 \to \mathcal{E}' \to p^*\mathcal{E} \to \mathcal{E}'' \to 0$ is generically zero then there exists a saturated subsheaf $\overline{\mathcal{E}}'$ of $\mathcal{E}$ such that $\mathcal{E}' = p^*\overline{\mathcal{E}}'$.

Proof. We follow the classical approach, originating in Van de Ven [VdV72], Grauert and Mülich [GM75] and Barth [Ba77] and clearly explained in Forster et al. [FHS80]. For another approach, see the proof of Prop. 1.11 in Flenner [Fl84].

Let Z be a closed subset of X, of codimension ≥ 2, such that $\mathcal{E}''|_{X\setminus Z}$ is locally free. Let r be the rank of $\mathcal{E}''$, let $\pi : \mathbb{G}^r(\mathcal{E}) \to Y$ be the relative Grassmannian of rank r quotients of $\mathcal{E}$, and let $f : X \setminus Z \to \mathbb{G}^r(\mathcal{E})$ be the Y-morphism defined by the epimorphism $p^*\mathcal{E}|_{X\setminus Z} \to \mathcal{E}''|_{X\setminus Z}$. From the hypothesis and from Lemma 2.1 it follows that the relative differential $\mathrm{d}f : f^*\Omega_{\mathbb{G}/Y} \to \Omega_{X/Y}|_{X\setminus Z}$ is *generically zero*.

Now, for a general point $y \in Y$, the fibre X_y is smooth (by generic smoothness), irreducible (by hypothesis), with $\mathrm{codim}(Z_y, X_y) \geq 2$, and with $\mathrm{d}f|_{X_y\setminus Z_y} = 0$ hence $f(X_y \setminus Z_y)$ consists of a single point. Let Y' be the closure of $f(X \setminus Z)$ in $\mathbb{G}^r(\mathcal{E})$. Since $f(X \setminus Z)$ contains an open dense subset of Y', one shows easily (using the well-known results about the dimension of the fibres of a morphism) that, for a general point $y \in Y$, the fibre Y'_y is the closure of $f(X_y \setminus Z_y)$, hence it consists of a single point. One deduces that $\pi|_{Y'} : Y' \to Y$ is *birational.* Now, Zariski's Main Theorem (in the elementary variant from Mumford [Mu99, III, § 9, Prop. 1]) implies that there exists an open subset $V = Y \setminus Z'$ of Y, with $\mathrm{codim}(Z', Y) \geq 2$, such that the restriction of $\pi : \pi^{-1}(V) \cap Y' \to V$ is an isomorphism. Let $\sigma : V \to \mathbb{G}^r(\mathcal{E})$ be the section over V of $\pi : \mathbb{G}^r(\mathcal{E}) \to Y$ defined by the inverse of this isomorphism. One has $f = \sigma \circ p$ over $X \setminus (Z \cup p^{-1}(Z'))$.

Let $\mathcal{A}$ be the kernel of the universal quotient $\pi^*\mathcal{E} \to \mathcal{B}$. $\sigma^*\mathcal{A}$ (which is a vector subbundle of $\mathcal{E}|_V$) can be extended to a saturated subsheaf $\overline{\mathcal{E}}'$ of $\mathcal{E}$. From the hypothesis on the dimensions of the fibres of p it follows, on one hand, that $\mathrm{codim}(p^{-1}(Z'), X) \geq 2$ and, on the other hand, that p is *flat* (see Hartshorne [Ha77, III, Ex. 10.9]). One deduces that $p^*\overline{\mathcal{E}}'$ is a saturated subsheaf of $p^*\mathcal{E}$. Finally, since $\mathcal{E}'$ and $p^*\overline{\mathcal{E}}'$ coincide over $X \setminus (Z \cup p^{-1}(Z'))$ they must coincide over X. □

Lemma 2.3. *Let $\mathbb{G}_d(\mathbb{P}^n)$ be the Grassmannian of d-dimensional linear subspaces of $\mathbb{P}^n$, $1 \leq d < n$, and consider the incidence diagram:*

$$\begin{array}{ccc} \mathbb{F}_{0,d}(\mathbb{P}^n) & \xrightarrow{\ q\ } & \mathbb{G}_d(\mathbb{P}^n) \\ {\scriptstyle p}\big\downarrow & & \\ \mathbb{P}^n & & \end{array}$$

For a point $\ell \in \mathbb{G}_d(\mathbb{P}^n)$, p maps isomorphically the fibre $q^{-1}(\ell)$ onto the corresponding linear subspace L of $\mathbb{P}^n$. Then $\Omega_{\mathbb{F}/\mathbb{P}}|_{q^{-1}(\ell)} \simeq \mathrm{T}_L(-1)^{\oplus\, n-d}$.

Proof. Consider the tautological exact sequences on $\mathbb{P}^n$ and $\mathbb{G}_d(\mathbb{P}^n)$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}}^{\oplus\, n+1} \longrightarrow \mathrm{T}_{\mathbb{P}}(-1) \longrightarrow 0$$
$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{O}_{\mathbb{G}}^{\oplus\, n+1} \longrightarrow \mathcal{B} \longrightarrow 0$$

(with $\mathrm{rk}\,\mathcal{A} = d+1$). The composite morphism $p^*\mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{F}}^{\oplus\, n+1} \to q^*\mathcal{B}$ is 0, hence it induces an epimorphism $p^*\mathrm{T}_{\mathbb{P}}(-1) \to q^*\mathcal{B}$. One can easily show that

$$\mathbb{F}_{0,d}(\mathbb{P}^n) \simeq \mathbb{G}^{n-d}(\mathrm{T}_{\mathbb{P}}(-1)) \text{ over } \mathbb{P}^n$$

such that the universal quotient of $p^*\mathrm{T}_{\mathbb{P}}(-1)$ corresponds to $q^*\mathcal{B}$. Let $\mathcal{A}'$ be the kernel of the epimorphism $p^*\mathrm{T}_{\mathbb{P}}(-1) \to q^*\mathcal{B}$. Restricting to $q^{-1}(\ell)$ the exact sequence:

$$0 \longrightarrow p^*\mathcal{O}_{\mathbb{P}}(-1) \longrightarrow q^*\mathcal{A} \longrightarrow \mathcal{A}' \longrightarrow 0$$

one gets an exact sequence $0 \to \mathcal{O}_L(-1) \to \mathcal{O}_L^{\oplus\, d+1} \to \mathcal{A}'_L \to 0$ from which one deduces that $\mathcal{A}'_L \simeq \mathrm{T}_L(-1)$. Now, from Lemma 2.1, $\Omega_{\mathbb{F}/\mathbb{P}} \simeq \mathcal{H}om_{\mathcal{O}_{\mathbb{F}}}(q^*\mathcal{B}, \mathcal{A}')$ hence

$$\Omega_{\mathbb{F}/\mathbb{P}}|_{q^{-1}(\ell)} \simeq \mathcal{H}om_{\mathcal{O}_L}(\mathcal{O}_L^{\oplus\, n-d}, \mathcal{A}'_L) \simeq \mathrm{T}_L(-1)^{\oplus\, n-d}. \qquad \square$$

Let us illustrate the way the Standard Construction method works by an easy example, of Grauert-Mülich type (cf. [ES84, Cor. 1.5]).

Proposition 2.4. *Let $\mathcal{E}$ be a semistable rank 2 reflexive sheaf on a smooth quadric hypersurface $Q = Q_{n-1} \subset \mathbb{P}^n$, $n \geq 3$, with $\det \mathcal{E} \simeq \mathcal{O}_Q(c_1)$, $c_1 \in \mathbb{Z}$. Then, for a general smooth conic $C \subset Q$ (avoiding, in particular, the singular points of $\mathcal{E}$), $\mathcal{E}|_C \simeq \mathcal{O}_{\mathbb{P}^1}(c_1)^{\oplus 2}$.*

Proof. We may assume that $c_1 = 0$ or -1. Consider the incidence diagram from the statement of Lemma 2.3 for $d = 2$. Let $X := p^{-1}(Q)$ and consider the induced diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ \overline{q}\ } & \mathbb{G}_2(\mathbb{P}^n) \\ {\scriptstyle \overline{p}}\big\downarrow & & \\ Q & & \end{array}$$

Let U be the open subset of $\mathbb{G} := \mathbb{G}_2(\mathbb{P}^n)$ consisting of the points ℓ for which the corresponding 2-plane L intersects Q transversally, along a conic C avoiding the singular points of $\mathcal{E}$. $\overline{p}$ maps $\overline{q}^{\,-1}(\ell)$ isomorphically onto C. By semicontinuity, there exists an integer $a \geq 0$ and a non-empty open subset U' of U such that $\mathcal{E}|_C \simeq \mathcal{O}_{\mathbb{P}^1}(c_1+a) \oplus \mathcal{O}_{\mathbb{P}^1}(c_1-a)$, $\forall \ell \in U'$. We want to show that $a = 0$. If $a \geq 1$ then the image of the canonical morphism $\overline{q}^*\overline{q}_*\overline{p}^*\mathcal{E}|_{\overline{q}^{\,-1}(U')} \to \overline{p}^*\mathcal{E}|_{\overline{q}^{\,-1}(U')}$ is a line subbundle $\mathcal{E}'$ of $\overline{p}^*\mathcal{E}|_{\overline{q}^{\,-1}(U')}$ such that $\mathcal{E}'|_{\overline{q}^{\,-1}(\ell)} \simeq \mathcal{O}_{\mathbb{P}^1}(c_1+a)$, $\forall \ell \in U'$. Applying Prop. 2.2 to the restriction of $\overline{p} : \overline{q}^{\,-1}(U') \to Q \setminus \mathrm{Sing}\,\mathcal{E}$ and to the exact sequence;

$$0 \longrightarrow \mathcal{E}' \longrightarrow \overline{p}^*\mathcal{E}|_{\overline{q}^{\,-1}(U')} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

(where $\mathcal{E}''$ is the cokernel of the left morphism) and taking into account the semistability of $\mathcal{E}$ one deduces that the second fundamental form $\mathcal{E}' \to (\Omega_{X/Q}|_{\overline{q}^{\,-1}(U')}) \otimes \mathcal{E}''$ has generically rank ≥ 1, hence its restriction to a general fibre $\overline{q}^{\,-1}(\ell)$, $\ell \in U'$, must have generically rank ≥ 1. Using Lemma 2.3, one deduces the existence of a non-zero morphism:

$$\mathcal{O}_{\mathbb{P}^1}(c_1+a) \longrightarrow (\mathrm{T}_L(-1)^{\oplus\, n-2}|_C) \otimes \mathcal{O}_{\mathbb{P}^1}(c_1-a).$$

Since $\mathrm{T}_L(-1)|_C \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$, one derives a contradiction. $\square$

Lemma 2.5. *Consider the incidence diagram from the statement of Lemma 2.3 with $d = n-1$, i.e., with $\mathbb{G}_d(\mathbb{P}^n) = \mathbb{P}^{n\vee}$. Let $\Sigma \subset \mathbb{P}^{n\vee}$ be a closed reduced and irreducible subscheme, of dimension $m \geq 2$, let $X := q^{-1}(\Sigma)$ and consider the induced incidence diagram:*

$$\begin{array}{ccc} X & \xrightarrow{\overline{q}} & \Sigma \\ {\scriptstyle \overline{p}}\downarrow & & \\ \mathbb{P}^n & & \end{array}$$

(i) *Let h be a nonsingular point of Σ and let $\mathrm{T}_h\Sigma \subset \mathbb{P}^{n\vee}$ be the tangent linear space of Σ at h. Let $H \subset \mathbb{P}^n$ be the hyperplane corresponding to h. One has $\mathrm{T}_h\Sigma = L^\vee$ for some linear subspace L of H with* $\mathrm{codim}(L, H) = m$. *Then one has an exact sequence:*

$$0 \longrightarrow \mathscr{O}_H(-1) \longrightarrow \mathscr{O}_H^{\oplus m} \longrightarrow \Omega_{X/\mathbb{P}}|_{\overline{q}^{-1}(h)} \longrightarrow 0$$

where the left morphism is the dual of an epimorphism $\mathscr{O}_H^{\oplus m} \to \mathscr{I}_{L,H}(1)$.

(ii) *If $x \in \mathbb{P}^n$ then the fibre $\overline{p}^{-1}(x)$ has pure dimension $m-1$, except in the case where $\Sigma = K^\vee$ for some linear subspace K of $\mathbb{P}^n$ of codimension $m+1$ and $x \in K$.*

(iii) *The set of points $x \in \mathbb{P}^n$ for which the fibre $\overline{p}^{-1}(x)$ is not irreducible and generically reduced is a closed subset of $\mathbb{P}^n$, of codimension $\geq m-1$.*

Proof. If $x \in \mathbb{P}^n$ and if $x^\vee$ is the hyperplane of $\mathbb{P}^{n\vee}$ consisting of the points h for which $H \ni x$, then $\overline{q}$ maps isomorphically the fibre $\overline{p}^{-1}(x)$ onto the scheme $x^\vee \cap \Sigma$. (ii) is, now, clear and (iii) follows from a recent result of O. Benoist [Be11, Thm. 0.5(i)] (see Remark 2.6 below).

(i) Let $\mathbb{F} := \mathbb{F}_{0,n-1}(\mathbb{P}^n)$. One has an exact sequence:

$$\overline{q}^*(\mathscr{I}_\Sigma/\mathscr{I}_\Sigma^2) \longrightarrow \Omega_{\mathbb{F}/\mathbb{P}}|_X \longrightarrow \Omega_{X/\mathbb{P}} \longrightarrow 0\,.$$

Restricting this exact sequence to $\overline{q}^{-1}(h)$ and taking into account Lemma 2.3, one gets an exact sequence:

$$\mathscr{O}_H^{\oplus\, n-m} \longrightarrow \mathrm{T}_H(-1) \longrightarrow \Omega_{X/\mathbb{P}}|_{\overline{q}^{-1}(h)} \longrightarrow 0\,.$$

But, if $x \in H \setminus L$, $x^\vee$ intersects transversally Σ at h. Identifying $\overline{q}^{-1}(h)$ with H via $\overline{p}$, one deduces that $\Omega_{X/\mathbb{P}}|_{\overline{q}^{-1}(h)}$ is locally free of rank $m-1$ over $H \setminus L$. One derives that the morphism $\mathscr{O}_H^{\oplus\, n-m} \to \mathrm{T}_H(-1)$ is a monomorphism. Using Euler's exact sequence on H:

$$0 \longrightarrow \mathscr{O}_H(-1) \longrightarrow \mathscr{O}_H^{\oplus\, n} \longrightarrow \mathrm{T}_H(-1) \longrightarrow 0$$

one gets an exact sequence as in the statement. The left morphism in this exact sequence degenerates only along L, hence it must be the dual of an epimorphism as in the statement. □

Remark 2.6. The result of O. Benoist quoted in the proof of Lemma 2.5 asserts that if Z is a reduced and irreducible closed subscheme of $\mathbb{P}^n$, of dimension $m \geq 2$, then the set of points $h \in \mathbb{P}^{n\vee}$ for which the scheme $H \cap Z$ is not generically reduced, irreducible, of dimension $m-1$ is a closed subset of $\mathbb{P}^{n\vee}$, of codimension $\geq m-1$. We shall, actually, use Lemma 2.5 only in the case where Σ is a hypersurface in $\mathbb{P}^{n\vee}$, hence we need the result of

Benoist only in the case where Z is a hypersurface in $\mathbb{P}^n$. In this particular case, the arguments used by Benoist become substantially simpler.

Indeed, if Z is a hypersurface of degree d, [Be11, Prop. 1.1] can be replaced by the following statement: let $\mathcal{E}$ be a locally free sheaf of rank n on a scheme T and let $\pi : \mathbb{P}(\mathcal{E}) \to T$ be the associated projective bundle. Let X be an effective relative Cartier divisor on $\mathbb{P}(\mathcal{E})/T$ such that, $\forall t \in T$, X_t is a hypersurface of degree d in $\mathbb{P}(\mathcal{E}(t))$. Then the set of the points $t \in T$ such that X_t is reduced and irreducible is an open subset of T.

The proof of this statement follows from the following easy fact: consider the polynomial ring $S := \mathbb{C}[X_0, \ldots, X_{n-1}]$. Then the set of the points $[f] \in \mathbb{P}(S_d)$ such that the polynomial f is irreducible is open because its complement is the union of the images of the morphisms $\mathbb{P}(S_e) \times \mathbb{P}(S_{d-e}) \to \mathbb{P}(S_d)$, $1 \leq e \leq d/2$.

Secondly, continuing to assume that Z is a hypersurface, one reduces the proof of the result of Benoist, as in [Be11, Prop. 2.5], to the case where Z is the cone over a reduced and irreducible plane curve, with vertex a linear subspace L of $\mathbb{P}^n$ of dimension $n - 3$.

Finally, in the case where Z is a cone as above, the set from the statement of the result of Benoist consists of the hyperplanes containing L.

3. The spinor bundle and lines on a quadric threefold

3.1. **The variety of lines on a quadric threefold.** We identify $Q = Q_3 \subset \mathbb{P}^4$ with a smooth hyperplane section $\mathbb{P}^4 \cap \mathbb{G}_2(\mathbb{C}^4)$ of the Plücker embedding into $\mathbb{P}^5$ of the Grassmannian of lines in $\mathbb{P}^3$, $\mathbb{G}_1(\mathbb{P}^3) = \mathbb{G}_2(\mathbb{C}^4)$. More precisely, let $U := \mathbb{C}^4$ and consider the Plücker embedding $\mathbb{G}_2(U) \hookrightarrow \mathbb{P}(\bigwedge^2 U) = \mathbb{P}^5$. The image of this embedding is the quadric 4-fold of $\mathbb{P}^5$ of equation $w \wedge w = 0$, $w \in \bigwedge^2 U$.

The restriction to $\mathbb{G}_2(U)$ of the universal skew-symmetric morphism on $\mathbb{P}^5$:

$$U^* \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^5}(-1) \longrightarrow U^* \otimes_{\mathbb{C}} \bigwedge^2 U \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^5} \longrightarrow U \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^5}$$

is a morphism of constant rank 2, whose image is the universal subbundle $\mathcal{A}$ of $U \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{G}}$ and whose cokernel is the universal quotient bundle $\mathcal{B}$. One has an isomorphism:

$$U^* = \mathrm{Hom}_{\mathcal{O}_{\mathbb{G}}}(U \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{G}}, \mathcal{O}_{\mathbb{G}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{A}, \mathcal{O}_{\mathbb{G}})$$

and the image of the morphism $\mathcal{A} \to \mathcal{O}_{\mathbb{G}}$ corresponding to $0 \neq \lambda \in U^*$ is the ideal sheaf of the 2-plane $\mathbb{P}(\bigwedge^2 (\mathrm{Ker}\, \lambda)) \subset \mathbb{G}_2(U)$. Let $\mathbb{F}_{0,1}(\mathbb{P}(U)) = \mathbb{F}_{1,2}(U)$ be the flag variety "point $\in$ line $\subset \mathbb{P}^3$" and consider the incidence diagram:

$$\begin{array}{ccc} \mathbb{F}_{1,2}(U) & \xrightarrow{\widetilde{q}} & \mathbb{P}(U) \\ {\scriptstyle \widetilde{p}} \downarrow & & \\ \mathbb{G}_2(U) & & \end{array}$$

$\widetilde{p}$ maps isomorphically the fibre $\widetilde{q}^{-1}([u])$ onto $\mathbb{P}(u \wedge U) \subset \mathbb{G}_2(U)$. Considering the tautological coKoszul sequence and the Euler sequence on $\mathbb{P}(U)$:

$$0 \to \mathcal{O}_{\mathbb{P}} \to U \otimes \mathcal{O}_{\mathbb{P}}(1) \to \overset{2}{\bigwedge} U \otimes \mathcal{O}_{\mathbb{P}}(2) \to \overset{3}{\bigwedge} U \otimes \mathcal{O}_{\mathbb{P}}(3) \to \overset{4}{\bigwedge} U \otimes \mathcal{O}_{\mathbb{P}}(4) \to 0$$
$$0 \to \mathcal{O}_{\mathbb{P}}(-1) \to U \otimes \mathcal{O}_{\mathbb{P}} \to \mathrm{T}_{\mathbb{P}}(-1) \to 0$$

one sees that $\mathbb{F}_{1,2}(U) \simeq \mathbb{P}(\mathrm{T}_{\mathbb{P}}(-2))$ over $\mathbb{P}(U)$ such that $\mathcal{O}_{\mathbb{P}(\mathrm{T}(-2))}(-1) \simeq \widetilde{p}^* \mathcal{O}_{\mathbb{G}}(-1)$. Moreover, $\mathbb{F}_{1,2}(U) \simeq \mathbb{P}(\mathcal{A})$ over $\mathbb{G}_2(U)$ such that $\mathcal{O}_{\mathbb{P}(\mathcal{A})}(-1) \simeq \widetilde{q}^* \mathcal{O}_{\mathbb{P}(U)}(-1)$.

Now, a linear form on $\mathbb{P}^5 = \mathbb{P}(\overset{2}{\bigwedge} U)$ can be identified with a skew-symmetric form $\omega : \overset{2}{\bigwedge} U \to \mathbb{C}$. The hyperplane $\mathbb{P}^4 \simeq K_\omega := \mathbb{P}(\mathrm{Ker}\,\omega) \subset \mathbb{P}(\overset{2}{\bigwedge} U)$ intersects transversally $\mathbb{G}_2(U)$ if and only if ω is non-degenerate. Assume that this is the case and put $Q = Q_\omega := K_\omega \cap \mathbb{G}_2(U)$ and $\mathcal{S} = \mathcal{S}_\omega := \mathcal{A} \,|\, Q_\omega$. $\mathcal{S}_\omega$ is the so-called *spinor bundle* on Q_ω. Let $\mathbb{F}_{0,1}(Q_\omega) := \widetilde{p}^{-1}(Q_\omega)$ and consider the incidence diagram:

$$\begin{array}{ccc} \mathbb{F}_{0,1}(Q_\omega) & \xrightarrow{\ q\ } & \mathbb{P}(U) \\ {\scriptstyle p}\downarrow & & \\ Q_\omega & & \end{array}$$

Of course, $\mathbb{F}_{0,1}(Q_\omega) \simeq \mathbb{P}(\mathcal{S}_\omega)$ over Q_ω. On the other hand, p maps isomorphically the fibre $q^{-1}([u])$ onto the line $K_\omega \cap \mathbb{P}(u \wedge U) \subset Q_\omega$. One can associate to ω a so-called *null correlation bundle* N_ω over $\mathbb{P}^3 = \mathbb{P}(U)$ defined as the cokernel of the composite morphism:

$$\mathcal{O}_{\mathbb{P}(U)}(-1) \xrightarrow{\omega \otimes \mathrm{id}} \overset{2}{\bigwedge} U^* \otimes \mathcal{O}_{\mathbb{P}(U)}(-1) \longrightarrow \Omega_{\mathbb{P}(U)}(1)\ .$$

It follows that the kernel of the composite morphism:

$$\mathrm{T}_{\mathbb{P}(U)}(-2) \longrightarrow \overset{2}{\bigwedge} U \otimes \mathcal{O}_{\mathbb{P}(U)} \xrightarrow{\omega \otimes \mathrm{id}} \mathcal{O}_{\mathbb{P}(U)}$$

is $N_\omega^*(-1)$, hence $\mathbb{F}_{0,1}(Q_\omega) \simeq \mathbb{P}(N_\omega^*(-1))$ over $\mathbb{P}(U)$.

3.2. **The spinor bundle.** Keeping the notation from par. 3.1, the restriction to Q_ω of the isomorphism $\omega \otimes \mathrm{id} : U \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{G}} \xrightarrow{\sim} U^* \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{G}}$ induces an isomorphism $\mathcal{A}|_{Q_\omega} \xrightarrow{\sim} \mathcal{B}^*|_{Q_\omega}$ hence one gets an exact sequence:

$$0 \longrightarrow \mathcal{S}_\omega \longrightarrow U \otimes_{\mathbb{C}} \mathcal{O}_Q \longrightarrow \mathcal{S}_\omega^* \longrightarrow 0\ . \tag{3.1}$$

Moreover, since $\det \mathcal{A} \simeq \mathcal{O}_{\mathbb{G}}(-1)$ one has $\det \mathcal{S}_\omega \simeq \mathcal{O}_Q(-1)$ hence $\mathcal{S}_\omega^* \simeq \mathcal{S}_\omega(1)$.

The morphism $\mathcal{A} \to \mathcal{O}_{\mathbb{G}}$ defined by a non-zero $\lambda \in U^*$ restricts to a morphism $\mathcal{S}_\omega \to \mathcal{O}_Q$ whose image is the ideal sheaf of the line $L := K_\omega \cap \mathbb{P}(\overset{2}{\bigwedge}(\mathrm{Ker}\,\lambda)) \subset Q_\omega$. One derives an exact sequence:

$$0 \longrightarrow \mathcal{O}_Q(-1) \longrightarrow \mathcal{S}_\omega \longrightarrow \mathcal{I}_L \longrightarrow 0 \tag{3.2}$$

from which one can, of course, compute the cohomology of $\mathcal{S}_\omega$ and of its twists.

Finally, let H be a hyperplane in $K_\omega \simeq \mathbb{P}^4$ which intersects Q_ω transversally. In this case, $H \cap Q_\omega$ is a smooth quadric in $H \simeq \mathbb{P}^3$. Let L be a line belonging to the first ruling $|\,\mathcal{O}_{H\cap Q}(1,0)\,|$ of $H\cap Q_\omega$. From (3.2) one gets an epimorphism $\mathcal{S}_\omega|_{H\cap Q}\to \mathcal{O}_{H\cap Q}(-L)\simeq \mathcal{O}_{H\cap Q}(-1,0)$. Since $\det \mathcal{S}_\omega \simeq \mathcal{O}_Q(-1)$, one deduces an exact sequence:

$$0\longrightarrow \mathcal{O}_{H\cap Q}(0,-1)\longrightarrow \mathcal{S}_\omega|_{H\cap Q}\longrightarrow \mathcal{O}_{H\cap Q}(-1,0)\longrightarrow 0\ .$$

This exact sequence splits because $\mathrm{H}^1(\mathcal{O}_{H\cap Q}(1,-1))=0$. Consequently:

$$\mathcal{S}_\omega|_{H\cap Q}\simeq \mathcal{O}_{H\cap Q}(-1,0)\oplus \mathcal{O}_{H\cap Q}(0,-1)\ . \tag{3.3}$$

3.3. **Lines on hyperplane sections of a quadric threefold.** We described in par. 3.1 the family $q:\mathbb{F}_{0,1}(Q)\to \mathbb{P}(U)=\mathbb{P}^3$ of lines on $Q=Q_\omega\subset K_\omega=\mathbb{P}^4$. For a point $\ell=[u]\in\mathbb{P}(U)$, let us denote by $L=\mathbb{P}(u\wedge u^\perp)$ the corresponding line on Q, where $u^\perp\subset U$ is the orthogonal of u with respect to ω. We want to describe the flag variety $\mathbb{F}_{1,2}(Q)\subset \mathbb{P}(U)\times\mathbb{P}^{4\vee}$ consisting of the pairs (ℓ,h) with $L\subset H\cap Q$.

One can identify $\mathbb{P}^{4\vee}$ with $\mathbb{P}(\mathrm{H}^0(\mathcal{O}_Q(1)))$. We saw that $\mathbb{F}_{0,1}(Q)\simeq \mathbb{P}(N_\omega^*(-1))$ over $\mathbb{P}(U)$ such that $\mathcal{O}_{\mathbb{P}(N^*(-1))}(-1)\simeq p^*\mathcal{O}_Q(-1)$. One deduces that $q_*p^*\mathcal{O}_Q(1)\simeq N_\omega(1)$, hence $\mathrm{H}^0(\mathcal{O}_Q(1))\simeq \mathrm{H}^0(N_\omega(1))$. Let $0\neq h\in \mathrm{H}^0(\mathcal{O}_Q(1))$. It corresponds to a section $s\in \mathrm{H}^0(N_\omega(1))$. Let $\ell=[u]\in\mathbb{P}(U)$. $p:\mathbb{F}_{0,1}(Q)\to Q$ maps $q^{-1}(\ell)$ isomorphically onto the corresponding line $L\subset Q$. It follows that:

$$h \text{ vanishes on } L\ \Leftrightarrow\ p^*h \text{ vanishes on } q^{-1}(\ell)\ \Leftrightarrow\ s=q_*p^*h \text{ vanishes in } \ell\,.$$

Consequently, $\mathbb{F}_{1,2}(Q)\subset \mathbb{P}(U)\times\mathbb{P}^{4\vee}$ can be identified with $Z\subset \mathbb{P}(U)\times \mathbb{P}(\mathrm{H}^0(N_\omega(1)))$ consisting of the pairs $(\ell,[s])$ with $s(\ell)=0$. Let M_ω be defined by the exact sequence:

$$0\longrightarrow M_\omega\longrightarrow \mathrm{H}^0(N_\omega(1))\otimes_{\mathbb{C}}\mathcal{O}_{\mathbb{P}(U)}\xrightarrow{\ \mathrm{ev}\ } N_\omega(1)\longrightarrow 0\ .$$

Consider the two projections:

$$\begin{array}{ccc}\mathbb{F}_{1,2}(Q) & \xrightarrow{\ q_1\ } & \mathbb{P}^{4\vee}\\ {\scriptstyle p_1}\big\downarrow & & \\ \mathbb{P}(U) & & \end{array}$$

From the above discussion, $\mathbb{F}_{1,2}(Q)\simeq \mathbb{P}(M_\omega)$ over $\mathbb{P}(U)$ and if $h\in\mathbb{P}^{4\vee}$ corresponds to $[s]\in \mathbb{P}(\mathrm{H}^0(N_\omega(1)))$ then p_1 maps isomorphically $q_1^{-1}(h)$ onto the scheme of zeroes $Z(s)$ of s. As it is well-known, if $h\in\mathbb{P}^{4\vee}\setminus Q^\vee$ then $Z(s)$ is the union of two disjoint lines (which correspond to the two rulings of $H\cap Q\simeq \mathbb{P}^1\times\mathbb{P}^1$), and if $h\in Q^\vee$ then $Z(s)$ is a double structure on a line (in this case, $H\cap Q$ is a quadratic cone).

4. Some auxiliary results on the quadric surface

Let $Q_2\subset\mathbb{P}^3$ be a smooth quadric, $Q_2\simeq\mathbb{P}^1\times\mathbb{P}^1$, and let $p_1,p_2:Q_2\to\mathbb{P}^1$ be the projections. For $a,b\in\mathbb{Z}$, we put $\mathcal{O}_{Q_2}(a,b):=p_1^*\mathcal{O}_{\mathbb{P}^1}(a)\otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(b)$. Throughout this section, F will denote a rank 2 vector bundle on Q_2, with $\det F\simeq \mathcal{O}_{Q_2}(c_1,c_1)$, $c_1=0$ or -1, and $c_2(F)=c_2\in\mathbb{Z}$. We shall denote by $\overline{\mathcal{S}}$ the direct sum $\mathcal{O}_{Q_2}(-1,0)\oplus\mathcal{O}_{Q_2}(0,-1)$ (if one views Q_2 as a hyperplane section of a quadric threefold $Q_3\subset\mathbb{P}^4$ then, according to (3.3), $\overline{\mathcal{S}}\simeq \mathcal{S}|_{Q_2}$).

Definition 4.1. We say that a vector bundle F as above (i.e., $\mathrm{rk}(F) = 2$ and $\det F \simeq \mathcal{O}_{Q_2}(c_1, c_1)$, $c_1 = 0$ or -1) satisfies the *Grauert-Mülich property* if, for any general line L on each of the two rulings of Q_2, $F|_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L(c_1)$.

Lemma 4.1. *Assume that F satisfies the Grauert-Mülich property. Then:*

(i) *F is stable if and only if $\mathrm{H}^0(F) = 0$ in the case $c_1 = 0$, and if and only if $\mathrm{Hom}(\overline{\mathcal{S}}, F) = 0$ in the case $c_1 = -1$.*

(ii) *If F is semistable but not stable and $c_2 > -c_1$ then $\mathrm{h}^0(F) = 1$ in the case $c_1 = 0$, and $\mathrm{hom}(\overline{\mathcal{S}}, F) = 1$ in the case $c_1 = -1$.*

Proof. Assume that F contains a saturated subsheaf (see Def. 2.2) of the form $\mathcal{O}_{Q_2}(a, b)$. Then one has an exact sequence:

$$0 \longrightarrow \mathcal{O}_{Q_2}(a, b) \longrightarrow F \longrightarrow \mathcal{I}_Z(c_1 - a, c_1 - b) \longrightarrow 0$$

where Z is a 0-dimensional subscheme of Q_2. Restricting this exact sequence to a general line, avoiding Z, on each of the two rulings of Q_2 and using the Grauert-Mülich property, one derives that $a \leq 0$ and $b \leq 0$. By definition, F is (semi)stable if and only if, for each saturated subsheaf of F of the form $\mathcal{O}_{Q_2}(a, b)$, one has $a + b$ $(\leq) < c_1$. If $a \leq 0$ and $b \leq 0$ and $a + b \geq c_1$ then $(a, b) = (0, 0)$ in the case $c_1 = 0$, and $(a, b) = (-1, 0)$ or $(0, -1)$ in the case $c_1 = -1$.

(i) is, now, clear.

(ii) If F is semistable but not stable then, in the case $c_1 = 0$, it can be realized as an extension:

$$0 \longrightarrow \mathcal{O}_{Q_2} \longrightarrow F \longrightarrow \mathcal{I}_Z \longrightarrow 0$$

with $\deg Z = c_2$, and, in the case $c_1 = -1$, it can be realized as an extension of one of the following two types:

$$0 \longrightarrow \mathcal{O}_{Q_2}(-1, 0) \longrightarrow F \longrightarrow \mathcal{I}_Z(0, -1) \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}_{Q_2}(0, -1) \longrightarrow F \longrightarrow \mathcal{I}_Z(-1, 0) \longrightarrow 0$$

with $\deg Z = c_2 - 1$. The condition $c_2 > -c_1$ implies that $Z \neq \emptyset$, and assertion (ii) follows. $\square$

Lemma 4.2. *Assume that F is semistable and let $L \subset Q_2$ be a line. Assume, to fix the ideas, that L belongs to the linear system $|\,\mathcal{O}_{Q_2}(1, 0)\,|$. Let a be the nonnegative integer such that $F|_L \simeq \mathcal{O}_L(a) \oplus \mathcal{O}_L(-a + c_1)$.*

Then $a \leq c_2 + c_1$ and if $a = c_2 + c_1$ then there exists a 0-dimensional subscheme Z of L such that F can be realized as an extension:

$$0 \longrightarrow \mathcal{O}_{Q_2} \longrightarrow F \longrightarrow \mathcal{I}_Z \longrightarrow 0$$

in the case $c_1 = 0$, and as an extension:

$$0 \longrightarrow \mathcal{O}_{Q_2}(-1, 0) \longrightarrow F \longrightarrow \mathcal{I}_Z(0, -1) \longrightarrow 0$$

in the case $c_1 = -1$. In particular, F is not stable in this case.

Proof. Assume, firstly, that $c_1 = 0$. In this case, $\mathrm{h}^0(F) \leq 1$, unless $F \simeq \mathcal{O}_{Q_2}^{\oplus 2}$. *Indeed*, since F is semistable, $\mathrm{H}^0(F(b, c)) = 0$ if $b + c < 0$. It follows that the zero scheme Z of a non-zero global section s of F has no divisorial components, i.e., it is 0-dimensional, hence F can be realized as an extension:

$$0 \longrightarrow \mathcal{O}_{Q_2} \xrightarrow{\ s\ } F \longrightarrow \mathcal{I}_Z \longrightarrow 0\,. \tag{4.1}$$

In particular, $\deg Z = c_2$. One derives that $\mathrm{h}^0(F) \leq 1$ unless $Z = \emptyset$ in which case $F \simeq \mathcal{O}_{Q_2}^{\oplus 2}$.

Assume, now, that F is not trivial. Tensorizing by F the short exact sequence:

$$0 \longrightarrow \mathcal{O}_{Q_2}(-1,0) \longrightarrow \mathcal{O}_{Q_2} \longrightarrow \mathcal{O}_L \longrightarrow 0 \tag{4.2}$$

one gets an exact sequence:

$$0 = \mathrm{H}^0(F(-1,0)) \longrightarrow \mathrm{H}^0(F) \longrightarrow \mathrm{H}^0(F|_L) \longrightarrow \mathrm{H}^1(F(-1,0))$$

from which one deduces that:

$$a + 1 = \mathrm{h}^0(F|_L) \leq \mathrm{h}^0(F) + \mathrm{h}^1(F(-1,0))\,.$$

But, by semistability, $\mathrm{H}^0(F(-1,0)) = 0$ and

$$\mathrm{H}^2(F(-1,0)) \simeq \mathrm{H}^0(F^*(-1,-2))^* \simeq \mathrm{H}^0(F(-1,-2))^* = 0\,,$$

hence $\mathrm{h}^1(F(-1,0)) = -\chi(F(-1,0)) = c_2$ (one may use the exact sequence (4.1) tensorized by $\mathcal{O}_{Q_2}(-1,0)$ to guess the Riemann-Roch formula in this case). One deduces that $a + 1 \leq 1 + c_2$, hence $a \leq c_2$. If $a = c_2$, one must have $\mathrm{h}^0(F) = 1$, and F can be realized as an extension (4.1). One has $s|_L \in \mathrm{H}^0(F|_L) = \mathrm{H}^0(\mathcal{O}_L(c_2) \oplus \mathcal{O}_L(-c_2))$, hence $\deg(Z \cap L) = c_2$. Since $\deg Z = c_2$, it follows that $Z \subset L$ as schemes.

Consider, now, the case $c_1 = -1$. F being semistable, $\mathrm{H}^0(F(b,c)) = 0$ if $b + c \leq 0$. We assert, firstly, that $\mathrm{h}^0(F(1,0)) \leq 1$. *Indeed*, as in the case $c_1 = 0$, if $F(1,0)$ has a non-zero global section s, then the zero scheme Z of s is 0-dimensional and F can be realized as an extension:

$$0 \longrightarrow \mathcal{O}_{Q_2}(-1,0) \xrightarrow{\ s\ } F \longrightarrow \mathcal{I}_Z(0,-1) \longrightarrow 0\,. \tag{4.3}$$

In particular, $\deg Z = c_2 - 1$. One derives that $\mathrm{h}^0(F(1,0)) \leq 1$.

Now, tensorizing by $F(1,0)$ the exact sequence (4.2), one gets an exact sequence:

$$0 = \mathrm{H}^0(F) \longrightarrow \mathrm{H}^0(F(1,0)) \longrightarrow \mathrm{H}^0(F(1,0)|_L) \longrightarrow \mathrm{H}^1(F)\,.$$

Since $\mathcal{O}_{Q_2}(1,0)|_L \simeq \mathcal{O}_L$, $F(1,0)|_L \simeq F|_L$. One deduces that:

$$a + 1 = \mathrm{h}^0(F|_L) = \mathrm{h}^0(F(1,0)|_L) \leq \mathrm{h}^0(F(1,0)) + \mathrm{h}^1(F)\,.$$

But, by semistability, $\mathrm{H}^0(F) = 0$ and $\mathrm{H}^2(F) \simeq \mathrm{H}^0(F^*(-2,-2))^* \simeq \mathrm{H}^0(F(-1,-1))^* = 0$, hence $\mathrm{h}^1(F) = -\chi(F) = c_2 - 1$. One deduces that $a+1 \leq 1+(c_2-1)$, hence $a \leq c_2 - 1$. If $a = c_2 - 1$ then $\mathrm{h}^0(F(1,0)) = 1$ and F can be realized as an extension (4.3). $s|_L \in \mathrm{H}^0(F(1,0)|_L) = \mathrm{H}^0(F|_L) = \mathrm{H}^0(\mathcal{O}_L(c_2 - 1) \oplus \mathcal{O}_L(-c_2))$ hence $\deg(Z \cap L) = c_2 - 1$. Since $\deg Z = c_2 - 1$, it follows that $Z \subset L$ as schemes. □

Corollary 4.3. *Assume that F is semistable and that $c_2 > -c_1$. If there is a line $L \subset Q_2$ such that $F|_L \simeq \mathcal{O}_L(c_2 + c_1) \oplus \mathcal{O}_L(-c_2)$ then, for any other line $L' \subset Q_2$, one has $F|_{L'} \simeq \mathcal{O}_{L'}(a') \oplus \mathcal{O}_{L'}(-a' + c_1)$ with $a' = 0$ or 1.*

Proof. If L' and L are on the same ruling of Q_2 then $L' \cap Z = \emptyset$ and if they are on different rulings then $\deg(L' \cap Z) \leq 1$. If s is the global section of F (resp., $F(1,0)$) defining the exact sequence from the last part of the statement of Lemma 4.2 then the zero scheme of $s|_{L'}$ is $L' \cap Z$. □

Corollary 4.4. *Assume that F is semistable. Then there exists a line $L \subset Q_2$ such that $F|_L \simeq \mathcal{O}_L(c_2 + c_1) \oplus \mathcal{O}_L(-c_2)$ if and only if $\mathrm{H}^0(F) \neq 0$ and $\hom(\overline{\mathcal{S}}, F) \geq 5$ in the case $c_1 = 0$, and if and only if $\mathrm{Hom}(\overline{\mathcal{S}}, F) \neq 0$ and $\mathrm{h}^0(F(1,1)) \geq 3$ in the case $c_1 = -1$.*

Remark 4.5. Assume that F is semistable. One can show that, in the case $c_1 = 0$, the condition $\hom(\overline{\mathcal{S}}, F) \geq 5$ implies that $\mathrm{H}^0(F) \neq 0$ and, in the case $c_1 = -1$ and $c_2 > 2$, the condition $\mathrm{h}^0(F(1,1)) \geq 3$ implies that $\mathrm{Hom}(\overline{\mathcal{S}}, F) \neq 0$.

On the other hand, let $\mathcal{F}$ be the rank 2 reflexive sheaf on $\mathbb{P}^3$ considered in the statement of Lemma 5.7 below. Assume that $x \notin Q_2$ and let $F := \mathcal{F}|_{Q_2}$. Then $\det F \simeq \mathcal{O}_{Q_2}(-1,-1)$, $c_2(F) = 2$, $\mathrm{h}^0(F(1,1)) = 3$ but, for every line $L \subset Q_2$, $F|_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L(-1)$.

Lemma 4.6. *If F is Gieseker-Maruyama stable then it is stable.*

Proof. F is, at least, semistable. If it were not stable then it would contain a saturated subsheaf of the form $\mathcal{O}_{Q_2}(a + c_1, -a)$ for some $a \in \mathbb{Z}$ and we would have an exact sequence:

$$0 \longrightarrow \mathcal{O}_{Q_2}(a + c_1, -a) \longrightarrow F \longrightarrow \mathcal{I}_Z(-a, a + c_1) \longrightarrow 0$$

with Z a 0-dimensional subscheme of Q_2. If $P(t)$ is the Hilbert polynomial of $\mathcal{O}_{Q_2}(a+c_1, -a)$ then the Hilbert polynomial of F is $P_F(t) = 2P(t) - \deg Z$. The existence of the subsheaf $\mathcal{O}_{Q_2}(a + c_1, -a)$ would thus contradict the Gieseker-Maruyama stability of F. □

5. Restrictions to nonsingular hyperplane sections

Definition 5.1. Let $\mathcal{E}$ be a rank 2 reflexive sheaf on a smooth quadric threefold $Q = Q_3 \subset \mathbb{P}^4$ with $\det \mathcal{E} \simeq \mathcal{O}_Q(c_1)$, $c_1 = 0$ or -1. We put:

$$\mathcal{U}(\mathcal{E}) := \mathbb{P}^{4\vee} \setminus (Q^\vee \cup \bigcup \{x^\vee \mid x \in \mathrm{Sing}\, \mathcal{E}\}) .$$

We also denote by $\mathcal{U}_{\mathrm{gm}}(\mathcal{E})$ (resp., $\mathcal{U}_{\mathrm{ss}}(\mathcal{E})$, resp., $\mathcal{U}_{\mathrm{s}}(\mathcal{E})$) the set of those $h \in \mathcal{U}(\mathcal{E})$ for which $\mathcal{E}|_{H \cap Q}$ has the Grauert-Mülich property (see Definition 4.1) (resp. it is semistable, resp., it is stable).

Lemma 5.1. *Consider the incidence diagram from par. 3.3 above and let Z be a closed subset of $\mathbb{P}(U) = \mathbb{P}^3$, $Z \neq \mathbb{P}^3$. Then the set of points $h \in \mathbb{P}^{4\vee}$ for which $Z \cap p_1(q_1^{-1}(h))$ is finite is an open subset of $\mathbb{P}^{4\vee}$ whose complement has codimension ≥ 2 in $\mathbb{P}^{4\vee}$.*

Proof. As we saw in par. 3.3, the restriction of $p_1 : q_1^{-1}(h) \to \mathbb{P}^3$ is a closed immersion, $\forall h \in \mathbb{P}^{4\vee}$. It follows that p_1 maps $p_1^{-1}(Z) \cap q_1^{-1}(h)$ isomorphically onto $Z \cap p_1(q_1^{-1}(h))$. Now, we may assume that Z is an irreducible surface in $\mathbb{P}^3$. As $\mathbb{F}_{1,2}(Q)$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^3$, $p_1^{-1}(Z)$ is irreducible of dimension 3. If $q_1(p_1^{-1}(Z))$ has dimension ≤ 2 then it is the complement of the set from the conclusion of the lemma. If $q_1(p_1^{-1}(Z))$ has dimension 3 then, by a well-known theorem of Chevalley (see, for example, [Mu99, I, § 8, Cor. 3]), the set W of the points $y \in p_1^{-1}(Z)$ which are isolated in the fibre $p_1^{-1}(Z) \cap q_1^{-1}(q_1(y))$ is open in $p_1^{-1}(Z)$ (and non-empty). In this case, the complement of the set from the conclusion of the lemma is $q_1(p_1^{-1}(Z) \setminus W)$. □

Lemma 5.2. *Consider the incidence diagram:*

$$\begin{array}{ccc} \mathbb{F}_{2,3}(\mathbb{P}^4) & \xrightarrow{\ g\ } & \mathbb{P}^{4\vee} \\ {\scriptstyle f}\big\downarrow & & \\ \mathbb{G}_2(\mathbb{P}^4) & & \end{array}$$

and let Z be a closed subset of $\mathbb{G}_2(\mathbb{P}^4)$, $Z \neq \mathbb{G}_2(\mathbb{P}^4)$. Then the set of points $h \in \mathbb{P}^{4\vee}$ for which $f(g^{-1}(h)) \subseteq Z$ is a closed subset of $\mathbb{P}^{4\vee}$ of codimension ≥ 2.

Proof. We have $f(g^{-1}(h)) \subseteq Z$ if and only if $g^{-1}(h) \subseteq f^{-1}(Z)$. We may assume that Z is an irreducible hypersurface in $\mathbb{G}_2(\mathbb{P}^4)$, hence that it has dimension 5. Since $\mathbb{F}_{2,3}(\mathbb{P}^4)$ is a $\mathbb{P}^1$-bundle over $\mathbb{G}_2(\mathbb{P}^4)$, $f^{-1}(Z)$ is irreducible of dimension 6. For $h \in \mathbb{P}^{4\vee}$, $f(g^{-1}(h))$ is a 3-plane in the Plücker embedding of $\mathbb{G}_2(\mathbb{P}^4)$ in $\mathbb{P}^9$. One deduces that a general fibre of the restriction of $g : f^{-1}(Z) \to \mathbb{P}^{4\vee}$ has dimension 2. The set T of the points $y \in f^{-1}(Z)$ for which $\dim_y(f^{-1}(Z) \cap g^{-1}(g(y))) \geq 3$ is a closed subset of $f^{-1}(Z)$, not equal to $f^{-1}(Z)$. The set from the conclusion of the lemma is exactly $f(T)$, which has dimension $\leq 5 - 3 = 2$. □

Proposition 5.3. *Let $\mathcal{E}$ be a semistable rank 2 reflexive sheaf on a quadric threefold $Q = Q_3 \subset \mathbb{P}^4$ with $\det \mathcal{E} \simeq \mathcal{O}_Q(c_1)$, $c_1 = 0$ or -1. Then:*

(i) *$\mathcal{U}_{\mathrm{gm}}(\mathcal{E})$ and $\mathcal{U}_{\mathrm{ss}}(\mathcal{E})$ are open subsets of $\mathcal{U}(\mathcal{E})$ and their complements in $\mathcal{U}(\mathcal{E})$ have codimension ≥ 2.*

(ii) *$\mathcal{U}_{\mathrm{s}}(\mathcal{E}) \cap \mathcal{U}_{\mathrm{gm}}(\mathcal{E})$ is an open subset of $\mathcal{U}(\mathcal{E})$.*

Proof. (i) Using the notation from par. 3.3, the set of points $\ell \in \mathbb{P}(U) = \mathbb{P}^3$ for which the corresponding line $L \subset Q$ passes through a singular point of $\mathcal{E}$ is a union of finitely many lines in $\mathbb{P}^3$. Let W be the complement of this union of lines. An argument of semicontinuity shows that the set of points $\ell \in W$ for which $\mathcal{E}|_L \simeq \mathcal{O}_L(a) \oplus \mathcal{O}_L(-a + c_1)$ with $a > 0$ is a closed subset of W and it is not equal to W by [ES84, Prop. 1.3]. One deduces, now, from Lemma 5.1, that $\mathcal{U}_{\mathrm{gm}}(\mathcal{E})$ is an open subset of $\mathcal{U}(\mathcal{E})$ and that its complement has codimension ≥ 2.

Secondly, if $h \in \mathcal{U}(\mathcal{E})$, then $\mathcal{E}|_{H\cap Q}$ is semistable if and only if $H \cap Q$ contains a smooth conic C such that $\mathcal{E}|_C \simeq \mathcal{O}_{\mathbb{P}^1}(c_1)^{\oplus 2}$ (by the case $n = 3$ of Prop. 2.4). The assertion about $\mathcal{U}_{\mathrm{ss}}(\mathcal{E})$ follows, now, from the case $n = 4$ of Prop. 2.4 and from Lemma 5.2.

(ii) follows from Lemma 4.1(i). □

Remark 5.4. $\mathcal{U}_{\mathrm{s}}(\mathcal{E})$ is, actually, itself an open subset of $\mathcal{U}(\mathcal{E})$. Indeed, this follows from Lemma 4.6 and from the openess of the Gieseker-Maruyama stability, a result due to Maruyama [Ma76, Thm. 2.8].

Proposition 5.5. *Let E be a stable rank 2 vector bundle on $Q = Q_3 \subset \mathbb{P}^4$ with $c_1(E) = c_1 = 0$ or -1 and with $c_2 > -c_1$. Let $\Sigma \subset \mathbb{P}^{4\vee}$ be an irreducible hypersurface, $\Sigma \neq Q^\vee$ and $\Sigma \neq x^\vee$, $\forall x \in Q$, such that $\mathcal{U}_{\mathrm{s}}(E) \cap \mathcal{U}_{\mathrm{gm}}(E) \cap \Sigma = \emptyset$. Let $H_0 \subset \mathbb{P}^4$ be a hyperplane intersecting Q transversally.*

If $E|_{H_0\cap Q}$ is unstable (i.e., not semistable) then $\Sigma = (H_0 \cap Q)^\vee$.

Proof. Let $\mathcal{O}_{H_0\cap Q}(a,b) \subset E|_{H_0\cap Q}$ be a maximal destabilizing subsheaf. One has $a+b > c_1$ and an exact sequance:

$$0 \longrightarrow \mathcal{O}_{H_0\cap Q}(a,b) \longrightarrow E|_{H_0\cap Q} \longrightarrow \mathcal{I}_{Z,H_0\cap Q}(c_1-a, c_1-b) \longrightarrow 0$$

where Z is a 0-dimensional subscheme of $H_0\cap Q$. Let $\mathcal{E}'$ be the rank 2 reflexive sheaf on Q defined by the exact sequence:

$$0 \longrightarrow \mathcal{E}'(c_1) \longrightarrow E \longrightarrow \mathcal{I}_{Z,H_0\cap Q}(c_1-a, c_1-b) \longrightarrow 0$$

One has $c_1' := c_1(\mathcal{E}') = -1$ if $c_1 = 0$, $c_1' = 0$ if $c_1 = -1$, $\mathrm{Sing}\,\mathcal{E}' = Z$, and E stable implies that $\mathrm{H}^0(E) = 0$, hence $\mathrm{H}^0(\mathcal{E}'(c_1)) = 0$ hence $\mathcal{E}'$ is semistable.

We will show that if one assumes that $\Sigma \neq (H_0\cap Q)^\vee$ then one gets a contradiction. Indeed, by Prop. 5.3(i), $\mathcal{U}_{\mathrm{ss}}(\mathcal{E}') \cap \Sigma \neq \emptyset$. Recall that, by definiton, $\mathcal{U}_{\mathrm{ss}}(\mathcal{E}')$ is an open subset of $\mathcal{U}(\mathcal{E}') = \mathbb{P}^{4\vee} \setminus (Q^\vee \cup \bigcup\{x^\vee \mid x \in Z\})$. If $\Sigma \neq (H_0\cap Q)^\vee$, choose a point

$$h \in (\mathcal{U}_{\mathrm{ss}}(\mathcal{E}') \cap \mathcal{U}_{\mathrm{ss}}(E) \cap \mathcal{U}_{\mathrm{gm}}(E) \cap \Sigma) \setminus (H_0\cap Q)^\vee\,.$$

If $H \subset \mathbb{P}^4$ is the corresponding hyperplane, then $H\cap H_0\cap Q$ is a smooth conic $C \simeq \mathbb{P}^1$, avoiding Z. Restricting to $H\cap Q$ the exact sequence defining $\mathcal{E}'$, one gets an exact sequence:

$$0 \longrightarrow \mathcal{E}'(c_1)|_{H\cap Q} \longrightarrow E|_{H\cap Q} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(2c_1-a-b) \longrightarrow 0\,.$$

By the proof of Lemma 4.1(ii), $E|_{H\cap Q}$ has a subsheaf $\mathcal{L}$ of the form $\mathcal{O}_{H\cap Q}$ in the case $c_1 = 0$ and of the form $\mathcal{O}_{H\cap Q}(-1,0)$ or $\mathcal{O}_{H\cap Q}(0,-1)$, in the case $c_1 = -1$. Since $\mathcal{L}|_C \simeq \mathcal{O}_{\mathbb{P}^1}(c_1)$, it follows that $\mathrm{Hom}_{\mathcal{O}_{H\cap Q}}(\mathcal{L}, \mathcal{O}_{\mathbb{P}^1}(2c_1 - a-b)) = 0$, hence $\mathcal{L} \subset \mathcal{E}'(c_1)|_{H\cap Q}$ and this contradicts the semistability of $\mathcal{E}'|_{H\cap Q}$. □

Corollary 5.6. *Under the hypothesis of Prop. 5.5, minus the assumption about* H_0:

(i) *For every line* $L \subset Q$ *one has* $E|_L \simeq \mathcal{O}_L(a) \oplus \mathcal{O}_L(-a+c_1)$, *with* $a \leq c_2 + c_1$.

(ii) *Assuming* $c_2 > -c_1 + 1$, *if* $L_1, L_2 \subset Q$ *are two distinct lines such that* $E|_{L_i} \simeq \mathcal{O}_{L_i}(c_2+c_1) \oplus \mathcal{O}_{L_i}(-c_2)$, $i = 1, 2$, *then* $L_1 \cap L_2 = \emptyset$.

Proof. By Prop. 5.5, the complement of $\mathcal{U}_{\mathrm{ss}}(E)$ in $\mathcal{U}(E) = \mathbb{P}^{4\vee} \setminus Q^\vee$ consists of one point or is empty. Let $L, L' \subset Q$ be two lines intersecting in a point x. The set of hyperplanes $H \subset \mathbb{P}^4$ containing $L \cup L'$ is a line in $\mathbb{P}^{4\vee}$ which intersects $Q^\vee$ in only one point (namely, the one corresponding to the tangent hyperplane to Q at x). It follows that, for a general H as above, $H\cap Q$ is nonsingular and $E|_{H\cap Q}$ is semistable. One can apply, now, Lemma 4.2 and Cor. 4.3. □

Lemma 5.7. *Let* $Q_2 \subset \mathbb{P}^3$ *be a smooth quadric, let* x *be a point of* $\mathbb{P}^3$ *and let* $\mathcal{F}$ *be a rank* 2 *reflexive sheaf on* $\mathbb{P}^3$ *defined by an exact sequence:*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 3} \longrightarrow \mathcal{F} \longrightarrow 0$$

where the left morphism is the dual of an epimorphism $\mathcal{O}_{\mathbb{P}^3}^{\oplus 3} \to \mathcal{I}_{\{x\}}(1)$.

(i) *If* $x \in Q_2$ *then one has an exact sequence:*

$$0 \longrightarrow \mathcal{F}|_{Q_2} \longrightarrow \mathcal{O}_{Q_2}(1,0) \oplus \mathcal{O}_{Q_2}(0,1) \longrightarrow \mathcal{O}_{\{x\}} \longrightarrow 0\,.$$

(ii) *If* $x \notin Q_2$ *then the zero scheme of any non-zero global section of* $\mathcal{F}|_{Q_2}$ *is* 0*-dimensional, of degree* 2*. Moreover,* $(\mathcal{F}|_{Q_2})(1,-1)$ *and* $(\mathcal{F}|_{Q_2})(-1,1)$

have only one non-zero global section (up to multiplication by scalars) and these sections vanish nowhere.

Proof. (i) Let p_1, $p_2 : Q_2 \to \mathbb{P}^1$ be the projections and put, as in Section 4, $\overline{\mathcal{S}} = \mathcal{O}_{Q_2}(-1,0) \oplus \mathcal{O}_{Q_2}(0,-1)$. Taking the direct sum of the exact sequences obtained by applying p_1^* and p_2^* to the exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0$$

one gets an exact sequence

$$0 \longrightarrow \overline{\mathcal{S}} \longrightarrow \mathcal{O}_{Q_2}^{\oplus 4} \longrightarrow \overline{\mathcal{S}}(1,1) \longrightarrow 0\,.$$

On the other hand, $\{x\}$ is the intersection of the lines $p_1^{-1}(p_1(x))$ and $p_2^{-1}(p_2(x))$, hence one has an exact sequence:

$$0 \longrightarrow \mathcal{O}_{Q_2}(-1,-1) \longrightarrow \overline{\mathcal{S}} \longrightarrow \mathcal{I}_{\{x\},Q_2} \longrightarrow 0\,.$$

Applying the Snake Lemma to the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{Q_2} & \longrightarrow & \mathcal{O}_{Q_2}^{\oplus 4} & \longrightarrow & \mathcal{O}_{Q_2}^{\oplus 3} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{Q_2} & \longrightarrow & \overline{\mathcal{S}}(1,1) & \longrightarrow & \mathcal{I}_{\{x\},Q_2}(1,1) & \longrightarrow & 0
\end{array}$$

(in which the right vertical morphism is the evaluation morphism) one deduces an exact sequence:

$$0 \longrightarrow \mathcal{O}_{Q_2}(-1,0) \oplus \mathcal{O}_{Q_2}(0,-1) \longrightarrow \mathcal{O}_{Q_2}^{\oplus 3} \longrightarrow \mathcal{I}_{\{x\},Q_2}(1,1) \longrightarrow 0\,.$$

Applying $\mathcal{H}om_{\mathcal{O}_{Q_2}}(-,\mathcal{O}_{Q_2})$ to this exact sequence one gets and exact sequence:

$$0 \longrightarrow \mathcal{O}_{Q_2}(-1,-1) \longrightarrow \mathcal{O}_{Q_2}^{\oplus 3} \longrightarrow \mathcal{O}_{Q_2}(1,0) \oplus \mathcal{O}_{Q_2}(0,1) \longrightarrow \mathcal{O}_{\{x\}} \longrightarrow 0$$

because $\mathcal{E}xt^1_{\mathcal{O}_{Q_2}}(\mathcal{I}_{\{x\},Q_2},\mathcal{O}_{Q_2}) \simeq \mathcal{O}_{\{x\}}$.

(ii) The zero scheme of any non-zero global section of $\mathcal{F}$ is a line L in $\mathbb{P}^3$ passing through x and one has an exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_L(1) \longrightarrow 0\,.$$

Such a line intersects Q_2 in a 0-dimensional scheme of degree 2. The first assertion from the second point of the conclusion is now clear. As for the last assertion, tensorizing by $\mathcal{O}_{Q_2}(1,-1)$ the exact sequence:

$$0 \longrightarrow \mathcal{O}_{Q_2}(-1,-1) \longrightarrow \mathcal{O}_{Q_2}^{\oplus 3} \longrightarrow \mathcal{F}|_{Q_2} \longrightarrow 0$$

and taking global sections, one deduces that $\mathrm{h}^0((\mathcal{F}|_{Q_2})(1,-1)) = 1$. Let s be a non-zero global section of $(\mathcal{F}|_{Q_2})(1,-1)$ and suppose that s vanishes in a point $y \in Q_2$. Let L be the line belonging to the linear system $|\,\mathcal{O}_{Q_2}(1,0)\,|$ passing through y. Since $x \notin L$, one sees easily that $\mathcal{F}|_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L(1)$, hence $(\mathcal{F}|_{Q_2})(1,-1)|_L \simeq \mathcal{O}_L(-1) \oplus \mathcal{O}_L$. Since s vanishes in $y \in L$, s must vanish on L. But:

$$(\mathcal{F}|_{Q_2})(1,-1) \otimes \mathcal{O}_{Q_2}(-L) \simeq (\mathcal{F}|_{Q_2})(0,-1)$$

and $\mathrm{H}^0((\mathcal{F}|_{Q_2})(0,-1)) = 0$, hence $s = 0$, a contradiction. □

The next result, in which we use the method described in Section 2, is the key point in the proof of Theorem 1.1.

Proposition 5.8. *Let $\mathcal{E}$ be a stable rank 2 reflexive sheaf on a nonsingular quadric threefold $Q = Q_3 \subset \mathbb{P}^4$, with $\det \mathcal{E} \simeq \mathcal{O}_Q(c_1)$, $c_1 = 0$ or -1, and with $c_2 > 2 + c_1$. Let $\Sigma \subset \mathbb{P}^{4\vee}$ be a reduced and irreducible hypersurface, $\Sigma \neq Q^\vee$ and $\Sigma \neq x^\vee$, $\forall x \in \operatorname{Sing} \mathcal{E}$.*

If $\mathcal{U}_{\mathrm{s}}(\mathcal{E}) \cap \mathcal{U}_{\mathrm{gm}}(\mathcal{E}) \cap \Sigma = \emptyset$ then, $\forall h \in \mathcal{U}_{\mathrm{ss}}(\mathcal{E}) \cap \Sigma$, there exists a line $L \subset H \cap Q$ such that $\mathcal{E}|_L \simeq \mathcal{O}_L(c_2 + c_1) \oplus \mathcal{O}_L(-c_2)$.

Proof. Consider the incidence diagram from Lemma 2.3 for $n = 4$ and $d = 3$, i.e, for $\mathbb{P}^n = \mathbb{P}^4$ and $\mathbb{G}_d(\mathbb{P}^n) = \mathbb{P}^{4\vee}$. Let Σ_{reg} be the set of nonsingular points of Σ, let $\Sigma' := \mathcal{U}_{\mathrm{ss}}(\mathcal{E}) \cap \mathcal{U}_{\mathrm{gm}}(\mathcal{E}) \cap \Sigma_{\mathrm{reg}}$, let $X' := p^{-1}(Q) \cap q^{-1}(\Sigma')$ and consider the induced diagram:

$$\begin{array}{ccc} X' & \xrightarrow{q'} & \Sigma' \\ {\scriptstyle p'}\big\downarrow & & \\ Q & & \end{array}$$

Proposition 5.3(i) implies that $\Sigma' \neq \emptyset$. By the proof of Lemma 4.1, it follows that, $\forall h \in \Sigma'$, one has, in the case $c_1 = 0$, an exact sequence:

$$0 \longrightarrow \mathcal{O}_{H\cap Q} \longrightarrow \mathcal{E}|_{H\cap Q} \longrightarrow \mathcal{I}_{Z,H\cap Q} \longrightarrow 0$$

where Z is a 0-dimensional subscheme of $H \cap Q$ with $\deg Z = c_2$, and, in the case $c_1 = -1$, one has an exact sequence of one of the forms:

$$0 \longrightarrow \mathcal{O}_{H\cap Q}(-1,0) \longrightarrow \mathcal{E}|_{H\cap Q} \longrightarrow \mathcal{I}_{Z,H\cap Q}(0,-1) \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}_{H\cap Q}(0,-1) \longrightarrow \mathcal{E}|_{H\cap Q} \longrightarrow \mathcal{I}_{Z,H\cap Q}(-1,0) \longrightarrow 0$$

where Z is a 0-dimensional subscheme of $H \cap Q$ with $\deg Z = c_2 - 1$.

It follows that, in the case $c_1 = 0$ (resp., $c_1 = -1$), $q'_*p'^*\mathcal{E}$ (resp., $q'_*p'^*\mathcal{H}om_{\mathcal{O}_Q}(\mathcal{S}, \mathcal{E})$) is a line bundle on Σ'. In the case $c_1 = 0$, the image of the canonical morphism $q'^*q'_*p'^*\mathcal{E} \to p'^*\mathcal{E}$ is a saturated subsheaf $\mathcal{E}'$ of $p'^*\mathcal{E}$ such that $\mathcal{E}'|_{q'^{-1}(h)} \simeq \mathcal{O}_{H\cap Q}$, $\forall h \in \Sigma'$. In the case $c_1 = -1$, the image of the composite morphism:

$$p'^*\mathcal{S} \otimes q'^*q'_*p'^*\mathcal{H}om(\mathcal{S}, \mathcal{E}) \longrightarrow p'^*\mathcal{S} \otimes p'^*\mathcal{H}om(\mathcal{S}, \mathcal{E}) \longrightarrow p'^*\mathcal{E}$$

is a saturated subsheaf $\mathcal{E}'$ of $p'^*\mathcal{E}$ such that $\mathcal{E}'|_{q'^{-1}(h)} \simeq \mathcal{O}_{H\cap Q}(-1,0)$ or to $\mathcal{O}_{H\cap Q}(0,-1)$, $\forall h \in \Sigma'$. Let $\mathcal{E}'' := p'^*\mathcal{E}/\mathcal{E}'$.

We have to consider two cases: 1) $\Sigma \neq x^\vee$, $\forall x \in Q$, and 2) $\Sigma = x_0^\vee$, for some $x_0 \in Q$. In Case 1, one applies Prop. 2.2 to $p' : X' \to Y := Q \setminus \operatorname{Sing} \mathcal{E}$ and to the exact sequence $0 \to \mathcal{E}' \longrightarrow p'^*\mathcal{E} \to \mathcal{E}'' \to 0$. In Case 2, one applies Prop. 2.2 to the restriction of $p' : X' \setminus p'^{-1}(x_0) \to Y \setminus \{x_0\}$ and to the above exact sequence restricted to $X' \setminus p'^{-1}(x_0)$. The hypothesis of Prop. 2.2 is satisfied by Lemma 2.5. Since there exists no saturated subsheaf $\overline{\mathcal{E}}'$ of $\mathcal{E}|_Y$ (resp., of $\mathcal{E}|_{Y\setminus\{x_0\}}$ in Case 2) such that $\overline{\mathcal{E}}'|_{H\cap Q}$ (resp., $\overline{\mathcal{E}}'|_{H\cap Q\setminus\{x_0\}}$ in Case 2) is isomorphic to $\mathcal{O}_{H\cap Q}$ for $c_1 = 0$ and to $\mathcal{O}_{H\cap Q}(-1,0)$ or to $\mathcal{O}_{H\cap Q}(0,-1)$ for $c_1 = -1$ (resp., to their restrictions to $H \cap Q \setminus \{x_0\}$ in Case 2), $\forall h \in \Sigma'$, one deduces, from Prop. 2.2, that the second fundamental form $\mathcal{E}' \to \Omega_{X'/Q} \otimes \mathcal{E}''$ has generically rank ≥ 1.

It follows that, for a general $h \in \Sigma'$, the restriction of the second fundamental form to $q'^{-1}(h)$ has generically rank ≥ 1. By Lemma 2.5(i), this restriction can be identified with a morphism of one of the forms:

$$\mathcal{O}_{H\cap Q} \longrightarrow (\mathcal{F}|_{H\cap Q})\otimes \mathcal{I}_{Z,H\cap Q} \quad (\text{for } c_1 = 0)$$
$$\mathcal{O}_{H\cap Q}(-1,0) \longrightarrow (\mathcal{F}|_{H\cap Q})\otimes \mathcal{I}_{Z,H\cap Q}(0,-1) \quad (\text{for } c_1 = -1)$$
$$\mathcal{O}_{H\cap Q}(0,-1) \longrightarrow (\mathcal{F}|_{H\cap Q})\otimes \mathcal{I}_{Z,H\cap Q}(-1,0) \quad (\text{for } c_1 = -1)$$

where $\mathcal{F}$ is the rank 2 reflexive sheaf on $H \simeq \mathbb{P}^3$ defined by an exact sequence:

$$0 \longrightarrow \mathcal{O}_H(-1) \longrightarrow \mathcal{O}_H^{\oplus 3} \longrightarrow \mathcal{F} \longrightarrow 0$$

with the left morphism equal to the dual of an epimorphism $\mathcal{O}_H^{\oplus 3} \to \mathcal{I}_{\{x\},H}(1)$, x being the point of H corresponding to the tangent hyperplane $\mathrm{T}_h\Sigma \subset \mathbb{P}^{4\vee}$. From the fact that the restriction of the second fundamental form to $q'^{-1}(h)$ has generically rank ≥ 1, one deduces that $(\mathcal{F}|_{H\cap Q})^{**}$ (for $c_1 = 0$) and $(\mathcal{F}|_{H\cap Q})^{**}(1,-1)$ or $(\mathcal{F}|_{H\cap Q})^{**}(-1,1)$ (for $c_1 = -1$) has a global section vanishing on Z. Using Lemma 5.7 and the fact that $c_2 > 2 + c_1$, one excludes the case where $x \notin H \cap Q$. It remains that $x \in H \cap Q$ and that $(\mathcal{F}|_{H\cap Q})^{**} \simeq \mathcal{O}_{H\cap Q}(1,0) \oplus \mathcal{O}_{H\cap Q}(0,1)$. One deduces, in the case $c_1 = 0$, that $\mathrm{H}^0(\mathcal{I}_{Z,H\cap Q}(1,0)) \neq 0$ or $\mathrm{H}^0(\mathcal{I}_{Z,H\cap Q}(0,1)) \neq 0$, and, in the case $c_1 = -1$, one deduces that $\mathrm{H}^0(\mathcal{I}_{Z,H\cap Q}(1,0)) \neq 0$ resp., that $\mathrm{H}^0(\mathcal{I}_{Z,H\cap Q}(0,1)) \neq 0$. It results that $H \cap Q$ contains a line L as in the statement.

We have proved, so far, that, for a general $h \in \mathcal{U}_{\mathrm{ss}}(\mathcal{E}) \cap \Sigma$, the conclusion of the Proposition is fulfilled. But, by Cor. 4.4, the set of the points $h \in \mathcal{U}_{\mathrm{ss}}(\mathcal{E}) \cap \Sigma$ satisfying the conclusion of the Proposition is closed in $\mathcal{U}_{\mathrm{ss}}(\mathcal{E}) \cap \Sigma$, hence it must be the whole of $\mathcal{U}_{\mathrm{ss}}(\mathcal{E}) \cap \Sigma$. □

Remark 5.9. Let $\mathcal{E}$ be a stable rank two reflexive sheaf on $Q = Q_3 \subset \mathbb{P}^4$ with $\det \mathcal{E} \simeq \mathcal{O}_Q(c_1)$, $c_1 = 0$ or -1. If $\mathcal{U}_{\mathrm{s}}(\mathcal{E}) \cap \mathcal{U}_{\mathrm{gm}}(\mathcal{E}) = \emptyset$ then $\mathcal{E} \simeq \mathcal{S}$.

Indeed, the case $c_1 = 0$ was settled in [ES84, Thm. 1.6]. Let us assume that $c_1 = -1$. Consider the incidence diagram from Lemma 2.3, for $n = 4$ and $d = 3$. It induces a diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ \overline{q}\ } & \mathbb{P}^{4\vee} \\ {\scriptstyle \overline{p}}\big\downarrow & & \\ Q & & \end{array}$$

where $X = p^{-1}(Q)$. It follows, from the first part of the proof of Prop. 5.8, that if $c_2(\mathcal{E}) = c_2[L]$, with $c_2 > 1$, then there exists a line bundle $\mathcal{E}'$ on $\overline{q}^{-1}(\mathcal{U}_{\mathrm{ss}}(\mathcal{E}) \cap \mathcal{U}_{\mathrm{gm}}(\mathcal{E}))$ such that $\mathcal{E}'|_{\overline{q}^{-1}(h)} \simeq \mathcal{O}_{H\cap Q}(-1,0)$ or to $\mathcal{O}_{H\cap Q}(0,-1)$, $\forall h \in \mathcal{U}_{\mathrm{ss}}(\mathcal{E}) \cap \mathcal{U}_{\mathrm{gm}}(\mathcal{E})$. But this is not possible because, X being a $\mathbb{P}^3$-bundle over Q, $\operatorname{Pic} X \simeq \overline{p}^* \operatorname{Pic} Q \oplus \overline{q}^* \operatorname{Pic} \mathbb{P}^{4\vee}$.

It remains that $c_2 = 1$ and that $\mathcal{E}|_{H\cap Q} \simeq \mathcal{O}_{H\cap Q}(-1,0) \oplus \mathcal{O}_{H\cap Q}(0,-1)$, $\forall h \in \mathcal{U}_{\mathrm{ss}}(\mathcal{E}) \cap \mathcal{U}_{\mathrm{gm}}(\mathcal{E})$. One uses, now, the fact, which can be verified as in the proof of [Ott88, Theorem 2.11(ii)], that if $\mathcal{F}$ is a rank 2 reflexive sheaf on Q such that $\mathcal{F}|_{H\cap Q} \simeq \mathcal{S}|_{H\cap Q}$ for some hyperplane $H \subset \mathbb{P}^4$ avoiding the singular points of $\mathcal{F}$ (but not necessarily cutting Q transversally) then $\mathcal{F} \simeq \mathcal{S}$.

Lemma 5.10. *Let $\mathcal{E}$ be a stable rank two reflexive sheaf on $Q = Q_3 \subset \mathbb{P}^4$ with $\det \mathcal{E} \simeq \mathcal{O}_Q(c_1)$, $c_1 = 0$ or -1, and with $c_2 > -c_1$. Then the set of points $\ell \in \mathbb{P}(U) = \mathbb{P}^3$ corresponding to the lines $L \subset Q$ which either* *pass through a singular point of* $\mathcal{E}$ *or have the property that* $\mathcal{E}|_L \simeq \mathcal{O}_L(a) \oplus \mathcal{O}_L(-a+c_1)$, $a \geq c_2 + c_1$, *is a closed subset of* $\mathbb{P}^3$, *of dimension* ≤ 1.

Proof. The set of points $\ell \in \mathbb{P}(U) = \mathbb{P}^3$ corresponding to the lines $L \subset Q$ passing through a singular point of $\mathcal{E}$ is a union of lines in $\mathbb{P}^3$ (see par. 3.1). Let $V \subseteq \mathbb{P}^3$ be the complement of this union of lines. By a semicontinuity argument, the set of points $\ell \in V$ such that $\mathcal{E}|_L \simeq \mathcal{O}_L(a) \oplus \mathcal{O}_L(-a+c_1)$, $a \geq c_2 + c_1$, is a closed subset of V. Consequently, the set Z from the statement of the lemma is a closed subset of $\mathbb{P}^3$.

Now, by [ES84, Thm. 1.6] (complete with Remark 5.9), for a general hyperplane $H \subset \mathbb{P}^4$, $H \cap Q$ is smooth, contains no singular point of $\mathcal{E}$ and $\mathcal{E}|_{H\cap Q}$ is stable. We have seen in par. 3.3 that the set of points $\ell \in \mathbb{P}^3$ such that $L \subset H \cap Q$ is a union of two disjoint line $\Lambda_1 \cup \Lambda_2$. By the last assertion of Lemma 4.2, $(\Lambda_1 \cup \Lambda_2) \cap Z = \emptyset$, hence $\dim Z \leq 1$. □

Proposition 5.11. *Let* $\mathcal{E}$ *be a stable rank two reflexive sheaf on* $Q = Q_3 \subset \mathbb{P}^4$ *with* $\det \mathcal{E} \simeq \mathcal{O}_Q(c_1)$, $c_1 = 0$ *or* -1, *and with* $c_2 > 2$. *Consider a point* $x_0 \in Q \setminus \operatorname{Sing} \mathcal{E}$.

Then, for a general hyperplane $H \subset \mathbb{P}^4$ *passing through* x_0, $\mathcal{E}|_{H\cap Q}$ *is stable.*

Proof. Let $\Sigma := x_0^\vee \subset \mathbb{P}^{4\vee}$. Assume that $\mathcal{U}_{\mathrm{s}}(\mathcal{E}) \cap \mathcal{U}_{\mathrm{gm}}(\mathcal{E}) \cap \Sigma = \emptyset$. Then Prop. 5.8 implies that, $\forall h \in \mathcal{U}_{\mathrm{ss}}(\mathcal{E}) \cap \Sigma$, $H \cap Q$ contains a line L such that $\mathcal{E}|_L \simeq \mathcal{O}_L(c_2 + c_1) \oplus \mathcal{O}_L(-c_2)$. Let $Z \subset \mathbb{P}(U) = \mathbb{P}^3$ be the set of lines in Q from Lemma 5.10. The lines in Q passing through x_0 correspond to a line $\Lambda_{x_0} \subset \mathbb{P}^3$.

We assert that $\Lambda_{x_0} \subseteq Z$. *Indeed*, otherwise $\Lambda_{x_0} \cap Z$ is a finite set. It would follow, from Lemma 5.10, that the set $\mathcal{H}$ of hyperplanes $H \subset \mathbb{P}^4$ containing x_0 and a line $L \subset Q$ corresponding to a point $\ell \in Z$, has dimension ≤ 2 (in $\mathbb{P}^{4\vee}$), which would contradict the fact that $\mathcal{U}_{\mathrm{ss}}(\mathcal{E}) \cap \Sigma \subseteq \mathcal{H}$.

Now, let $h \in \mathcal{U}_{\mathrm{ss}}(\mathcal{E}) \cap \Sigma$. Let L_1, L_2 be the two lines in $H \cap Q$ passing through x_0. Since $\Lambda_{x_0} \subseteq Z$, it follows that $\mathcal{E}|_{L_i} \simeq \mathcal{O}_{L_i}(a_i) \oplus \mathcal{O}_{L_i}(-a_i + c_1)$, with $a_i \geq c_2 + c_1$, $i = 1, 2$. Since $c_2 + c_1 > 1$, this contradicts Cor. 4.3.

It thus remains that $\mathcal{U}_{\mathrm{s}}(\mathcal{E}) \cap \mathcal{U}_{\mathrm{gm}}(\mathcal{E}) \cap x_0^\vee \neq \emptyset$. □

The next proposition concludes the proof of Thm. 1.1 (taking into account Prop. 5.11 and the results quoted in the Introduction, before the statement of Thm. 1.1).

Proposition 5.12. *Let* E *be a stable rank* 2 *vector bundle on* $Q = Q_3 \subset \mathbb{P}^4$ *with* $\det E \simeq \mathcal{O}_Q(c_1)$, $c_1 = 0$ *or* -1, *and with* $c_2 > 2$. *Let* $\Sigma \subset \mathbb{P}^{4\vee}$ *be an irreducible hypersurface,* $\Sigma \neq Q^\vee$ *and* $\Sigma \neq x^\vee$, $\forall x \in Q$.

If $\mathcal{U}_{\mathrm{s}}(\mathcal{E}) \cap \mathcal{U}_{\mathrm{gm}}(\mathcal{E}) \cap \Sigma = \emptyset$ *then* E *is as in the statement of Thm. 1.1*(ii).

Proof. Prop. 5.8 implies that, $\forall h \in \mathcal{U}_{\mathrm{ss}}(\mathcal{E}) \cap \Sigma$, $H \cap Q$ contains a line L such that $E|_L \simeq \mathcal{O}_L(c_2 + c_1) \oplus \mathcal{O}_L(-c_2)$. Cor. 5.6(i) implies that the set Z of the points $\ell \in \mathbb{P}(U) = \mathbb{P}^3$ corresponding to the lines $L \subset Q$ such that $E|_L \simeq \mathcal{O}_L(c_2 + c_1) \oplus \mathcal{O}_L(-c_2)$ is a closed subset of $\mathbb{P}^3$. Since $\dim(\mathcal{U}_{\mathrm{ss}}(\mathcal{E}) \cap \Sigma) = 3$, one must have $\dim Z = 1$ (taking into account Lemma 5.10). Choose two distinct points $\ell_1, \ell_2 \in Z$. By Cor. 5.6(ii), $L_1 \cap L_2 = \emptyset$. Let $H_0 \subset \mathbb{P}^4$ be the hyperplane spanned by L_1 and L_2. H_0 intersects Q transversally (because $H_0 \cap Q$ contains two disjoint lines) and, by Cor. 4.3, $E|_{H_0 \cap Q}$ is unstable.

From Prop. 5.5 it follows that $\Sigma = (H_0 \cap Q)^\vee$. Let $\ell \in Z$ be another point and let $L \subset Q$ be the corresponding line. As above, $L \cap L_1 = \emptyset$ and if $H \subset \mathbb{P}^4$ is the hyperplane spanned by L and L_1 then $\Sigma = (H \cap Q)^\vee$. One deduces that $H = H_0$.

Consequently, the points of Z correspond to the lines from one ruling of $H_0 \cap Q$. Assume that this ruling is the linear system $|\,\mathcal{O}_{H_0 \cap Q}(1,0)\,|$. Then there exists an integer a such that $E|_{H_0 \cap Q}$ can be realized as an extension:

$$0 \longrightarrow \mathcal{O}_{H_0\cap Q}(a, c_2 + c_1) \longrightarrow E|_{H_0\cap Q} \longrightarrow \mathcal{O}_{H_0\cap Q}(-a + c_1, -c_2) \longrightarrow 0\,.$$

Computing Chern classes, one gets that:

$$c_2 = c_2(E|_{H_0\cap Q}) = -ac_2 + (c_2 + c_1)(-a + c_1) = -(2c_2 + c_1)a + (c_2 + c_1)c_1\,,$$

hence $(2c_2 + c_1)a = (c_2 + c_1)c_1 - c_2$. It follows that $c_1 = -1$ and $a = -1$. Let E' be the rank 2 vector bundle on Q defined by the exact sequence:

$$0 \longrightarrow E'(-1) \longrightarrow E \longrightarrow \mathcal{O}_{H_0\cap Q}(0, -c_2) \longrightarrow 0\,. \tag{5.1}$$

One has $\det E' \simeq \mathcal{O}_Q$. Let $c_2' \in \mathbb{Z}$ be defined by $c_2(E') = c_2'[L]$. Let $H \subset \mathbb{P}^4$ be a general hyperplane intersecting Q transversally and such that $H \cap H_0 \cap Q$ is a smooth conic $C \simeq \mathbb{P}^1$. Restricting to $H \cap Q$ the exact sequence (5.1) one gets an exact sequence:

$$0 \longrightarrow E'(-1)|_{H\cap Q} \longrightarrow E|_{H\cap Q} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-c_2) \longrightarrow 0\,.$$

By Riemann-Roch on $H \cap Q$, $\chi(E|_{H\cap Q}) = -c_2 + 1$, and $\chi(E'(-1)|_{H\cap Q}) = -c_2(E'|_{H\cap Q}) = -c_2'$. Since $\chi(\mathcal{O}_{\mathbb{P}^1}(-c_2)) = -c_2 + 1$, one deduces that $c_2' = 0$. E being stable, E' is semistable, hence $E' \simeq \mathcal{O}_Q^{\oplus 2}$. Dualizing the exact sequence (5.1) and twisting by -1, one gets an exact sequence:

$$0 \longrightarrow E \longrightarrow \mathcal{O}_Q^{\oplus 2} \xrightarrow{\ \varepsilon\ } \mathcal{O}_{H_0\cap Q}(0, c_2) \longrightarrow 0\,.$$

The epimorphism ε is defined by two global sections $\varepsilon_1, \varepsilon_2$ of $\mathcal{O}_{H_0\cap Q}(0, c_2)$ generating this sheaf. For a general choice of constants $\alpha_1, \alpha_2 \in \mathbb{C}$, the divisor Y on $H_0 \cap Q$ associated to the global section $\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2$ of $\mathcal{O}_{H_0\cap Q}(0, c_2)$ consists of c_2 disjoint lines belonging to the linear system $|\,\mathcal{O}_{H_0\cap Q}(0,1)\,|$. One sees easily that the image of the composite morphism

$$E \longrightarrow \mathcal{O}_Q^{\oplus 2} \xrightarrow{(-\alpha_2, \alpha_1)} \mathcal{O}_Q$$

is exactly the ideal sheaf $\mathcal{I}_Y$ of Y in Q. Consequently E can be realized as an extension:

$$0 \longrightarrow \mathcal{O}_Q(-1) \longrightarrow E \longrightarrow \mathcal{I}_Y \longrightarrow 0\,. \qquad \square$$

Remark. We take this opportunity to point out some simplifications in the proof of the main result of Coandă [Co92].

(i) The approach to the Standard Construction used in Section 2 of the present paper clarifies, hopefully, the proof of [Co92, Lemma 1].

(ii) The proof of [Co92, Lemma 2] is too complicated. *Indeed*, as it was noticed by Vallès [Va95], one can use the following easy argument: let L be a line in $\mathbb{P}^3$ such that $E|_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L(c_1)$. Let $L^\vee$ be the line in $\mathbb{P}^{3\vee}$ consisting of the planes $H \subset \mathbb{P}^3$ containing L. Then, obviously, $L^\vee \cap W_1(E) = \emptyset$, hence $\dim W_1(E) \leq 1$.

(iii) More important, one can get rid of the Case B in the proof of [Co92, Prop. 2] and, consequently, there's no need of [Co92, Lemma 6]. *Indeed*, under the hypothesis of [Co92, Prop. 2], the first part of the proof of [Co92, Prop. 2] (on page 105) and [Co92, Lemma 5] show that E is a *mathematical instanton*. We *claim* that, in this case, the set Γ of points of $\mathbb{G}_1(\mathbb{P}^3)$ corresponding to the lines $L \subset \mathbb{P}^3$ such that $E|_L \simeq \mathcal{O}_L(c_2) \oplus \mathcal{O}_L(-c_2)$ is a linear section of $\mathbb{G}_1(\mathbb{P}^3)$ in its Plücker embedding in $\mathbb{P}^5$.

Assume the claim, for the moment. Then, by [Co92, Lemma 3] and [Co92, Remark 2], $\dim \Gamma = 1$ hence Γ is a smooth conic, the union of two lines or a line. But, by [Co92, Lemma 4(b)], the lines in $\mathbb{P}^3$ corresponding to the points of Γ are mutually disjoint. Since the lines in $\mathbb{P}^3$ corresponding to a line in $\mathbb{G}_1(\mathbb{P}^3)$ are contained in a fixed plane and pass through a fixed point in that plane, it follows that Γ is a smooth conic, i.e., the lines in $\mathbb{P}^3$ corresponding to the points of Γ form one ruling of a nonsingular quadric surface. Consequently, only the Case A from the proof of [Co92, Prop. 2] can occur.

Finally, let us *prove the above claim* (which is, actually, a well known fact). One has $\mathbb{P}^3 = \mathbb{P}(U)$, where $U = \mathbb{C}^4$. Let $L \subset \mathbb{P}^3$ be a line corresponding to a 2-dimensional vector subspace $\mathbb{C}u \oplus \mathbb{C}v$ of U. Using the coKoszul and Euler exact sequences recalled in par. 3.1, one sees easily that the image of the composite morphism:

$$U \otimes \mathcal{O}_{\mathbb{P}} \longrightarrow \overset{2}{\bigwedge} U \otimes \mathcal{O}_{\mathbb{P}}(1) \xrightarrow{u\wedge v\wedge} \overset{4}{\bigwedge} U \otimes \mathcal{O}_{\mathbb{P}}(1)$$

is $\overset{4}{\bigwedge} U \otimes \mathcal{I}_L(1)$, hence one gets an exact sequence:

$$0 \longrightarrow (\mathbb{C}u \oplus \mathbb{C}v) \otimes \mathcal{O}_{\mathbb{P}} \longrightarrow \mathrm{T}_{\mathbb{P}}(-1) \longrightarrow \overset{4}{\bigwedge} U \otimes \mathcal{I}_L(1) \longrightarrow 0\,.$$

Tensorizing the Euler sequence by $E(-2)$ and taking into account that E is a mathematical instanton, one gets that $\mathrm{H}^1(\mathrm{T}_{\mathbb{P}}(-3) \otimes E) \xrightarrow{\sim} \mathrm{H}^2(E(-3))$ and $\mathrm{H}^2(\mathrm{T}_{\mathbb{P}}(-3) \otimes E) = 0$. Then, tensorizing the last exact sequence by $E(-2)$ one gets that:

$$\mathrm{H}^1(\mathrm{T}_{\mathbb{P}}(-3) \otimes E) \xrightarrow{\sim} \overset{4}{\bigwedge} U \otimes \mathrm{H}^1(\mathcal{I}_L \otimes E(-1)) \;\text{ and }\; \mathrm{H}^2(\mathcal{I}_L \otimes E(-1)) = 0\,.$$

Finally, tensorizing by $\overset{4}{\bigwedge} U \otimes_{\mathbb{C}} E(-1)$ the short exact sequence $0 \to \mathcal{I}_L \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_L \to 0$, one deduces an exact sequence:

$$0 \to \overset{4}{\bigwedge} U \otimes \mathrm{H}^0(E_L(-1)) \to \mathrm{H}^1(\mathrm{T}_{\mathbb{P}}(-3) \otimes E) \xrightarrow{\psi(\ell)} \overset{4}{\bigwedge} U \otimes \mathrm{H}^1(E(-1)) \to \overset{4}{\bigwedge} U \otimes \mathrm{H}^1(E_L(-1)) \to 0$$

where $\psi(\ell)$ is the composite morphism:

$$\mathrm{H}^1(\mathrm{T}_{\mathbb{P}}(-3) \otimes E) \longrightarrow \overset{2}{\bigwedge} U \otimes \mathrm{H}^1(E(-1)) \xrightarrow{u\wedge v\wedge} \overset{4}{\bigwedge} U \otimes \mathrm{H}^1(E(-1))\,.$$

Let ψ be the composite morphism on $\mathbb{P}^5 = \mathbb{P}(\overset{2}{\bigwedge} U)$:

$$\mathcal{O}_{\mathbb{P}^5}(-1) \otimes \mathrm{H}^1(\mathrm{T}_{\mathbb{P}}(-3) \otimes E) \longrightarrow \mathcal{O}_{\mathbb{P}^5} \otimes \overset{2}{\bigwedge} U \otimes \mathrm{H}^1(T_{\mathbb{P}}(-3) \otimes E) \longrightarrow$$
$$\longrightarrow \mathcal{O}_{\mathbb{P}^5} \otimes \overset{2}{\bigwedge} U \otimes \overset{2}{\bigwedge} U \otimes \mathrm{H}^1(E(-1)) \longrightarrow \mathcal{O}_{\mathbb{P}^5} \otimes \overset{4}{\bigwedge} U \otimes \mathrm{H}^1(E(-1))\,.$$

Since $\mathrm{h}^1(E(-1)) = c_2 = \mathrm{h}^2(E(-3))$, one deduces that the above defined set Γ is the intersection of $\mathbb{G}_1(\mathbb{P}^3)$ with the zero scheme of the morphism ψ.

6. Restrictions to singular hyperplane sections

In this section we prove the second theorem from the Introduction. We explicitate, firstly, the notion of stability for a rank 2 vector bundle on a singular hyperplane section of a quadric threefold $Q \subset \mathbb{P}^4$ (which is a quadratic cone) in terms of the pull-back of the bundle on the desingularization of the cone. Then, we consider a simultaneous desingularization of the family of singular hyperplane sections of Q and describe its sheaf of relative Kähler differentials over Q. These are preparatory results for the Standard Construction which we, finally, apply in order to get a proof of Theorem 1.2.

6.1. **Stability on the quadratic cone.** We use the notation from par. 3.1. Recall, also, that we follow *the classical convention for projective bundles.* Let y be a point of Q and let $Y := Q \cap \mathrm{T}_yQ$. One, usually, desingularizes Y by blowing-up its vertex y. In our situation, one can obtain, geometrically, this desingularization as it follows. We viewed Q as the intersection of the Grassmannian $\mathbb{G}_1(\mathbb{P}^3) \hookrightarrow \mathbb{P}^5$ by a hyperplane $\mathbb{P}^4 \subset \mathbb{P}^5$. Recall the incidence diagram:

$$\begin{array}{ccc} \mathbb{F}_{0,1}(Q) & \xrightarrow{\;q\;} & \mathbb{P}^3 \\ {\scriptstyle p}\big\downarrow & & \\ Q & & \end{array} \tag{6.1}$$

If $\ell \in \mathbb{P}^3$ then p maps $q^{-1}(\ell)$ isomorphically onto a line $L \subset Q$ (and, in this way, one gets all the lines contained in Q) and if $x \in Q$ then q maps $p^{-1}(x)$ isomorphically onto a line $L_x \subset \mathbb{P}^3$ (and, in this way, one gets all the jumping lines of the null-correlation bundle N_ω on $\mathbb{P}^3$). Put $\widetilde{Y} := q^{-1}(L_y)$. From diagram (6.1) one deduces an incidence diagram:

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\;\pi_y\;} & L_y \simeq \mathbb{P}^1 \\ {\scriptstyle \sigma_y}\big\downarrow & & \\ Y & & \end{array}$$

Since $\mathbb{F}_{0,1}(Q) \simeq \mathbb{P}(N_\omega^*(-1))$ over $\mathbb{P}^3$ and since L_y is a jumping line for the null-correlation bundle N_ω, it follows that $\widetilde{Y} \simeq \mathbb{P}(\mathcal{O}_{L_y} \oplus \mathcal{O}_{L_y}(-2))$ over L_y (such that $\mathcal{O}_{\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-2))}(-1) \simeq \sigma_y^*\mathcal{O}_Y(-1)$). π maps isomorphically $C_y := \sigma_y^{-1}(y) = p^{-1}(y)$ onto L_y. As $\sigma_y^*\mathcal{O}_Y(-1)|_{C_y} \simeq \mathcal{O}_{C_y}$, it follows that C_y can be identified with $\mathbb{P}(\mathcal{O}_{L_y}) \subset \mathbb{P}(\mathcal{O}_{L_y} \oplus \mathcal{O}_{L_y}(-2))$. One deduces that:

$$\mathcal{O}_{\widetilde{Y}}(C_y) \simeq \sigma_y^*\mathcal{O}_Y(1) \otimes \pi_y^*\mathcal{O}_{L_y}(-2)$$

(*in general*, if E is a vector bundle over a scheme T, if $f : \mathbb{P}(E) \to T$ is the associated projective bundle, and if E' is a vector subbundle of E then $\mathbb{P}(E') \subset \mathbb{P}(E)$ is the zero scheme of the composite morphism $\mathcal{O}_{\mathbb{P}(E)}(-1) \hookrightarrow f^*E \to f^*(E/E')$). Moreover, the restriction of $\sigma_y : \widetilde{Y} \setminus C_y \to Y \setminus \{y\}$ is an isomorphism. As Y is a normal variety, one deduces easily (see, for example, the proof of [Ha77, III, Cor. 11.4]) that $\mathcal{O}_Y \xrightarrow{\sim} \sigma_{y*}\mathcal{O}_{\widetilde{Y}}$. Besides, since $\mathcal{I}_{C_y}/\mathcal{I}_{C_y}^2 \simeq \mathcal{O}_{\mathbb{P}^1}(2)$, one derives, as in the proof of [Ha77, V, Prop. 3.4], that $\mathrm{R}^1\sigma_{y*}\pi_y^*\mathcal{O}_{L_y}(a) = 0$ for $a \geq -1$.

Lemma 6.1. *Let ℓ be a point of L_y and put $\widetilde{L} := \pi_y^{-1}(\ell) \subset \widetilde{Y}$. σ_y maps $\widetilde{L}$ isomorphically onto a line $L \subset Y$. Then $\sigma_{y*}\pi_y^*\mathcal{O}_{L_y}(-1) \simeq \mathcal{I}_{L,Y}$, $\sigma_{y*}\pi_y^*\mathcal{O}_{L_y}(1) \simeq \mathcal{I}_{L,Y}(1)$, and $\mathcal{I}_{L,Y}$ is a rank 1 reflexive sheaf on Y.*

Proof. Applying σ_{y*} to the short exact sequence:

$$0 \longrightarrow \pi_y^*\mathcal{O}_{L_y}(-1) \longrightarrow \mathcal{O}_{\widetilde{Y}} \longrightarrow \mathcal{O}_{\widetilde{L}} \longrightarrow 0$$

one gets an exact sequence:

$$0 \longrightarrow \sigma_{y*}\pi_y^*\mathcal{O}_{L_y}(-1) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_L \longrightarrow 0$$

from which one deduces that $\sigma_{y*}\pi_y^*\mathcal{O}_{L_y}(-1) \simeq \mathcal{I}_{L,Y}$. In order to prove the second isomorphism, one tensorizes by $\pi_y^*\mathcal{O}_{L_y}(1)$ the exact sequence:

$$0 \longrightarrow \mathcal{O}_{\widetilde{Y}} \longrightarrow \mathcal{O}_{\widetilde{Y}}(C_y) \longrightarrow \mathcal{O}_{\widetilde{Y}}(C_y)|_{C_y} \longrightarrow 0$$

and one applies, then, σ_{y*}. Since $(\mathcal{O}_{\widetilde{Y}}(C_y) \otimes \pi_y^*\mathcal{O}_{L_y}(1))|_{C_y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$, one deduces that:

$$\sigma_{y*}\pi_y^*\mathcal{O}_{L_y}(1) \simeq \sigma_{y*}(\mathcal{O}_{\widetilde{Y}}(C_y) \otimes \pi_y^*\mathcal{O}_{L_y}(1)) \simeq \sigma_{y*}(\sigma_y^*\mathcal{O}_Y(1) \otimes \pi_y^*\mathcal{O}_{L_y}(-1)) \simeq \mathcal{O}_Y(1) \otimes \sigma_{y*}\pi_y^*\mathcal{O}_{L_y}(-1) \simeq \mathcal{I}_{L,Y}(1)\,.$$

Finally, applying $\sigma_{y*}\pi_y^*$ to the exact sequence:

$$0 \longrightarrow \mathcal{O}_{L_y}(-1) \longrightarrow \mathcal{O}_{L_y}^{\oplus 2} \longrightarrow \mathcal{O}_{L_y}(1) \longrightarrow 0\,,$$

one gets an exact sequence:

$$0 \longrightarrow \mathcal{I}_{L,Y} \longrightarrow \mathcal{O}_Y^{\oplus 2} \longrightarrow \mathcal{I}_{L,Y}(1) \longrightarrow 0\,.$$

Since $\mathcal{I}_{L,Y}(1)$ is torsion free one deduces that $\mathcal{I}_{L,Y}$ is reflexive (see [Ha80, Prop. 1.1]). □

Lemma 6.2. *Let $L \subset Y$ be a line. If $\mathcal{L}$ is a rank 1 reflexive sheaf on Y then there exists $a \in \mathbb{Z}$ such that $\mathcal{L} \simeq \mathcal{O}_Y(a)$ or $\mathcal{L} \simeq \mathcal{I}_{L,Y}(a)$. Moreover, the dual of $\mathcal{I}_{L,Y}$ is $\mathcal{I}_{L,Y}(1)$.*

Proof. One deduces from [Ha80, Prop. 1.6] that the map $\mathcal{L} \mapsto \mathcal{L}|_{Y\setminus\{y\}}$ is a bijection between the set of isomorphism classes of rank 1 reflexive sheaves on Y and $\mathrm{Pic}(Y \setminus \{y\})$ (the inverse bijection being $M \mapsto j_*M$, where $j : Y \setminus \{y\} \to Y$ is the inclusion map). On the other hand, if $C \subset Y \setminus \{y\}$ is a smooth conic then, applying [Ha77, II, Prop. 6.5] and [Ha77, III, Ex. 12.5] to $\widetilde{Y}$ and C_y, one deduces that the restriction map $\mathrm{Pic}(Y \setminus \{y\}) \to \mathrm{Pic}(C)$ is an isomorphism. We notice, now, that $\mathcal{O}_Y(1)|_C \simeq \mathcal{O}_{\mathbb{P}^1}(2)$, that $\mathcal{I}_{L,Y}|_C \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$ and that $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_{L,Y}, \mathcal{O}_Y)|_C \simeq \mathcal{I}_{L,Y}(1)|_C$. Fixing an isomorphism $\mathbb{P}^1 \xrightarrow{\sim} C$, one gets that if $\mathcal{L}|_C \simeq \mathcal{O}_{\mathbb{P}^1}(2a)$ then $\mathcal{L} \simeq \mathcal{O}_Y(a)$ and if $\mathcal{L}|_C \simeq \mathcal{O}_{\mathbb{P}^1}(2a-1)$ then $\mathcal{L} \simeq \mathcal{I}_{L,Y}(a)$. □

Proposition 6.3. *Let F be a rank 2 vector bundle on Y, with* $\det F \simeq \mathcal{O}_Y(c_1)$*, $c_1 = 0$ or -1. Assume that, for a general line $L \subset Y$, $F|_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L(c_1)$ and that, for a general conic $C \subset Y \setminus \{y\}$, $F|_C \simeq \mathcal{O}_{\mathbb{P}^1}(c_1)^{\oplus 2}$.*

(i) *If $c_1 = 0$ and F is not stable then σ_y^*F can be realized as an extension:*

$$0 \longrightarrow \mathcal{O}_{\widetilde{Y}} \longrightarrow \sigma_y^*F \longrightarrow \mathcal{I}_Z \longrightarrow 0$$

where Z is a 0-dimensional (or empty) subscheme of $\widetilde{Y}$ with $Z \cap C_y = \emptyset$.

(ii) *If* $c_1 = -1$ *and* F *is not stable then* $\sigma_y^* F$ *can be realized as an extension:*

$$0 \longrightarrow \pi_y^* \mathcal{O}_{L_y}(-1) \longrightarrow \sigma_y^* F \longrightarrow \mathcal{I}_Z \otimes \sigma_y^* \mathcal{O}_Y(-1) \otimes \pi_y^* \mathcal{O}_{L_y}(1) \longrightarrow 0$$

where Z *is a* 0*-dimensional (or empty) subscheme of* $\widetilde{Y}$ *such that* $Z \cap C_y = \emptyset$ *or* $Z \cap C_y = 1$ *simple point.*

Proof. By stability we understand, of course, the Mumford-Takemoto stability with respect to $\mathcal{O}_Y(1)$. A general member of the linear system $|\,\mathcal{O}_Y(1)\,|$ is a smooth conic $C \subset Y \setminus \{y\}$.

(i) Since F is not stable it must contain a rank 1 reflexive sheaf $\mathcal{L}$ with $\deg(\mathcal{L}|_C) = 0$. By Lemma 6.2, $\mathcal{L} \simeq \mathcal{O}_Y$ hence $\mathrm{H}^0(F) \neq 0$. As $\sigma_{y*}\sigma_y^* F \simeq F$, it follows that $\mathrm{H}^0(\sigma_y^* F) \neq 0$. Let s be a non-zero global section of $\sigma_y^* F$ and let $Z \subset \widetilde{Y}$ be its zero scheme.

We show, firstly, that $Z \cap C_y = \emptyset$. Indeed, let ℓ be a point of L_y and $\widetilde{L} := \pi_y^{-1}(\ell) \subset \widetilde{Y}$. If ℓ is general then $s|_{\widetilde{L}} \neq 0$. Since, by hypothesis, $\sigma_y^* F|_{\widetilde{L}} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$, it follows that s vanishes at no point of $\widetilde{L}$. It particular, it does not vanish at $\widetilde{L} \cap C_y$. Since $\sigma_y^* F|_{C_y} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$, one deduces that s vanishes at no point of C_y, hence $Z \cap C_y = \emptyset$.

Next, we show that $\dim Z \leq 0$. Indeed, assume that there exists an effective irreducible divisor $D \subset \widetilde{Y}$ such that s vanishes on D. As divisors, $D \sim aC_y + b\widetilde{L}$. Since $D \cap \widetilde{L} = \emptyset$, $a = (D \cdot \widetilde{L}) = 0$. Since $D \cap C_y = \emptyset$, $b = b - 2a = (D \cdot C_y) = 0$. It follows that $D \sim 0$, a contradiction.

Consequently, $\sigma_y^* F$ can be realized as an extension:

$$0 \longrightarrow \mathcal{O}_{\widetilde{Y}} \xrightarrow{\ s\ } \sigma_y^* F \longrightarrow \mathcal{I}_Z \longrightarrow 0\,.$$

(ii) It follows, as in (i), that F has a subsheaf isomorphic to $\mathcal{I}_{L,Y}$, L being a (any) line contained in Y. From Lemma 6.1 and the last part of Lemma 6.2:

$$\sigma_{y*}(\sigma_y^* F \otimes \pi_y^* \mathcal{O}_{L_y}(1)) \simeq F \otimes \mathcal{I}_{L,Y}(1) \simeq \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_{L,Y}, F)\,.$$

One deduces that $\mathrm{H}^0(\sigma_y^* F \otimes \pi_y^* \mathcal{O}_{L_y}(1)) \neq 0$. Let s be a non-zero global section of $\sigma_y^* F \otimes \pi_y^* \mathcal{O}_{L_y}(1)$ and $Z \subset \widetilde{Y}$ its zero scheme.

We show, firstly, that $Z \cap C_y$ is empty or consists of one simple point. Indeed, for a general $\ell \in L_y$, the restriction of s to $\widetilde{L} := \pi_y^{-1}(\ell)$ is non-zero. Since

$$\sigma_y^* F \otimes \pi_y^* \mathcal{O}_{L_y}(1)|_{\widetilde{L}} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

it follows that s vanishes at no point of $\widetilde{L}$. In particular, it does not vanish at $\widetilde{L} \cap C_y$. As:

$$\sigma_y^* F \otimes \pi_y^* \mathcal{O}_{L_y}(1)|_{C_y} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$$

one deduces that $Z \cap C_y$ is empty or consists of one simple point.

Next, we show that Z is 0-dimensional or empty. Indeed, assume that there exists an effective irreducible divisor $D \subset \widetilde{Y}$ such that s vanishes on D. As a divisor, $D \sim aC_y + b\widetilde{L}$. $D \cap \widetilde{L} = \emptyset$ implies that $a = (D \cdot \widetilde{L}) = 0$. Since $D \cap C_y$ is empty or consists of one simple point, it follows that $b = b - 2a = (D \cdot C_y) \in \{0, 1\}$. One deduces that D must be a fibre of $\pi_y : \widetilde{Y} \to L_y$.

But this would imply that $\sigma_y^* F \simeq (\sigma_y^* F \otimes \pi_y^* \mathcal{O}_{L_y}(1)) \otimes \mathcal{O}_{\widetilde{Y}}(-D)$ has a non-zero global section, which contradicts the fact that $\mathrm{H}^0(F) = 0$ (because $F|_C \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$).

Consequently, $\sigma_y^* F$ can be realized as an extension:

$$0 \longrightarrow \pi_y^* \mathcal{O}_{L_y}(-1) \xrightarrow{\ s\ } \sigma_y^* F \longrightarrow \mathcal{I}_Z \otimes \sigma_y^* \mathcal{O}_Y(-1) \otimes \pi_y^* \mathcal{O}_{L_y}(1) \longrightarrow 0 . \quad \square$$

Corollary 6.4. *Under the hypothesis of Prop. 6.3*(ii)*, one has* $\mathrm{h}^0(\sigma_y^* F \otimes \pi_y^* \mathcal{O}_{L_y}(1)) = 1$.

Proof. One tensorizes the exact sequence from the conclusion of Prop. 6.3(ii) by $\pi_y^* \mathcal{O}_{L_y}(1)$ and one uses the fact that $\sigma_y^* \mathcal{O}_Y(-1) \otimes \pi_y^* \mathcal{O}_{L_y}(2) \simeq \mathcal{O}_{\widetilde{Y}}(-C_y)$. $\square$

Remark. For general results concerning the structure of rank 2 vector bundles on ruled surfaces, one may consult the papers of Brossius [Bro83] and Brînzănescu [Br91].

6.2. **Desingularization of the family of singular hyperplane sections.** Consider the subset $X \subset Q \times Q$ defined by $X := \{(x, y) \,|\, \overline{xy} \subset Q\}$, where $\overline{xy}$ is the linear span of $\{x, y\}$ in $\mathbb{P}^4$. If the equation of $Q \subset \mathbb{P}^4$ is $\mathbf{q}(x, x) = 0$, where $\mathbf{q} : \mathbb{C}^5 \times \mathbb{C}^5 \to \mathbb{C}$ is a non-degenerate symmetric bilinear form, then:

$$X = (Q \times Q) \cap \{(x, y) \in \mathbb{P}^4 \times \mathbb{P}^4 \,|\, \mathbf{q}(x, y) = 0\} .$$

Let p_1, $p_2 : X \to Q$ be the restrictions of the canonical projections. If $y \in Q$ then $p_1 : X \to Q$ maps $p_2^{-1}(y)$ isomorphically onto $Y := Q \cap \mathrm{T}_y Q$. One parametrizes, in this way, the singular hyperplane sections of Q in $\mathbb{P}^4$.

Recalling the diagram (6.1), consider the fibre product $\widetilde{X} := \mathbb{F}_{0,1}(Q) \times_{\mathbb{P}^3} \mathbb{F}_{0,1}(Q)$. Since

$$\mathbb{F}_{0,1}(Q) = \{(x, \ell) \in Q \times \mathbb{P}^3 \,|\, x \in L\} ,$$

it follows that

$$\widetilde{X} = \{(x, y, \ell) \in Q \times Q \times \mathbb{P}^3 \,|\, x \in L,\, y \in L\} .$$

Let $\widetilde{p}_1$, $\widetilde{p}_2 : \widetilde{X} \to Q$ be the restrictions of the canonical projections $\mathrm{pr}_1, \mathrm{pr}_2 : Q \times Q \times \mathbb{P}^3 \to Q$, and let $\pi : \widetilde{X} \to \mathbb{P}^3$ be the restriction of the canonical projection $\mathrm{pr}_3 : Q \times Q \times \mathbb{P}^3 \to \mathbb{P}^3$. The canonical projection $\mathrm{pr}_{12} : Q \times Q \times \mathbb{P}^3 \to Q \times Q$ induces a morphism $\sigma : \widetilde{X} \to X$. If Δ is the diagonal of $Q \times Q$ then $\sigma^{-1}(\Delta) = \{(x, x, \ell) \,|\, x \in L\} \simeq \mathbb{F}_{0,1}(Q)$ and σ induces an isomorphism $\widetilde{X} \setminus \sigma^{-1}(\Delta) \xrightarrow{\sim} X \setminus \Delta$. Finally, the canonical projections $\mathrm{pr}_{13}, \mathrm{pr}_{23} : Q \times Q \times \mathbb{P}^3 \to Q \times \mathbb{P}^3$ induce morphisms $p_{13}, p_{23} : \widetilde{X} \to \mathbb{F}_{0,1}(Q)$ and one gets a diagram:

$$(6.2) \qquad \begin{array}{ccccc}
\widetilde{X} & \xrightarrow{\ p_{23}\ } & \mathbb{F}_{0,1} & \xrightarrow{\ p\ } & Q \\
{\scriptstyle p_{13}}\downarrow & & \downarrow{\scriptstyle q} & & \\
\mathbb{F}_{0,1}(Q) & \xrightarrow{\ q\ } & \mathbb{P}^3 & & \\
{\scriptstyle p}\downarrow & & & & \\
Q & & & &
\end{array}$$

having a cartesian square and with $\widetilde{p}_1 = p \circ p_{13} = p_1 \circ \sigma$, $\widetilde{p}_2 = p \circ p_{23} = p_2 \circ \sigma$, $\pi = q \circ p_{13} = q \circ p_{23}$.

Now, let y be a point of Q. $p_{13} : \widetilde{X} \to \mathbb{F}_{0,1}(Q)$ maps $\widetilde{p}_2^{-1}(y)$ isomorphically onto the desingularization $\widetilde{Y}$ of $Y := Q \cap \mathrm{T}_y Q$ considered in par. 6.1. Under this identification, the restriction of $\sigma : \widetilde{X} \to X$ (resp., $\pi : \widetilde{X} \to \mathbb{P}^3$) to $\widetilde{p}_2^{-1}(y)$ is identified with the morphism $\sigma_y : \widetilde{Y} \to Y$ (resp., $\pi_y : \widetilde{Y} \to L_y$) from par. 6.1.

Let $\Omega_{\widetilde{p}_1}$ be the sheaf of relative Kähler differentials of the morphism $\widetilde{p}_1 : \widetilde{X} \to Q$. We want to describe, in this paragraph, the restriction of $\Omega_{\widetilde{p}_1}$ to $\widetilde{p}_2^{-1}(y) \simeq \widetilde{Y}$.

Lemma 6.5. *Let $H := \mathrm{T}_y Q \simeq \mathbb{P}^3$ and let $\mathcal{F}_y$ be the $\mathcal{O}_H$-module defined by the exact sequence:*

$$0 \longrightarrow \mathcal{O}_H(-1) \longrightarrow \mathcal{O}_H^{\oplus 3} \longrightarrow \mathcal{F}_y \longrightarrow 0$$

where the left morphism is the dual of an epimorphism $\mathcal{O}_H^{\oplus 3} \to \mathcal{I}_{\{y\},H}(1)$. Then $\Omega_{\widetilde{p}_1}|_{\widetilde{Y} \setminus C_y}$ can be identified, via the isomorphism $\widetilde{Y} \setminus C_y \xrightarrow{\sim} Y \setminus \{y\}$ induced by σ_y, with $\mathcal{F}_y|_{Y \setminus \{y\}}$.

Proof. $\Omega_{\widetilde{p}_1}|_{\widetilde{Y} \setminus C_y}$ can be identified with $\Omega_{p_1}|_{Y \setminus \{y\}}$ and, by Lemma 2.5(i), $\Omega_{p_1}|_Y$ can be identified with $\mathcal{F}_y|_Y$. □

Remark 6.6. Let T be a scheme, E a vector bundle on T and $f : \mathbb{P}(E) \to T$ the associated (classical) projective bundle. If L is a line bundle on T then $\mathrm{R}^i f_*(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes f^* L) = 0$ for $i \geq 1$. One deduces that the canonical morphism:

$$\mathrm{H}^1(E^* \otimes L) \simeq \mathrm{H}^1(f_*(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes f^* L)) \longrightarrow \mathrm{H}^1(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes f^* L)$$

is an isomorphism.

Now, let E' be a vector subbundle of E and let $f' : \mathbb{P}(E') \to T$ be the associated projective bundle. One has

$$(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes f^* L)|_{\mathbb{P}(E')} \simeq \mathcal{O}_{\mathbb{P}(E')}(1) \otimes f'^* L$$

and, from the above observation, the restriction morphism:

$$\mathrm{H}^1(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes f^* L) \longrightarrow \mathrm{H}^1(\mathcal{O}_{\mathbb{P}(E')}(1) \otimes f'^* L)$$

can be identified with the morphism $\mathrm{H}^1(E^* \otimes L) \to \mathrm{H}^1(E'^* \otimes L)$.

Proposition 6.7. *One has a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\widetilde{Y}} & \longrightarrow & \pi_y^*(\mathcal{O}_{L_y}(1)^{\oplus 2}) & \longrightarrow & \pi_y^* \mathcal{O}_{L_y}(2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \scriptstyle{\mathrm{id}} & & \\ 0 & \longrightarrow & \mathcal{O}_{\widetilde{Y}}(C_y) & \longrightarrow & \Omega_{\widetilde{p}_1}|_{\widetilde{p}_2^{-1}(y)} & \longrightarrow & \pi_y^* \mathcal{O}_{L_y}(2) & \longrightarrow & 0 \end{array}$$

where the left vertical morphism is the canonical one.

Proof. Using the diagram (6.2), one gets an exact sequence:

$$0 \longrightarrow p_{13}^* \Omega_p \longrightarrow \Omega_{\widetilde{p}_1} \longrightarrow \Omega_{p_{13}} \longrightarrow 0$$

and an isomorphism $\Omega_{p_{13}} \simeq p_{23}^* \Omega_q$. Since $\mathbb{F}_{0,1}(Q) \simeq \mathbb{P}(\mathcal{S})$ over Q and $\mathbb{F}_{0,1}(Q) \simeq \mathbb{P}(N_\omega^*(-1))$ over $\mathbb{P}^3$, it follows, from [Ha77, III, Ex. 8.4(b)] (recall that Hartshorne uses Grothendieck's convention for projective bundles), that $\Omega_p \simeq p^*\mathcal{O}_Q(1) \otimes q^*\mathcal{O}_{\mathbb{P}^3}(-2)$ and $\Omega_q \simeq q^*\mathcal{O}_{\mathbb{P}^3}(2) \otimes p^*\mathcal{O}_Q(-2)$. One deduces an exact sequence:

$$0 \longrightarrow \widetilde{p}_1^*\mathcal{O}_Q(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^3}(-2) \longrightarrow \Omega_{\widetilde{p}_1} \longrightarrow \pi^*\mathcal{O}_{\mathbb{P}^3}(2) \otimes \widetilde{p}_2^*\mathcal{O}_Q(-2) \longrightarrow 0$$

which restricted to $\widetilde{p}_2^{-1}(y)$ gives us an exact sequence:

$$(6.3) \quad 0 \longrightarrow \sigma_y^*\mathcal{O}_Y(1) \otimes \pi_y^*\mathcal{O}_{L_y}(-2) \longrightarrow \Omega_{\widetilde{p}_1}|_{\widetilde{p}_2^{-1}(y)} \longrightarrow \pi_y^*\mathcal{O}_{L_y}(2) \longrightarrow 0\,.$$

This extension of line bundles correspondes to an element $\varepsilon \in \mathrm{H}^1(\sigma_y^*\mathcal{O}_Y(1)\otimes \pi_y^*\mathcal{O}_{L_y}(-4))$. Recalling that $\widetilde{Y} \simeq \mathbb{P}(\mathcal{O}_{L_y} \oplus \mathcal{O}_{L_y}(-2))$ over L_y, one gets, from Remark 6.6, a canonical isomorphism:

$$\mathrm{H}^1(\sigma_y^*\mathcal{O}_Y(1) \otimes \pi_y^*\mathcal{O}_{L_y}(-4)) \simeq \mathrm{H}^1(\mathcal{O}_{L_y}(-4) \oplus \mathcal{O}_{L_y}(-2))\,.$$

Now, $C_y \subset \widetilde{Y}$ can be (and was) identified with the projective subbundle $\mathbb{P}(\mathcal{O}_{L_y})$ of $\mathbb{P}(\mathcal{O}_{L_y} \oplus \mathcal{O}_{L_y}(-2))$. One deduces, from the second part of Remark 6.6, that the image of ε by the canonical morphism $\mathrm{H}^1(\mathcal{O}_{L_y}(-4) \oplus \mathcal{O}_{L_y}(-2)) \to \mathrm{H}^1(\mathcal{O}_{L_y}(-4))$ corresponds to the restriction of the extension (6.3) to C_y. But $C_y := \sigma_y^{-1}(y) = \widetilde{p}_1^{-1}(y) \cap \widetilde{p}_2^{-1}(y)$ hence:

$$\Omega_{\widetilde{p}_1}|_{C_y} \simeq (\Omega_{\widetilde{p}_1}|_{\widetilde{p}_1^{-1}(y)})|_{C_y} \simeq \Omega_{\widetilde{p}_1^{-1}(y)}|_{C_y}\,.$$

Since $\pi : \widetilde{X} \to \mathbb{P}^3$ maps $\widetilde{p}_1^{-1}(y)$ onto L_y and induces an isomorphism $C_y \overset{\sim}{\to} L_y$, it follows that the inclusion $C_y \hookrightarrow \widetilde{p}_1^{-1}(y)$ admits a left inverse $\widetilde{p}_1^{-1}(y) \to C_y$ which implies that $\Omega_{C_y} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$ is a direct summand of $\Omega_{\widetilde{p}_1^{-1}(y)}|_{C_y}$. One deduces that the restriction to C_y of the exact sequence (6.3) *splits*, hence the image of ε into $\mathrm{H}^1(\mathcal{O}_{L_y}(-4))$ is 0. The kernel of the canonical morphism $\mathrm{H}^1(\mathcal{O}_{L_y}(-4) \oplus \mathcal{O}_{L_y}(-2)) \to \mathrm{H}^1(\mathcal{O}_{L_y}(-4))$ is a 1-dimensional $\mathbb{C}$-vector space. Consequently, it remains only to decide whether the exact sequence (6.3) splits or not.

Let $C \subset Y \setminus \{y\}$ be a smooth conic and let $\widetilde{C} := \sigma_y^{-1}(C) \subset \widetilde{Y}$. By Lemma 6.5, $\Omega_{\widetilde{p}_1}|_{\widetilde{C}} \simeq \mathcal{F}_y|_C$. Let $H := \mathrm{T}_yQ$ and let $P \subset H$ be the 2-plane spanned by C. One has $\mathcal{F}_y|_P \simeq \mathrm{T}_P(-1)$ and $\mathrm{T}_P(-1)|_C \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ (because $\mathrm{H}^0(\mathrm{T}_P(-2)|_C) = 0$) hence the restriction of (6.3) to $\widetilde{C}$ *does not split.*

Recalling that $\sigma_y^*\mathcal{O}_Y(1) \otimes \pi_y^*\mathcal{O}_{L_y}(-2) \simeq \mathcal{O}_{\widetilde{Y}}(C_y)$, one concludes that the exact sequence (6.3) is obtained by pushing-forward the exact sequence:

$$0 \longrightarrow \mathcal{O}_{\widetilde{Y}} \longrightarrow \pi_y^*(\mathcal{O}_{L_y}(1)^{\oplus 2}) \longrightarrow \pi_y^*\mathcal{O}_{L_y}(2) \longrightarrow 0$$

via a non-zero morphism $\mathcal{O}_{\widetilde{Y}} \to \mathcal{O}_{\widetilde{Y}}(C_y)$. $\square$

Corollary 6.8. *Let F denote the rank 2 vector bundle $\Omega_{\widetilde{p}_1}|_{\widetilde{p}_2^{-1}(y)}$ on $\widetilde{p}_2^{-1}(y) \simeq \widetilde{Y}$.*

(a) *If s is a non-zero global section of F and $Z \subset \widetilde{Y}$ its zero scheme then $Z \cap (\widetilde{Y} \setminus C_y) = \emptyset$ or there exists a fibre $\widetilde{L} = \pi_y^{-1}(\ell)$ of $\pi_y : \widetilde{Y} \to L_y$ such that $Z \cap (\widetilde{Y} \setminus C_y) = \widetilde{L} \setminus C_y$ as closed subschemes of $\widetilde{Y} \setminus C_y$.*

(b) $\mathrm{h}^0(F \otimes \mathcal{O}_{\widetilde{Y}}(-C_y)) = 1$ *and a non-zero global section s of $F \otimes \mathcal{O}_{\widetilde{Y}}(-C_y)$ vanishes nowhere.*

Proof. (a) Applying the Snake Lemma to the commutative diagram in the statement of Prop. 6.7 one gets an exact sequence:

$$0 \longrightarrow \pi_y^*(\mathcal{O}_{L_y}(1)^{\oplus 2}) \longrightarrow F \longrightarrow \mathcal{O}_{\widetilde{Y}}(C_y)|_{C_y} \longrightarrow 0\,.$$

Since $\mathcal{O}_{\widetilde{Y}}(C_y)|_{C_y} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$, s comes from a global section s' of $\pi_y^*(\mathcal{O}_{L_y}(1)^{\oplus 2})$. The assertion from the statement becomes, now, obvious.

(b) Tensorizing the bottom line of the diagram in the statement of Prop. 6.7 by $\mathcal{O}_{\widetilde{Y}}(-C_y) \simeq \sigma_y^*\mathcal{O}_{\widetilde{Y}}(-1) \otimes \pi_y^*\mathcal{O}_{L_y}(2)$ one gets an exact sequence:

$$0 \longrightarrow \mathcal{O}_{\widetilde{Y}} \longrightarrow F \otimes \mathcal{O}_{\widetilde{Y}}(-C_y) \longrightarrow \sigma_y^*\mathcal{O}_{\widetilde{Y}}(-1) \otimes \pi_y^*\mathcal{O}_{L_y}(4) \longrightarrow 0\,.$$

Since $\pi_{y*}(\sigma_y^*\mathcal{O}_{\widetilde{Y}}(-1)\otimes\pi_y^*\mathcal{O}_{L_y}(4)) = 0$ it follows that $\mathrm{H}^0(\sigma_y^*\mathcal{O}_{\widetilde{Y}}(-1)\otimes\pi_y^*\mathcal{O}_{L_y}(4)) = 0$ and the assertion from the statement becomes, now, obvious. □

6.3. **Use of the Standard Construction.** We are, now, ready to give the

Proof of Theorem 1.2. By [ES84, Prop. 1.3], for a general line $L \subset Q$, one has $\mathcal{E}|_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L(c_1)$. By [ES84, Cor. 1.5] (see, also, Prop. 2.4), for a general conic $C \subset Q$, one has $\mathcal{E}|_C \simeq \mathcal{O}_{\mathbb{P}^1}(c_1)^{\oplus 2}$. Moreover, every line $L \subset Q$ and every conic $C \subset Q$ is contained in a singular hyperplane section of Q. Assume now that, for a general point $y \in Q$ as in the statement, the restriction of $\mathcal{E}$ to $Y := Q \cap \mathrm{T}_yQ$ is not stable. Then, by Prop. 6.3, there exists a non-empty open subset $\mathcal{V}$ of $Q \setminus \bigcup_{x\in \mathrm{Sing}\,\mathcal{E}} \mathrm{T}_xQ$ such that, for $y \in \mathcal{V}$, the rank 2 vector bundle $\sigma_y^*\mathcal{E}$ on $\widetilde{Y}$ can be realized, in the case $c_1(\mathcal{E}) = 0$, as an extension:

$$0 \longrightarrow \mathcal{O}_{\widetilde{Y}} \longrightarrow \sigma_y^*\mathcal{E} \longrightarrow \mathcal{I}_Z \longrightarrow 0$$

where Z is a 0-dimensional subscheme of $\widetilde{Y}$, of length $c_2(\mathcal{E})$, and such that $Z \cap C_y = \emptyset$, and, in the case $c_1(\mathcal{E}) = -1$, it can be realized as an extension:

$$0 \longrightarrow \pi_y^*\mathcal{O}_{L_y}(-1) \longrightarrow \sigma_y^*\mathcal{E} \longrightarrow \mathcal{I}_Z \otimes \sigma_y^*\mathcal{O}_Y(-1) \otimes \pi_y^*\mathcal{O}_{L_y}(1) \longrightarrow 0$$

where Z is a 0-dimensional subscheme of $\widetilde{Y}$, of length $c_2(\mathcal{E}) - 1$, and such that $Z \cap C_y = \emptyset$ or $Z \cap C_y$ consists of a simple point. Consider the diagram:

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\overline{p}_2} & \mathcal{V} \\ {\scriptstyle \overline{p}_1}\big\downarrow & & \\ Q \setminus \mathrm{Sing}\,\mathcal{E} & & \end{array}$$

where $\overline{X} := \widetilde{p}_2^{-1}(\mathcal{V}) \subset \widetilde{X}$ and $\overline{p}_i$ is the restriction of $\widetilde{p}_i$, $i = 1, 2$. In the case $c_1(\mathcal{E}) = 0$, $\overline{p}_1^*\mathcal{E}$ can be realized as an extension:

$$0 \longrightarrow \mathcal{L} \longrightarrow \overline{p}_1^*\mathcal{E} \longrightarrow \mathcal{I}_{\mathcal{Z}} \otimes \mathcal{L}^{-1} \longrightarrow 0$$

where $\mathcal{L}$ is the line bundle $\overline{p}_2^*\overline{p}_{2*}\overline{p}_1^*\mathcal{E}$ and $\mathcal{Z}$ is a closed subscheme of $\overline{X}$ of codimension ≥ 2. In the case $c_1(\mathcal{E}) = -1$, $\overline{p}_1^*\mathcal{E}$ can be realized as an extension:

$$0 \longrightarrow (\pi^*\mathcal{O}_{\mathbb{P}^3}(-1)|_{\overline{X}})\otimes\mathcal{L} \longrightarrow \overline{p}_1^*\mathcal{E} \longrightarrow \mathcal{I}_{\mathcal{Z}}\otimes\overline{p}_1^*\mathcal{O}_Q(-1)\otimes(\pi^*\mathcal{O}_{\mathbb{P}^3}(1)|_{\overline{X}})\otimes\mathcal{L}^{-1} \longrightarrow 0$$

where $\mathcal{L}$ is the line bundle $\overline{p}_2^*\overline{p}_{2*}(\overline{p}_1^*\mathcal{E} \otimes (\pi^*\mathcal{O}_{\mathbb{P}^3}(1)|_{\overline{X}}))$ (see Cor. 6.4) and $\mathcal{Z}$ is a closed subscheme of $\overline{X}$ of codimension ≥ 2.

Since there exists no saturated subsheaf $\overline{\mathcal{E}}'$ of $\mathcal{E}|_{Q\setminus\mathrm{Sing}\,\mathcal{E}}$ such that, for every $y \in \mathcal{V}$, $\sigma_y^*(\overline{\mathcal{E}}'|_Y) \simeq \mathcal{O}_{\widetilde{Y}}$ in the case $c_1(\mathcal{E}) = 0$, and such that $\sigma_y^*(\overline{\mathcal{E}}'|_Y) \simeq$

$\pi_y^*\mathcal{O}_{L_y}(-1)$ in the case $c_1(\mathcal{E}) = -1$, one deduces, from Prop. 2.2, the existence of a non-zero morphism:

$$\mathcal{L} \longrightarrow (\Omega_{\widetilde{p}_1}|_{\overline{X}}) \otimes \mathcal{I}_{\mathcal{Z}} \otimes \mathcal{L}^{-1} \quad (c_1 = 0)$$

$$(\pi^*\mathcal{O}_{\mathbb{P}^3}(-1)|_{\overline{X}}) \otimes \mathcal{L} \longrightarrow (\Omega_{\widetilde{p}_1}|_{\overline{X}}) \otimes \mathcal{I}_{\mathcal{Z}} \otimes \overline{p}_1^*\mathcal{O}_Q(-1) \otimes (\pi^*\mathcal{O}_{\mathbb{P}^3}(1)|_{\overline{X}}) \otimes \mathcal{L}^{-1} \quad (c_1 = -1)$$

The restriction of such a morphism to a general fibre of $\overline{p}_2 : \overline{X} \to \mathcal{V}$ is generically non-zero. One deduces, for a general $y \in \mathcal{V}$, the existence of a non-zero morphism:

$$\mathcal{O}_{\widetilde{Y}} \longrightarrow (\Omega_{\widetilde{p}_1}|_{\widetilde{p}_2^{-1}(y)}) \otimes \mathcal{I}_{\mathcal{Z}_y} \quad (\text{for } c_1(\mathcal{E}) = 0)$$

$$\pi_y^*\mathcal{O}_{L_y}(-1) \longrightarrow (\Omega_{\widetilde{p}_1}|_{\widetilde{p}_2^{-1}(y)}) \otimes \mathcal{I}_{\mathcal{Z}_y} \otimes \sigma_y^*\mathcal{O}_Y(-1) \otimes \pi_y^*\mathcal{O}_{L_y}(1) \quad (\text{for } c_1(\mathcal{E}) = -1)$$

This means that, in the case $c_1(\mathcal{E}) = 0$, the rank 2 vector bundle $\Omega_{\widetilde{p}_1}|_{\widetilde{p}_2^{-1}(y)}$ on $\widetilde{p}_2^{-1}(y) \simeq \widetilde{Y}$ has a non-zero global section vanishing on the scheme $\mathcal{Z}_y$, and that, in the case $c_1(\mathcal{E}) = -1$, the vector bundle $(\Omega_{\widetilde{p}_1}|_{\widetilde{p}_2^{-1}(y)}) \otimes \sigma_y^*\mathcal{O}_Y(-1) \otimes \pi_y^*\mathcal{O}_{L_y}(2) \simeq (\Omega_{\widetilde{p}_1}|_{\widetilde{p}_2^{-1}(y)}) \otimes \mathcal{O}_{\widetilde{Y}}(-C_y)$ has a global section vanishing on the scheme $\mathcal{Z}_y$.

One uses, now, Cor. 6.8. In the case $c_1(\mathcal{E}) = 0$, it follows that $\mathcal{Z}_y$ is a subscheme of a fibre $\widetilde{L} = \pi_y^{-1}(\ell)$ of $\pi_y : \widetilde{Y} \to L_y$ (recall that $\mathcal{Z}_y \cap C_y = \emptyset$). $\sigma_y : \widetilde{Y} \to Y$ maps $\widetilde{L}$ isomorphically onto a line $L \subset Y$. Since $\mathcal{Z}_y$ has length $c_2 := c_2(\mathcal{E})$, one deduces that $\mathcal{E}|_L \simeq \mathcal{O}_L(c_2) \oplus \mathcal{O}_L(-c_2)$. Consequently, through any general point $y \in Q$ passes a line $L \subset Q$ such that $\mathcal{E}|_L \simeq \mathcal{O}_L(c_2) \oplus \mathcal{O}_L(-c_2)$. But this contradicts Lemma 5.10.

Finally, in the case $c_1(\mathcal{E}) = -1$, it follows that $\mathcal{Z}_y = \emptyset$, i.e., that $\sigma_y^*\mathcal{E}$ can be realized as an extension:

$$0 \longrightarrow \pi_y^*\mathcal{O}_{L_y}(-1) \longrightarrow \sigma_y^*\mathcal{E} \longrightarrow \sigma_y^*\mathcal{O}_Y(-1) \otimes \pi_y^*\mathcal{O}_{L_y}(1) \longrightarrow 0\,.$$

On the other hand, if L is a line contained in Y then, using the fact that there is an epimorphism $\mathcal{S} \to \mathcal{I}_L$, one gets an exact sequence:

$$0 \longrightarrow \mathcal{I}_{L,Y} \longrightarrow \mathcal{S}|_Y \longrightarrow \mathcal{I}_{L,Y} \longrightarrow 0\,.$$

One derives, as in the proof of Prop. 6.3(ii), the existence of an exact sequence:

$$0 \longrightarrow \pi_y^*\mathcal{O}_{L_y}(-1) \longrightarrow \sigma_y^*\mathcal{S} \longrightarrow \sigma_y^*\mathcal{O}_Y(-1) \otimes \pi_y^*\mathcal{O}_{L_y}(1) \longrightarrow 0\,.$$

Using the first part of Remark 6.6 one gets that:

$$\mathrm{H}^1(\sigma_y^*\mathcal{O}_Y(1) \otimes \pi_y^*\mathcal{O}_{L_y}(-2)) \simeq \mathrm{H}^1(\mathcal{O}_{L_y}(-2) \oplus \mathcal{O}_{L_y}) \simeq \mathbb{C}\,.$$

It follows that $\sigma_y^*\mathcal{E} \simeq \sigma_y^*\mathcal{S}$ hence $\mathcal{E}|_Y \simeq \mathcal{S}|_Y$. This implies, as in the last part of the proof of Remark 5.9, that $\mathcal{E}$ is isomorphic to the spinor bundle $\mathcal{S}$. □

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