LINEAR SPACES OF MATRICES OF CONSTANT RANK AND INSTANTON BUNDLES

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Abstract. We present a new method to study 4-dimensional linear spaces of skew-symmetric matrices of constant co-rank 2, based on rank 2 vector bundles on $\mathbb{P}^3$ and derived category tools. The method allows one to prove the existence of new examples of size 10×10 and 14×14 via instanton bundles of charge 2 and 4 respectively, and it provides an explanation for what used to be the only known example (Westwick 1996). We also give an algorithm to construct explicitly a matrix of size 14 of this type.

1. Introduction

Given two vector spaces V and W of dimension n and m respectively, we consider a d-dimensional vector subspace A of $V \otimes W \simeq \mathrm{Hom}(V^*, W)$. Fixing bases, we can either write down A as an $m \times n$ matrix whose entries are linear forms in d variables, or think of A as a matrix whose coefficients are linear forms, i.e. a map $V^* \otimes \mathcal{O}_{\mathbb{P}^{d-1}} \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}^{d-1}}(1)$. We say that A has constant rank if every non-zero element of A has the same rank, or equivalently if the map $V^* \otimes \mathcal{O}_{\mathbb{P}^{d-1}} \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}^{d-1}}(1)$, evaluated at every point of $\mathbb{P}^{d-1}$, has the same rank.

The interest for this kind of matrices, founded on the classical work by Kronecker and Weierstrass, bears to different contexts: linear algebra, theory of degeneracy of vector bundles, study of varieties with degenerate dual, to cite only a few. When analysing these matrices, algebraic geometry appears as a natural tool, as perhaps first observed by J. Sylvester [Syl86]; for instance characteristic classes of vector bundles prove to be very useful. Indeed, since A has constant rank, say r, we obtain a sequence of vector bundles as kernel and cokernel of A:

$$0 \to K_A \to V^* \otimes \mathcal{O}_{\mathbb{P}^{d-1}} \xrightarrow{A} W \otimes \mathcal{O}_{\mathbb{P}^{d-1}}(1) \to N_A \to 0. \tag{1.1}$$

The computation of Chern classes of these bundles yields restrictions on the values that r, n, m and d can attain. Nevertheless, the problem of finding an optimal upper bound on the dimension of the linear system A, given the size and the (constant) rank of the matrices involved, is widely open in many cases. For a more extensive introduction to the topic we refer to [IL99] and the numerous references therein, such as [Wes96, Wes87, EH88]. In this work we deal with the special case when A is skew-symmetric. Very little is known about it, contrary to its symmetric counterpart. In our setting the rank r is even and the maximal dimension $l(r,n)$ of a linear subspace A as above is comprised between $n-r+1$ and $2(n-r)+1$. Under the particular hypothesis $r = n-2$, then $3 \le l(r, r+2) \le 5$, and both the kernel and cokernel bundles K_A and N_A have rank 2. The initial cases with $n \le 8$ have been studied in [MM05, FM11]. It turns out that $l(4,6) = l(6,8) = 3$.

Here, we are mostly interested in the case $d = 4$. When we began our research, the only known 4-dimensional space of this kind was one with $r = 8$, presented in [Wes96] without any

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explanation. We reproduce it in formula (5.1). The analysis of this "mysterious" example has been the starting point of our investigation.

For $d = 4$, given a skew-symmetric matrix A as above of size $(r+2)$ and constant rank r, one has that N_A and K_A are isomorphic respectively to $E(\frac{r}{4}+1)$ and $E(-\frac{r}{4})$, where E is a vector bundle of rank 2 with $c_1(E) = 0$ and $c_2(E) = \frac{r(r+4)}{48}$. The (twisted) sequence (1.1) then reads:

$$0 \to E(-\tfrac{r}{4}-2) \to \mathcal{O}_{\mathbb{P}^3}(-2)^{r+2} \xrightarrow{A} \mathcal{O}_{\mathbb{P}^3}(-1)^{r+2} \to E(\tfrac{r}{4}-1) \to 0. \tag{1.2}$$

The main question treated in this paper is how to reverse the construction, namely, how to start from a vector bundle E of rank 2 on $\mathbb{P}^3$ and obtain a skew-symmetric matrix A of linear forms on $\mathbb{P}^3$ having constant co-rank 2. (A will then have E as kernel, up to a twist by a line bundle.) In fact, besides E, one more ingredient is needed, namely a class $\beta \in \mathrm{Ext}^2(E(\frac{r}{4}-1), E(-\frac{r}{4}-2))$ corresponding to an extension of type (1.2). This is where the first new tool from algebraic geometry comes into play: derived category theory. The main idea is that, given E with the allowed Chern classes $c_1(E) = 0$ and $c_2(E) = \frac{r(r+4)}{48}$, and a class β as above, we can obtain a 2-term complex $\mathcal{C}$ as cone of β, interpreted as a morphism $E(\frac{r}{4}-1) \to E(-\frac{r}{4}-2)[2]$ in the derived category $\mathrm{D}^b(\mathbb{P}^3)$. The next step entails using Beilinson's theorem to show that, under some non-degeneracy conditions of the maps $\mu_t^p : \mathrm{H}^p(E(\frac{r}{4}-1+t)) \to \mathrm{H}^{p+2}(E(-\frac{r}{4}-2+t))$ induced by β, the complex $\mathcal{C}$ is of the form $\mathcal{O}_{\mathbb{P}^3}(-2)^{r+2} \to \mathcal{O}_{\mathbb{P}^3}(-1)^{r+2}$, sitting in degrees -2 and -1. The desired matrix A then appears as differential of $\mathcal{C}$. To complete the argument, we show that A is necessarily skew-symmetrizable. This is the content of the main Theorem 3.3. For the reader not familiar with the subject, Section 3 also contains a brief review of the most important features of derived category theory, with emphasis on the tools and techniques that are used in this paper.

Our next results aim at reducing the number of requirements on the maps μ_t^p's by imposing convenient assumptions on E. First, in Theorem 3.6 we show that these conditions can be significantly simplified if E has natural cohomology, i.e. $\mathrm{H}^p(E(t)) \neq 0$ for at most one p, for all t. Here is where the next algebro-geometric ingredient comes in, namely instanton bundles ("k-instantons" if $c_2(E) = k$), first introduced in [AHDM78]. Indeed, general instantons have natural cohomology, see [HH82]. Next, we use the description of the minimal graded free resolution of a general instanton E and of its cohomology module $\bigoplus_{t\in\mathbb{Z}} \mathrm{H}^2(E(t))$ (cf. [Rah97, Dec90]) to further reduce the requirements on the μ_t^p's to a *single condition*. This is done in Theorem 4.1.

Let us outline the main applications we draw from this method.

(1) Westwick's example is given an explanation in terms of 2-instantons. Taking advantage of the extensive literature (for example [Har78, New81, CO02]) on the moduli space $M_{\mathbb{P}^3}(2;0,2)$ of rank 2 stable vector bundles on $\mathbb{P}^3$ with Chern classes $c_1 = 0$ and $c_2 = 2$, we determine that the "Westwick instanton" belongs to the most special orbit of $M_{\mathbb{P}^3}(2;0,2)$ under the natural action of SL(4), see Theorem 5.1.
(2) We show the existence of a continuous family of examples of 10×10 matrices of rank 8, all non-equivalent to Westwick's one. These are obtained by showing that all 2-instantons have classes β satisfying the required condition, see Theorem 5.2.
(3) We show the existence of a continuous family of new examples of 14×14 matrices of rank 12, starting with general 4-instantons, see Theorem 6.1.
(4) We exhibit an explicit example of a 14×14 skew-symmetric matrix of constant rank 12 in 4 variables, together with an algorithm capable of constructing infinitely many of them, see the Appendix.

In all instances above, β is a general element of $\mathrm{Ext}^2(E(\frac{r}{4}-1), E(-\frac{r}{4}-2))$. This is not at all surprising: the non-degeneracy conditions on the μ_t^p's are open, so once an element β_0 satisfies the requirements, the same will hold for the general element β. Still, the construction of examples of higher size does not seem straightforward. However, we believe that this method is promising, and that it can be potentially applied to other varieties than $\mathbb{P}^3$, and to other linear subspaces than the one we considered. We would also like to develop algorithmic methods to find explicit expressions of our new examples, generalising what is done in the Appendix. We are currently working on these further applications.

The paper is organised as follows: in Section 2 we introduce the problem and the state of the art, focusing on the case of 4-dimensional linear subspaces of skew-symmetric matrices of constant co-rank 2. Section 3 contains our main results. Here we characterise these spaces in terms of necessary and sufficient conditions to impose on the maps μ_t^p introduced above. In Section 4 we show how and why the most natural candidates for our purposes are general instantons. The last Sections 5 and 6 are devoted to show the existence of new examples. Finally in the Appendix we give an algorithmic method that explicitly gives a new matrix of size 14.

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2. General set-up and the co-rank two case

Here we introduce some preliminary material. We work over the field $\mathbb{C}$ of complex numbers, although all the constructions seem to carry over smoothly to any algebraically closed field of characteristic other than 2. Given a vector space W over $\mathbb{C}$, of dimension d, we denote by $W^* = \mathrm{Hom}(W, \mathbb{C})$ its dual, and we fix a determinant form so that $W \simeq W^*$. The projective space $\mathbb{P}^{d-1} = \mathbb{P}W$ is the space of lines through 0, thus $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}W}(1)) = W^*$.

Any claim about a *general element* in a given parameter space means that the claim holds for all elements away from a countable union of Zariski closed subsets of the parameters.

2.1. **General set-up.** Let V be a vector space of dimension n over $\mathbb{C}$, and let $A \subseteq V \otimes V$ be a linear subspace of dimension d. We say that A has *constant rank* r if every non-zero element of A has rank r. Given a basis for V, we can write down A as a $n \times n$ matrix of linear forms in d variables, that we denote by the same letter A.

The matrix A can be viewed naturally as a map $V^* \otimes \mathcal{O}_{\mathbb{P}A} \xrightarrow{A} V \otimes \mathcal{O}_{\mathbb{P}A}(1)$, inducing the following exact sequence, where K_A and N_A denote the kernel and cokernel of A, and $\mathcal{E}_A$ its image:

$$0 \longrightarrow K_A \longrightarrow V^* \otimes \mathcal{O}_{\mathbb{P}A} \xrightarrow{\quad A \quad} V \otimes \mathcal{O}_{\mathbb{P}A}(1) \longrightarrow N_A \longrightarrow 0 \,. \tag{2.1}$$

$$0 \longrightarrow \mathcal{E}_A \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}A}(1), \qquad V^* \otimes \mathcal{O}_{\mathbb{P}A} \longrightarrow \mathcal{E}_A \longrightarrow 0$$

From the assumption that A has constant rank r we deduce that K_A, N_A and $\mathcal{E}_A$ are vector bundles on $\mathbb{P}A$. (That are of rank $n-r$, $n-r$ and r respectively.)

A computation of invariants of these vector bundles shows that there is a bound on the maximal dimension $l(r,n)$ that such a subspace $A \subseteq V \otimes V$ can attain. Westwick [Wes87] proved that for $2 \le r \le n$:

$$n - r + 1 \le l(r,n) \le 2(n-r) + 1. \tag{2.2}$$

Moreover for a given $d = \dim A$ with $n - r + 1 \le d \le 2(n-r) + 1$, only some values of r are allowed. We refer to [IL99, Wes87, Wes96] for further details.

We now focus on the special case $r = n-2$. In this case K_A and N_A are vector bundles of rank 2, and the bounds (2.2) on the maximal dimension of A become $3 \le l(r, r+2) \le 5$.

Specialising even more, let us now suppose that the subspace A lies either in $\wedge^2 V$ or in $S^2 V$. Then the skew-symmetry (or the symmetry) of the matrix yields a symmetry of the exact sequence (2.1). In particular: $N_A \simeq K_A^*(1)$, and $\mathcal{E}_A^* \simeq \mathcal{E}_A(1)$. The same computation of invariants as above entails that the first Chern class $c_1(\mathcal{E}) = c_1(K^*) = \frac{r}{2}$, and thus the rank r is even. Moreover from the rank 2 hypothesis we get that $K_A^* \simeq K_A(\frac{r}{2})$.

Remark 2.1. The natural action of $\mathrm{SL}(n)$ on V extends to an action on the linear subspaces of $\wedge^2 V$ (or $S^2 V$). Let $g \in \mathrm{SL}(n)$ act on a subspace A, and denote $A' = gA$. Then we have a commutative diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & K_A & \longrightarrow & V^* \otimes \mathcal{O}_{\mathbb{P}A} & \xrightarrow{A} & V \otimes \mathcal{O}_{\mathbb{P}A}(1) & \longrightarrow & K_A^*(1) & \longrightarrow 0 \\
 & \uparrow{\scriptstyle g^*} & & \uparrow{\scriptstyle g^*} & & \uparrow{\scriptstyle g^*} & & \uparrow{\scriptstyle g^*} & \\
0 \longrightarrow & K_{A'} & \longrightarrow & V^* \otimes \mathcal{O}_{\mathbb{P}A'} & \xrightarrow{A'} & V \otimes \mathcal{O}_{\mathbb{P}A'}(1) & \longrightarrow & K_{A'}^*(1) & \longrightarrow 0
\end{array}$$

which shows that the kernel bundles K_A and $K_{A'}$ are isomorphic. Similarly, if $h \in \mathrm{SL}(d)$, then $h : A \to A$ induces:

$$\begin{array}{ccccccccc}
0 \longrightarrow & K_A & \longrightarrow & V^* \otimes \mathcal{O}_{\mathbb{P}A} & \xrightarrow{A} & V \otimes \mathcal{O}_{\mathbb{P}A}(1) & \longrightarrow & K_A^*(1) & \longrightarrow 0 \\
 & \uparrow{\scriptstyle h^*} & & \uparrow{\scriptstyle h^*} & & \uparrow{\scriptstyle h^*} & & \uparrow{\scriptstyle h^*} & \\
0 \longrightarrow & K_{(hA)} & \longrightarrow & V^* \otimes \mathcal{O}_{\mathbb{P}(hA)} & \xrightarrow{hA} & V \otimes \mathcal{O}_{\mathbb{P}(hA)}(1) & \longrightarrow & K_{(hA)}^*(1) & \longrightarrow 0
\end{array}$$

so $K_A = h^*(K_{hA})$.

Altogether, there is an action of $\mathrm{SL}(n) \times \mathrm{SL}(d)$ on the linear spaces of skew-symmetric (or symmetric) matrices of constant rank. If two matrices A and B are equivalent under this action, the corresponding vector bundles K_A and K_B will belong to the same orbit under the action of $\mathrm{SL}(d)$.

2.2. **Skew-symmetric matrices of constant co-rank two.** We are interested in the case where $A \subseteq \wedge^2 V$, i.e. linear spaces of skew-symmetric matrices of constant co-rank 2. The case 6×6 is treated in [MM05], and the case 8×8 in [FM11]. In both instances the maximal dimension of A is 3.

The state of the art when we began working on the subject—at the best of our knowledge—was the following: many examples were known for $d = 3$, no examples for $d = 5$, and only one example for $d = 4$, appearing in [Wes96]. The example has $r = 8$, which is the smallest value allowed for dimension $d = 4$.

From now on we work on the case $d = 4$. A will always denote a skew-symmetric matrix of linear forms in 4 variables, having size $r+2$ and constant rank r, kernel K_A and cokernel $K_A^*(1)$.

Lemma 2.1. *Let A be as above. Then $K_A \simeq E(-\frac{r}{4})$, where E is an indecomposable rank 2 vector bundle on $\mathbb{P}^3$ with Chern classes:*

$$c_1(E) = 0 \quad \textit{and} \quad c_2(E) = \frac{r(r+4)}{48}.$$

It follows that r is of the form $12s$ or $12s-4$, for some $s \in \mathbb{N}$.

Proof. We have already remarked that the condition $c_1(K_A^*) = \frac{r}{2}$ is entailed by the invariants of the bundles in (2.1). Imposing the further condition that the cokernel is isomorphic to $K_A^*(1)$ forces the Chern polynomials to satisfy the equality $c_t(K_A^*(1)) = (1+t)^{r+2} c_t(K_A)$. From this we get $c_2(K_A) = \frac{r(r+1)}{12}$, and a direct computation then shows that $E = K_A(\frac{r}{4})$ has the desired

Chern classes. Moreover since the Chern polynomial of E is irreducible over $\mathbb{Z}$, we deduce that E is indecomposable. □

The exact sequence (2.1) can then be written as follows:

$$0 \to E(-\tfrac{r}{4}) \to \mathcal{O}_{\mathbb{P}^3}^{r+2} \xrightarrow{A} \mathcal{O}_{\mathbb{P}^3}(1)^{r+2} \to E(\tfrac{r}{4}+1) \to 0. \tag{2.3}$$

It will be useful to have Riemann-Roch formula at hand, see for instance [Har77, Appendix A]. It implies that the algebraic Euler characteristic $\chi(E) = \sum_i (-1)^i \operatorname{h}^i(E)$ of a vector bundle E of rank 2 on $\mathbb{P}^3$ with Chern classes c_1, c_2 is:

$$\chi(E) = \frac{1}{6}c_1^3 - \frac{1}{2}c_1c_2 + c_1^2 - 2c_2 + \frac{11}{6}c_1 + 2.$$

3. Main construction

In this section we will state and prove our main result, Theorem 3.3. This result establishes a necessary and sufficient condition for a rank-2 vector bundle on $\mathbb{P}^3$, together with a certain cohomology class, to give a skew-symmetric matrix of constant co-rank 2. The results of this section continue to hold if we replace $\mathbb{C}$ with an algebraically closed field of characteristic different from 2.

3.1. **Necessary conditions.** We work in the setting described in Section 2.2: assume that $A \subseteq \wedge^2 V$ is a 4-dimensional linear subspace of skew-symmetric matrices of size $r+2$ and constant rank r. Take the sequence (2.3) and tensor it by $\mathcal{O}_{\mathbb{P}^3}(-2)$:

$$\begin{array}{ccccccccc} 0 \longrightarrow E(-\tfrac{r}{4}-2) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-2)^{r+2} & & \xrightarrow{\quad A \quad} & & \mathcal{O}_{\mathbb{P}^3}^{r+2}(-1) \longrightarrow E(\tfrac{r}{4}-1) \longrightarrow 0 \\ & \searrow & & \nearrow & \\ & & \mathcal{E}(-2) & & \\ & \nearrow & & \searrow & \\ 0 & & & & 0 \end{array} \tag{3.1}$$

The 4-term sequence above corresponds to an element:

$$\beta \in \operatorname{Ext}^2(E(\tfrac{r}{4}-1), E(-\tfrac{r}{4}-2)).$$

For all integers t, p, the composition of boundary maps in the cohomology long exact sequence of (3.1) gives maps:

$$\mu_t^p : \operatorname{H}^p(E(\tfrac{r}{4}-1+t)) \to \operatorname{H}^{p+2}(E(-\tfrac{r}{4}-2+t)).$$

The maps μ_t^p can also be thought of as the cup product with the cohomology class β in the Yoneda product:

$$\operatorname{Ext}^2(E(\tfrac{r}{4}-1), E(-\tfrac{r}{4}-2)) \otimes \operatorname{Ext}^p(\mathcal{O}_{\mathbb{P}^3}(-t), E(\tfrac{r}{4}-1)) \to \operatorname{Ext}^{p+2}(\mathcal{O}_{\mathbb{P}^3}(-t), E(-\tfrac{r}{4}-2)).$$

Lemma 3.1. *The exact sequence* (3.1) *gives the following:*

$$(\star) \qquad \begin{cases} \mu_0^p : \operatorname{H}^p(E(\tfrac{r}{4}-1)) \simeq \operatorname{H}^{p+2}(E(-\tfrac{r}{4}-2)), & p = 0,1; \\ \operatorname{H}^q(E(\tfrac{r}{4}-1)) = \operatorname{H}^{q-2}(E(-\tfrac{r}{4}-2)) = 0, & q = 2,3. \end{cases}$$

Proof. This is a direct consequence of the fact that the cohomology of both $\mathcal{O}_{\mathbb{P}^3}(-2)$ and $\mathcal{O}_{\mathbb{P}^3}(-1)$ vanishes in all degrees. The vanishing $\operatorname{H}^0(E(-\tfrac{r}{4}-2)) = \operatorname{H}^3(E(\tfrac{r}{4}-1)) = 0$ are immediate. The same holds for $\operatorname{H}^0(\mathcal{E}(-2)) = \operatorname{H}^3(\mathcal{E}(-2)) = 0$, which in turn implies $\operatorname{H}^1(E(-\tfrac{r}{4}-2)) = \operatorname{H}^2(E(\tfrac{r}{4}-1)) = 0$. From the rightmost short exact sequence in (3.1) we deduce the isomorphisms $\operatorname{H}^p(E(\tfrac{r}{4}-1)) \simeq \operatorname{H}^{p+1}(\mathcal{E}(-2))$, for $p = 0,1$, while from the leftmost one we get, for the same values of p, the isomorphism $\operatorname{H}^{p+2}(E(-\tfrac{r}{4}-2)) \simeq \operatorname{H}^{p+1}(\mathcal{E}(-2))$. □

Lemma 3.2. *The exact sequence* (3.1) *gives the following:*

$$(\star\star)\qquad \begin{cases} \mu_1^0 : \mathrm{H}^0(E(\frac{r}{4})) \twoheadrightarrow \mathrm{H}^2(E(-\frac{r}{4}-1)); \\ \mu_1^1 : \mathrm{H}^1(E(\frac{r}{4})) \simeq \mathrm{H}^3(E(-\frac{r}{4}-1)); \\ \mathrm{H}^2(E(\frac{r}{4})) = \mathrm{H}^3(E(\frac{r}{4})) = \mathrm{H}^0(E(-\frac{r}{4}-1)) = 0. \end{cases}$$

Proof. We apply the argument of the previous Lemma to (2.3) twisted by $\mathcal{O}_{\mathbb{P}^3}(-1)$ instead of $\mathcal{O}_{\mathbb{P}^3}(-2)$. The vanishing $\mathrm{H}^2(E(\frac{r}{4})) = \mathrm{H}^3(E(\frac{r}{4})) = \mathrm{H}^0(E(-\frac{r}{4}-1)) = 0$ is immediate, as in the previous proof. Moreover the cohomology of the sequence on the left-hand side of (3.1) yields $\mathrm{H}^i(E(-\frac{r}{4}-1)) \simeq \mathrm{H}^{i-1}(\mathcal{E}(-1))$ for $i = 1, 2$ and 3, and the sequence on the right-hand side this time gives an isomorphism $\mathrm{H}^1(E(\frac{r}{4})) \simeq \mathrm{H}^2(\mathcal{E}(-1))$, and also an exact sequence:

$$0 \to \mathrm{H}^0(\mathcal{E}(-1)) \to \mathbb{C}^{r+2} \to \mathrm{H}^0(E(\tfrac{r}{4})) \to \mathrm{H}^1(\mathcal{E}(-1)) \to 0.$$

This concludes the proof. □

3.2. **The conditions are sufficient.** Our first main result is that conditions ($\star$) and ($\star\star$) are not only necessary, but also sufficient.

Theorem 3.3. *Let r be a fixed integer number of the form $12s$ or $12s-4$, $s \in \mathbb{N}$. Let E be a rank 2 vector bundle on $\mathbb{P}^3$, with $c_1(E) = 0$ and $c_2(E) = \frac{r(r+4)}{48}$. There exists a skew-symmetric matrix A of linear forms, having size $r+2$, constant rank r, and $E(-\frac{r}{4}-2)$ as its kernel, if and only if there exists $\beta \in \mathrm{Ext}^2(E(\frac{r}{4}-1), E(-\frac{r}{4}-2))$ that induces* ($\star$) *and* ($\star\star$).

The main idea in the proof of this result is the following. First, look at the element $\beta \in \mathrm{Ext}^2(E(\frac{r}{4}-1), E(-\frac{r}{4}-2))$ as a morphism in $\mathrm{D}^b(\mathbb{P}^3)$, the derived category of $\mathbb{P}^3$, so $\beta : E(\frac{r}{4}-1) \to E(-\frac{r}{4}-2)[2]$. Second, use Beilinson Theorem to show that the cone of the morphism β is a 2-term complex of the form $\partial : \mathcal{O}_{\mathbb{P}^3}(-2)^{r+2} \to \mathcal{O}_{\mathbb{P}^3}(-1)^{r+2}$, having $E(-\frac{r}{4}-2)$ as kernel and $E(\frac{r}{4}-1)$ as cokernel. Third, prove that the differential ∂ is skew-symmetric, so ∂ is nothing but the matrix A we are looking for.

The advantage of working with derived categories consists in allowing us to deal more comfortably with complexes, which is precisely what we need, since A appears as differential of a 2-term complex. As a by-product, derived categories allow cleaner and more conceptual statements, avoiding cumbersome tracking of indexes in spectral sequences.

3.2.1. *A quick tour of derived categories I. General features.* For the reader's convenience, we give a quick account of the language of derived categories, trying to avoid excessive technicalities. For more precise definitions and results, we refer to the excellent book [Huy06], in particular Chapters 1-3 and 8. We use some additional remarks from [GM96] and [ML95]. For a short but enlightening introduction to derived categories, see also [Căl05].

We give some elements of the definition of the derived category $\mathrm{D}^b(X)$ of bounded complexes of coherent sheaves over a smooth projective variety X, defined over $\mathbb{C}$. As an auxiliary tool, we also describe quickly the homotopical category $\mathrm{K}(X)$.

The first ingredient of a category are objects. For this, denote by $\mathrm{Kom}(X)$ the abelian, $\mathbb{C}$-linear category of complexes of coherent sheaves on X. The objects in $\mathrm{D}(X)$ and $\mathrm{K}(X)$ are the same as those in $\mathrm{Kom}(X)$, i.e., complexes $\mathcal{F}$ of the following form:

$$\mathcal{F}: \qquad \ldots \to F_{i-1} \xrightarrow{\partial_{i-1}} F_i \xrightarrow{\partial_i} F_{i+1} \to \ldots$$

where the F_i's are coherent sheaves on X, and $\partial_{i+1} \circ \partial_i = 0$ for all $i \in \mathbb{Z}$. We say that $\mathcal{F}$ lies in $\mathrm{D}^b(X)$ if $\mathcal{F}$ is bounded in both directions. We have a *shift functor* $\mathcal{F} \mapsto \mathcal{F}[1]$, given by shifting degrees of one place to the left, so $F[1]_i := F_{i+1}$. For the differential of $\mathcal{F}[1]$, as well as for other sign conventions, we follow the standard agreement (or at least the same as [Huy06]), so $\partial_i^{\mathcal{F}[1]} = -\partial_{i+1}^{\mathcal{F}}$.

Let us now turn to morphisms in $\mathrm{K}(X)$. One starts with morphisms in $\mathrm{Kom}(X)$, i.e. *chain maps* $f : \mathcal{F} \to \mathcal{G}$, that is, collections of maps $f_i : F_i \to G_i$ such that the obvious squares commute. Then, one considers *homotopically equivalent* morphisms $f, g : \mathcal{F} \to \mathcal{G}$, i.e. such that there exists a collection of morphisms $h_i : F_i \to G_{i-1}$, for all i, such that $f_i - g_i = h_{i+1} \circ \partial_i^{\mathcal{F}} + \partial_{i-1}^{\mathcal{G}} \circ h_i$. Denote by $\mathrm{K}(X)$ the *homotopy category* of X: morphisms in $\mathrm{K}(X)$ are chain maps of complexes, modulo homotopy equivalence.

An important feature of $\mathrm{K}(X)$ is that it has the structure of a triangulated category. This means that there is a collection, modeled on exact sequences, of *distinguished triangles*, i.e. triples of complexes and morphisms as in the diagram:

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{E} \xrightarrow{h} \mathcal{F}[1], \tag{3.2}$$

satisfying certain axioms (see [Huy06, Chapter 1]). A triangle is distinguished in $\mathrm{K}(X)$ if and only if it is isomorphic in $\mathrm{K}(X)$ to the cone triangle associated with $f : \mathcal{F} \to \mathcal{G}$, where the complex $\mathcal{E} = \mathcal{C}(f)$ is defined by:

$$E_i := F_{i+1} \oplus G_i, \text{ with differential } \partial_i^{\mathcal{E}} := \begin{bmatrix} -\partial_{i+1}^{\mathcal{F}} & 0 \\ f_{i+1} & \partial_i^{\mathcal{G}} \end{bmatrix},$$

and with the obvious maps $g : \mathcal{G} \to \mathcal{E}$ and $h : \mathcal{E} \to \mathcal{F}[1]$. Given a complex $\mathcal{F}$, its *cohomology sheaves* $\mathcal{H}^i(\mathcal{F})$ are defined as:

$$\mathcal{H}^i(\mathcal{F}) := \mathrm{Ker}(\partial^i)/\,\mathrm{Im}(\partial^{i-1}).$$

Any distinguished triangle (3.2) gives a long cohomology sequence:

$$\cdots \to \mathcal{H}^i(\mathcal{F}) \to \mathcal{H}^i(\mathcal{E}) \to \mathcal{H}^i(\mathcal{G}) \to \mathcal{H}^{i+1}(\mathcal{F}) \to \cdots.$$

A chain map $f : \mathcal{F} \to \mathcal{G}$ induces maps $\mathcal{H}^i(f) : \mathcal{H}^i(\mathcal{F}) \to \mathcal{H}^i(\mathcal{G})$, and f is called a *quasi-isomorphism* if, for all i, the map $\mathcal{H}^i(f)$ is an isomorphism.

We are now ready to introduce morphisms $\mathcal{F} \to \mathcal{G}$ in the derived category $\mathrm{D}(X)$. These are equivalence classes of diagrams of the form:

$$\begin{array}{ccc} & \mathcal{E} & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ \mathcal{F} & \cdots\cdots\cdots\to & \mathcal{G} \end{array} \tag{3.3}$$

where both arrows f and g represent morphisms in $\mathrm{K}(X)$ and $f : \mathcal{E} \to \mathcal{F}$ is a quasi-isomorphism. Such diagrams are called *roofs*. In other words, we formally invert quasi-isomorphisms, so a roof $\mathcal{F} \to \mathcal{G}$ as above can be thought of as g/f, with a convenient formalism. This process is called *localisation*, by analogy with the process of localisation of rings along a multiplicative system. The category $\mathrm{D}(X)$ inherits from $\mathrm{K}(X)$ the structure of a triangulated category; in particular the notions of shift, cone of a morphism, and distinguished triangle are well-defined. Note that the cone of a morphism in $\mathrm{D}(X)$ is defined up to an isomorphism which is not unique in general.

Coherent sheaves on X are elements of $\mathrm{D}(X)$, concentrated in a single degree. We will usually take this degree to be zero, following the standard convention. Morphisms of coherent sheaves can be seen as complexes whose cohomology is concentrated in two consecutive degrees. Indeed, given a complex $\mathcal{F}$ having cohomology in degrees -2 and -1 only, we can replace $\mathcal{F}$ with:

$$\mathcal{F}' : \qquad \cdots \to F_{-3} \to F_{-2} \to \mathrm{Ker}(\partial_{-1})$$

where the map $F_{-2} \to \mathrm{Ker}(\partial_{-1})$ is induced by the composition $F_{-2} \to \mathrm{Im}(\partial_{-2}) \to \mathrm{Ker}(\partial_{-1})$. The induced chain map $\mathcal{F}' \to \mathcal{F}$ is a quasi-isomorphism. Then, one replaces $\mathcal{F}'$ with:

$$\mathcal{F}'' : \qquad \mathrm{Coker}(\partial_{-3}) \to \mathrm{Ker}(\partial_{-1}),$$

where the differential is the composition of the surjection $\mathrm{Coker}(\partial_{-3}) \to \mathrm{Im}(\partial_{-2})$ with the injection $\mathrm{Im}(\partial_{-2}) \to \mathrm{Ker}(\partial_{-1})$. This time, we obtain a chain map $\mathcal{F}'' \to \mathcal{F}'$, which is again a quasi-isomorphism. Altogether, we get a roof $\mathcal{F} \leftarrow \mathcal{F}' \to \mathcal{F}''$, so $\mathcal{F}$ is quasi-isomorphic to a complex with two terms only.

Having introduced derived categories, let us now briefly introduce derived functors. In the derived category, a coherent sheaf $\mathcal{F}$ is the same thing as any resolution of $\mathcal{F}$. (We think essentially of injective resolutions, with a slight abuse of terminology since we have to rely on quasi-coherent sheaves as well to perform this.) Then, taking global sections of (an injective resolution of) $\mathcal{F}$ results in a complex whose i-th cohomology is $\mathrm{H}^i(\mathcal{F})$. Likewise, an object $\mathcal{F}$ in $\mathrm{D}(X)$ is equivalent to the total complex attached to a resolution of each of the F_i. By taking global sections of this total complex, we get a complex of vector spaces: the i-th cohomology of this complex is called the *hypercohomology* $\mathrm{H}^i(\mathcal{F})$ of $\mathcal{F}$. Given an exact triangle (3.2), we have the hypercohomology long exact sequence (see [Huy06, Section 2.2]):

$$\cdots \to \mathrm{H}^i(\mathcal{F}) \to \mathrm{H}^i(\mathcal{E}) \to \mathrm{H}^i(\mathcal{G}) \to \mathrm{H}^{i+1}(\mathcal{F}) \to \cdots$$

Moreover, given complexes $\mathcal{F}$ and $\mathcal{G}$ in $\mathrm{D}^b(X)$, we will have to consider the groups $\mathrm{Ext}^i(\mathcal{F},\mathcal{G})$ in the category $\mathrm{Kom}(X)$, namely the i-th cohomology of the total complex obtained by applying $\mathrm{Hom}(-,\mathcal{G})$ to (an injective resolution of) $\mathcal{F}$. It turns out that this amounts to compute morphisms of shifted complexes in the derived category (see [Huy06, Rmk 2.57]):

$$\mathrm{Ext}^i(\mathcal{F},\mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{F},\mathcal{G}[i]). \tag{3.4}$$

Also in this case, we get a long cohomology exact sequence by applying $\mathrm{Hom}_{\mathrm{D}^b(X)}(-,\mathcal{G})$ to a distinguished triangle. The same considerations apply to other classical functors, such as tensor product, local cohomology, higher direct images, and so forth.

3.2.2. *A quick tour of derived categories II. Beilinson theorem.* We now focus our attention on $\mathrm{D}^b(\mathbb{P}^n)$, the bounded derived category of the projective space. Its main feature is Beilinson Theorem, which states that $\mathrm{D}^b(\mathbb{P}^n)$ is generated by the exceptional collection $\langle \mathcal{O}_{\mathbb{P}^n}(-n), \mathcal{O}_{\mathbb{P}^n}(-n+1), \ldots, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n}\rangle$, with dual collection $\langle \mathcal{O}_{\mathbb{P}^n}(-1), \Omega^{n-1}_{\mathbb{P}^n}(n-1), \ldots, \Omega^1_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}\rangle$, cf. [Huy06, Coroll. 8.29]. We need the following version:

Proposition 3.4 (Beilinson Theorem)**.** *Let $\mathcal{F}$ be a bounded complex of coherent sheaves on $\mathbb{P}^n$. Then there exists a complex $\mathcal{L}$, whose factors are $L_k := \bigoplus_{s-j=k} \mathrm{H}^s(\mathcal{F}\otimes \Omega^j_{\mathbb{P}^n}(j)) \otimes \mathcal{O}_{\mathbb{P}^n}(-j)$, which is quasi-isomorphic to $\mathcal{F}$.*

We call $\mathcal{L}$ the *decomposition of* $\mathcal{F}$, and, for fixed j, we call the terms $\mathrm{H}^s(\mathcal{F}\otimes\Omega^j_{\mathbb{P}^n}(j))\otimes\mathcal{O}_{\mathbb{P}^n}(-j)$ the *components* of $\mathcal{F}$ along $\mathcal{O}_{\mathbb{P}^n}(-j)$. The theorem of Beilinson was proved in [Beĭ78], see also [Huy06, Prop. 8.28], [Căl05]. Although our statement is slightly more general than in [Huy06], since we take into account complexes and not just sheaves, the proof goes through verbatim. It will be useful to have a graphic description of Beilinson Theorem. Consider the $(n+1)\times(n+1)$ square diagram, sometimes referred to as the *Beilinson table* of $\mathcal{F}$:

(3.5)

$\mathrm{h}^n(\mathcal{F}\otimes\Omega^j_{\mathbb{P}^n}(j))$	$\mathrm{h}^n(\mathcal{F}(-1))$	$\cdots$	$\mathrm{h}^n(\mathcal{F}\otimes\Omega^1_{\mathbb{P}^n}(1))$	$\mathrm{h}^n(\mathcal{F})$
$\mathrm{h}^{n-1}(\mathcal{F}\otimes\Omega^j_{\mathbb{P}^n}(j))$	$\mathrm{h}^{n-1}(\mathcal{F}(-1))$	$\cdots$	$\mathrm{h}^{n-1}(\mathcal{F}\otimes\Omega^1_{\mathbb{P}^n}(1))$	$\mathrm{h}^{n-1}(\mathcal{F})$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\mathrm{h}^1(\mathcal{F}\otimes\Omega^j_{\mathbb{P}^n}(j))$	$\mathrm{h}^1(\mathcal{F}(-1))$	$\cdots$	$\mathrm{h}^1(\mathcal{F}\otimes\Omega^1_{\mathbb{P}^n}(1))$	$\mathrm{h}^1(\mathcal{F})$
$\mathrm{h}^0(\mathcal{F}\otimes\Omega^j_{\mathbb{P}^n}(j))$	$\mathrm{h}^0(\mathcal{F}(-1))$	$\cdots$	$\mathrm{h}^0(\mathcal{F}\otimes\Omega^1_{\mathbb{P}^n}(1))$	$\mathrm{h}^0(\mathcal{F})$
	$j=n$		$j=1$	$j=0$

The terms L_k of the complex $\mathcal{L}$ can be computed by taking the direct sum of all the terms on the "NW-SE" diagonals, the main diagonal corresponding to $k=0$, the first subdiagonal to

$k = -1$, and so on. For $j = 0, \dots, n$, each sheaf $\mathcal{O}_{\mathbb{P}^n}(-j)$ must be taken with the multiplicity given by the corresponding integer in the Beilinson table, which is exactly $\mathrm{h}^s(\mathcal{F} \otimes \Omega^j_{\mathbb{P}^n}(j))$.

3.2.3. *Proof of the main theorem I. Getting the matrix.* We need to show that conditions $(\star)$ and $(\star\star)$ are sufficient in order to get a matrix $\mathcal{O}_{\mathbb{P}^3}(-2)^{r+2} \to \mathcal{O}_{\mathbb{P}^3}(-1)^{r+2}$, whose kernel is $E(-\frac{r}{4}-2)$ and whose cokernel is $E(\frac{r}{4}-1)$. In this step we are not interested in the skew-symmetry of the map: we deal with it in the next subsection. The proof is divided into two claims.

The distinguished element $\beta \in \mathrm{Ext}^2(E(\frac{r}{4}-1), E(-\frac{r}{4}-2))$ corresponds to a 2-term extension:

$$0 \to E(-\tfrac{r}{4}-2) \to P_2 \xrightarrow{\partial} P_1 \to E(\tfrac{r}{4}-1) \to 0. \tag{3.6}$$

Via the isomorphism (3.4), β can be seen as an element of $\mathrm{Hom}_{\mathrm{D}^b(\mathbb{P}^3)}(E(\frac{r}{4}-1), E(-\frac{r}{4}-2)[2])$. Let $\mathcal{C} := \mathcal{C}(\beta)$ be the cone of this morphism, so $\mathcal{C}$ lies in the exact triangle:

$$E(\tfrac{r}{4}-1) \to E(-\tfrac{r}{4}-2)[2] \to \mathcal{C} \to E(\tfrac{r}{4}-1)[1]. \tag{3.7}$$

By taking the cohomology sequence induced by the exact triangle (3.7), we see that $\mathcal{C}$ has cohomology $E(-\frac{r}{4}-2)$ in degree -2 and $E(\frac{r}{4}-1)$ in degree -1. We have seen that $\mathcal{C}$ is thus a 2-term complex, non-zero in degree -2 and -1 only. In fact the triangle is nothing but (3.6), and we have $C_{-2} = P_2$, $C_{-1} = P_1$, and ∂ as differential.

We apply Beilinson Theorem 3.4 to the complex $\mathcal{C} = P_2 \xrightarrow{\partial} P_1$, decomposing it with respect to the collection $\langle \mathcal{O}_{\mathbb{P}^3}(-3), \mathcal{O}_{\mathbb{P}^3}(-2), \mathcal{O}_{\mathbb{P}^3}(-1), \mathcal{O}_{\mathbb{P}^3} \rangle$. Recall that the components of $\mathcal{C}$ along the term $\mathcal{O}_{\mathbb{P}^3}(-j)$ are computed by $\mathrm{h}^s(\mathcal{C} \otimes \Omega^j(j))$.

Claim 1. *The terms $\mathcal{O}_{\mathbb{P}^3}$ and $\mathcal{O}_{\mathbb{P}^3}(-3)$ do not occur in the decomposition of $\mathcal{C}$.*

Proof of Claim 1. We have to show that the components of $\mathcal{C}$ along $\mathcal{O}_{\mathbb{P}^3}$ and $\mathcal{O}_{\mathbb{P}^3}(-3)$ are zero. We show that this is a direct consequence of $(\star)$.

To check the statement regarding $\mathcal{O}_{\mathbb{P}^3}$, we need $\mathrm{H}^i(\mathcal{C}) = 0$ for all i. For this, take hypercohomology of (3.7). The vanishing $\mathrm{H}^q(E(\frac{r}{4}-1)) = \mathrm{H}^{q-2}(E(-\frac{r}{4}-2)) = 0$ for $q = 2, 3$ tells us that the only groups $\mathrm{H}^i(\mathcal{C})$ that are not trivially zero fit in the exact sequence:

$$0 \longrightarrow \mathrm{H}^{-1}(\mathcal{C}) \longrightarrow \mathrm{H}^0(E(\tfrac{r}{4}-1)) \longrightarrow \mathrm{H}^2(E(-\tfrac{r}{4}-2)) \longrightarrow \mathrm{H}^0(\mathcal{C}) \longrightarrow$$

$$\longrightarrow \mathrm{H}^1(E(\tfrac{r}{4}-1)) \longrightarrow \mathrm{H}^3(E(-\tfrac{r}{4}-2)) \longrightarrow \mathrm{H}^1(\mathcal{C}) \longrightarrow 0.$$

From the isomorphisms $\mathrm{H}^p(E(\frac{r}{4}-1)) \simeq \mathrm{H}^{p+2}(E(-\frac{r}{4}-2))$, $p = 0, 1$, entailed by $(\star)$, we deduce the desired vanishing.

To check that $\mathcal{O}_{\mathbb{P}^3}(-3)$ does not occur, we need $\mathrm{H}^i(\mathcal{C}(-1)) = 0$ for all i. By Serre duality $\mathrm{H}^i(E(\frac{r}{4}-2)) \simeq \mathrm{H}^{3-i}(E(-\frac{r}{4}-2))^*$ and $\mathrm{H}^i(E(-\frac{r}{4}-3)) \simeq \mathrm{H}^{3-i}(E(\frac{r}{4}-1))^*$, thus the same argument as above applies, and Claim 1 is proved. ☐

Now we show that in the decomposition of $\mathcal{C}$ the terms that we have not yet considered appear concentrated in one degree. After Claim 1, the Beilinson table (3.5) of $\mathcal{C}$ looks like this:

(3.8)

$\mathrm{h}^3(\mathcal{C} \otimes \Omega^j_{\mathbb{P}^3}(j))$	0	♢	♢	0
$\mathrm{h}^2(\mathcal{C} \otimes \Omega^j_{\mathbb{P}^3}(j))$	0	♢	♢	0
$\mathrm{h}^1(\mathcal{C} \otimes \Omega^j_{\mathbb{P}^3}(j))$	0	♢	♢	0
$\mathrm{h}^0(\mathcal{C} \otimes \Omega^j_{\mathbb{P}^3}(j))$	0	♦	♦	0
	$j = 3$	$j = 2$	$j = 1$	$j = 0$

Claim 2. *In the Beilinson table (3.8), we have ♢ $= 0$ and ♦ $= r + 2$*

Proof of Claim 2. We show that this is a consequence of $(\star\star)$. Let us start with the term $\mathcal{O}_{\mathbb{P}^3}(-1)$. We need $\mathrm{h}^i(\mathcal{C} \otimes \Omega^1_{\mathbb{P}^3}(1)) = 0$ for all $i \neq 0$ and $\mathrm{h}^0(\mathcal{C} \otimes \Omega^1_{\mathbb{P}^3}(1)) = r+2$. Taking the Euler sequence tensored by $\mathcal{C}$, since all terms of the Euler sequence are vector bundles, we obtain a distinguished triangle:

$$\mathcal{C} \otimes \Omega^1_{\mathbb{P}^3}(1) \to \mathcal{C}^4 \to \mathcal{C}(1) \to \mathcal{C} \otimes \Omega^1_{\mathbb{P}^3}(1)[1],$$

and we compute its hypercohomology. The vanishing $\mathrm{H}^i(\mathcal{C}) = 0$ for all i that we proved in Claim 1 implies that $\mathrm{H}^i(\mathcal{C} \otimes \Omega^1_{\mathbb{P}^3}(1)) \simeq \mathrm{H}^{i-1}(\mathcal{C}(1))$, hence what we want is $\mathrm{H}^i(\mathcal{C}(1)) = 0$ for all $i \neq -1$ and $\mathrm{h}^{-1}(\mathcal{C}(1)) = r+2$. So let us compute hypercohomology of (3.7) twisted by $\mathcal{O}_{\mathbb{P}^3}(1)$:

$$E(\tfrac{r}{4}) \to E(-\tfrac{r}{4}-1)[2] \to \mathcal{C}(1) \to E(\tfrac{r}{4})[1].$$

Analogously to what happened in the previous case, the vanishing $\mathrm{H}^2(E(\frac{r}{4})) = \mathrm{H}^3(E(\frac{r}{4})) = \mathrm{H}^0(E(-\frac{r}{4}-1)) = 0$ implies that the only groups $\mathrm{H}^i(\mathcal{C}(1))$ that are not trivially zero fit in the exact sequence:

$$0 \longrightarrow \mathrm{H}^1(E(-\tfrac{r}{4}-1)) \longrightarrow \mathrm{H}^{-1}(\mathcal{C}(1)) \longrightarrow \mathrm{H}^0(E(\tfrac{r}{4})) \longrightarrow \mathrm{H}^2(E(-\tfrac{r}{4}-1)) \longrightarrow \mathrm{H}^0(\mathcal{C}(1)) \longrightarrow$$
$$\longrightarrow \mathrm{H}^1(E(\tfrac{r}{4})) \longrightarrow \mathrm{H}^3(E(-\tfrac{r}{4}-1)) \longrightarrow \mathrm{H}^1(\mathcal{C}(1)) \longrightarrow 0.$$

Now the surjection $\mathrm{H}^0(E(\frac{r}{4})) \twoheadrightarrow \mathrm{H}^2(E(-\frac{r}{4}))$ and the isomorphism $\mathrm{H}^1(E(\frac{r}{4})) \simeq \mathrm{H}^3(E(-\frac{r}{4}-1))$ guarantee that $\mathrm{H}^i(\mathcal{C}(1)) = 0$ for $i = 0$ and 1. From Riemann-Roch we get:

$$\mathrm{h}^{-1}(\mathcal{C}(1)) = \chi\Big(E\Big(\frac{r}{4}\Big)\Big) - \chi\Big(E\Big(-\frac{r}{4}-1\Big)\Big) = \frac{(r+4)(r+8)}{16} - \frac{r(r-4)}{16} = r+2.$$

Finally we deal with the term $\mathcal{O}_{\mathbb{P}^3}(-2)$. We need to show that $\mathrm{H}^i(\mathcal{C} \otimes \Omega^2_{\mathbb{P}^3}(2)) = 0$ for all $i \neq 0$ and $\mathrm{h}^0(\mathcal{C} \otimes \Omega^2_{\mathbb{P}^3}(2)) = r+2$. Notice that $\Omega^2_{\mathbb{P}^3}(2) \simeq \mathrm{T}_{\mathbb{P}^3}(-2)$. Then using again the (dual) Euler sequence tensored by $\mathcal{C}(-1)$:

$$\mathcal{C}(-2) \to \mathcal{C}(-1)^4 \to \mathcal{C} \otimes \mathrm{T}_{\mathbb{P}^3}(-2) \to \mathcal{C}(-2)[1],$$

we see that the vanishing $\mathrm{H}^i(\mathcal{C}(-1)) = 0$ for all i proved in Claim 1 implies that $\mathrm{H}^i(\mathcal{C} \otimes \Omega^2_{\mathbb{P}^3}(2)) \simeq \mathrm{H}^i(\mathcal{C} \otimes \mathrm{T}_{\mathbb{P}^3}(-2)) \simeq \mathrm{H}^{i+1}(\mathcal{C}(-2))$. We are thus left to prove $\mathrm{H}^i(\mathcal{C}(-2)) = 0$ for all $i \neq 1$ and $\mathrm{h}^1(\mathcal{C}(-2)) = r+2$.

We take cohomology of (3.7) twisted by -2, and notice that by Serre duality $\mathrm{H}^i(E(\frac{r}{4}-3)) \simeq \mathrm{H}^{3-i}(E(-\frac{r}{4}-1))^*$. Hence the conditions required by $(\star\star)$ and the same argument as above yield that the only non-vanishing group is $\mathrm{H}^1(\mathcal{C}(-2))$. Moreover, again by Serre duality, $\mathrm{h}^1(\mathcal{C}(-2)) = \mathrm{h}^{-1}(\mathcal{C}(1)) = r+2$, so Claim 2 is proved. □

3.2.4. *Proof of the main theorem II. Skew-symmetrising the matrix.* So far, we have shown that we can decompose the cone $\mathcal{C}$ explicitly as a map $\partial : \mathcal{O}_{\mathbb{P}^3}(-2)^{r+2} \to \mathcal{O}_{\mathbb{P}^3}(-1)^{r+2}$, i.e. the differential ∂ is a matrix A of size $r+2$ and constant rank r, that by construction will fit in a 2-term extension of type (3.1). What is left to prove is that:

Claim 3. *The matrix A is skew-symmetrizable.*

The claim means that A is skew-symmetric in an appropriate basis, i.e., that up to composing A with isomorphisms on the right and on the left, we get an honest skew-symmetric matrix. To prove this claim, we need a homological algebra lemma, that we state in greater generality for future reference. Let F be a vector bundle on a smooth projective variety X over $\mathbb{C}$, L be a line bundle on X. We have the canonical decomposition:

$$\mathrm{Ext}^k(F, F^* \otimes L) \cong \mathrm{H}^k(\wedge^2 F^* \otimes L) \oplus \mathrm{H}^k(S^2 F^* \otimes L).$$

We say that $\beta \in \mathrm{Ext}^k(F, F^* \otimes L)$ is *symmetric* if it belongs to $\mathrm{H}^k(S^2 F^* \otimes L)$, and *skew-symmetric* if it lies in $\mathrm{H}^k(\wedge^2 F^* \otimes L)$.

Let $\mathcal{P}$ be a bounded complex of coherent sheaves on X (i.e., an object of $\mathrm{D}^b(X)$). The complex $\mathcal{P}$ corresponds to an element of $\mathrm{Ext}^k(F, F^* \otimes L)$ if we have an exact complex:

$$\mathcal{P} : 0 \to F^* \otimes L = P_{k+1} \xrightarrow{\partial_k} P_k \to \cdots \to P_2 \xrightarrow{\partial_1} P_1 \xrightarrow{\partial_0} P_0 = F \to 0. \tag{3.9}$$

Lemma 3.5. *In the above setting, let $\mathcal{P}$ be a complex of $k+2$ vector bundles corresponding to $\beta \in \mathrm{Ext}^k(F, F^* \otimes L)$, and assume that β is symmetric. Moreover assume:*

$$\mathrm{Ext}^{>0}(P_i, P_j^* \otimes L) = 0, \qquad \textit{for all } i, j; \tag{3.10}$$

$$\mathrm{Hom}(P_i, P_{k-i}^* \otimes L) = 0, \qquad \textit{for } i \leq \lfloor \tfrac{k}{2} \rfloor. \tag{3.11}$$

Then, up to isomorphism:

i) if $k \equiv 0 \mod 4$, the middle map of $\mathcal{P}$ is symmetric;
ii) if $k \equiv 1 \mod 4$, the middle term of $\mathcal{P}$ has a skew-symmetric duality;
iii) if $k \equiv 2 \mod 4$, the middle map of $\mathcal{P}$ is skew-symmetric;
iv) if $k \equiv 3 \mod 4$, the middle term of $\mathcal{P}$ has a symmetric duality.

If β is skew-symmetric, all signs in the above 4 cases must be reversed.

Proof of Lemma 3.5. We treat the symmetric case, the skew-symmetric one being analogous.

We dualise the expression (3.9) of $\mathcal{P}$ and we twist by L (we can do this with no harm since the P_i's are locally free). We denote the resulting complex by $\mathcal{P}'$. In view of the standard sign convention that we adopted for dual complexes, $\mathcal{P}'$ reads:

$$\mathcal{P}' : 0 \to F^* \otimes L \simeq P_0^* \otimes L \xrightarrow{\partial_0^{\intercal}} P_1^* \otimes L \xrightarrow{-\partial_1^{\intercal}} P_2^* \otimes L \to \cdots \to P_k^* \otimes L \xrightarrow{(-1)^k \partial_k^{\intercal}} P_{k+1}^* \otimes L \simeq F \to 0.$$

Since β is symmetric, the class in $\mathrm{Ext}^k(F, F^* \otimes L)$ corresponding to $\mathcal{P}'$ is again β, so the two extensions are equivalent. Even though this does not imply the existence of an isomorphism $\mathcal{P} \to \mathcal{P}'$ in general, but it does under our hypothesis. Indeed, from [ML95, Ex. 5 Chapter III.6] we learn that there exists a complex $\mathcal{Q}$ of $k+2$ terms, equipped with maps $\mathcal{Q} \to \mathcal{P}$ and $\mathcal{Q} \to \mathcal{P}'$ lifting the identity over the terms $F^* \otimes L$ and F at the two ends of $\mathcal{P}$ and $\mathcal{P}'$. In other words, the cones of the maps $\beta : F \to F^* \otimes L[k]$ and $\beta^{\intercal} : F \to F^* \otimes L[k]$ are quasi-isomorphic, hence isomorphic in $\mathrm{D}^b(X)$ cf. [Huy06, Page 32]. The situation is described in the following diagram:

$$\begin{array}{lccccccccccc}
\mathcal{P}: & 0 & \longrightarrow & F^* \otimes L & \xrightarrow{\partial_k} & P_k & \xrightarrow{\partial_{k-1}} & \cdots & \longrightarrow & P_1 & \xrightarrow{\partial_0} & F \longrightarrow 0 \\
& & & \| & & \uparrow & & & & \uparrow & & \| \\
\mathcal{Q}: & 0 & \longrightarrow & F^* \otimes L & \longrightarrow & Q_k & \longrightarrow & \cdots & \longrightarrow & Q_1 & \longrightarrow & F \longrightarrow 0 \\
& & & \| & & \downarrow & & & & \downarrow & & \| \\
\mathcal{P}': & 0 & \longrightarrow & F^* \otimes L & \xrightarrow[\partial_0^{\intercal}]{} & P_1^* \otimes L & \xrightarrow[-\partial_1^{\intercal}]{} & \cdots & \longrightarrow & P_k^* \otimes L & \xrightarrow[(-1)^k \partial_k^{\intercal}]{} & F \longrightarrow 0
\end{array}$$

Under the condition $\mathrm{Ext}^p(P_i, P_j^* \otimes L) = 0$ for $p > 0$ appearing in (3.10), the complexes $\mathcal{P}$ and $\mathcal{P}'$ are actually homotopic, see [Kap88, Lemma 1.6]. The further condition (3.11) implies that the homotopy maps $P_i \to P_{k-i}^* \otimes L$ are zero, hence $\mathcal{P}$ and $\mathcal{P}'$ are isomorphic complexes.

So let $\varphi : \mathcal{P} \to \mathcal{P}'$ be an isomorphism lifting the identity over F and $F^* \otimes L$. For all $0 \leq i \leq k+1$, we have isomorphisms $\varphi_i : P_i \to P_{k+1-i}^* \otimes L$, with $\varphi_0 = \mathrm{id}_F$ and $\varphi_{k+1} = \mathrm{id}_{F^* \otimes L}$, such that the following diagrams commute:

$$(D_i)\qquad \begin{array}{ccc} P_{k+1-i} & \xrightarrow{\ \partial_{k-i}\ } & P_{k-i} \\ {\scriptstyle \varphi_{k+1-i}}\downarrow & & \downarrow{\scriptstyle \varphi_{k-i}} \\ P_i^*\otimes L & \xrightarrow[(-1)^i\partial_i^{\mathsf{T}}]{} & P_{i+1}^*\otimes L \end{array}$$

Let us now look at case (i), so $k = 4h$. For $2h+1 \le i \le k+1$, we can replace φ_i with $\varphi_{k+1-i}^{\mathsf{T}}$. We obtain squares (D_i') analogous to the (D_i)'s above, and the diagram has the form:

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & P_{4h+1-i} & \xrightarrow{\partial_{4h-i}} & P_{4h-i} & \longrightarrow \cdots \longrightarrow & P_{2h+1-i} & \xrightarrow{\partial_{2h-i}} & P_{2h-i} & \longrightarrow & \cdots \\ & & {\scriptstyle \varphi_i^{\mathsf{T}}}\downarrow & (D_i') & \downarrow{\scriptstyle \varphi_{i+1}^{\mathsf{T}}} & & {\scriptstyle \varphi_{2h+1-i}}\downarrow & (D_i) & \downarrow{\scriptstyle \varphi_{2h-i}} & & \\ \cdots & \longrightarrow & P_i^*\otimes L & \xrightarrow[(-1)^i\partial_i^{\mathsf{T}}]{} & P_{i+1}^*\otimes L & \longrightarrow \cdots \longrightarrow & P_{2h+i}^*\otimes L & \xrightarrow[(-1)^i\partial_{2h+i}^{\mathsf{T}}]{} & P_{2h+1+i}^*\otimes L & \longrightarrow & \cdots \end{array}$$

An easy computation shows that the squares in the diagram above still commute, thanks to the good behavior of sign changes. The diagram is symmetric with respect to the middle square (D_{2h}'), that looks like this:

$$(D_{2h}')\qquad \begin{array}{ccc} P_{2h+1} & \xrightarrow{\ \partial_{2h}\ } & P_{2h} \\ {\scriptstyle \varphi_{2h}^{\mathsf{T}}}\downarrow & & \downarrow{\scriptstyle \varphi_{2h}} \\ P_{2h}^*\otimes L & \xrightarrow[\partial_{2h}^{\mathsf{T}}]{} & P_{2h+1}^*\otimes L \end{array}$$

So up to isomorphism (i.e. up to replacing ∂_{2h} with $\varphi_{2h}\circ\partial_{2h}$), the middle differential of $\mathcal{P}$ is a symmetric map.

Case (iii) is similar. Indeed, if $k = 4h+2$, we obtain new commuting diagrams $(D_i)'$ as above by replacing φ_i with $\varphi_{k+1-i}^{\mathsf{T}}$ for $2h+2 \le i \le k+1$, and the middle diagram is (D_{2h+1}'), that yields:

$$\varphi_{2h+1}\circ\partial_{2h+1} = -\partial_{2h+1}^{\mathsf{T}}\circ\varphi_{2h+1}^{\mathsf{T}},$$

so in this case the middle differential of $\mathcal{P}$ is skew-symmetric (up to isomorphism).

Let us now look at case (ii), so $k = 4h+1$. This time sign changes do not behave as well as before. To cope with this, we replace φ_{k+1-i} with $(-1)^i\varphi_i^{\mathsf{T}}$, for $0 \le i \le 2h$. We obtain a diagram of the form:

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & P_{4h+2-i} & \xrightarrow{\partial_{4h+1-i}} & P_{4h+1-i} & \longrightarrow \cdots\cdots \longrightarrow & P_{2h+1-i} & \xrightarrow{\partial_{2h-i}} & P_{2h-i} & \longrightarrow & \cdots \\ & & {\scriptstyle (-1)^i\varphi_i^{\mathsf{T}}}\downarrow & (D_i') & \downarrow{\scriptstyle (-1)^{i+1}\varphi_{i+1}^{\mathsf{T}}} & & {\scriptstyle \varphi_{2h+1-i}}\downarrow & (D_i) & \downarrow{\scriptstyle \varphi_{2h-i}} & & \\ \cdots & \longrightarrow & P_i^*\otimes L & \xrightarrow[(-1)^i\partial_i^{\mathsf{T}}]{} & P_{i+1}^*\otimes L & \longrightarrow \cdots\cdots \longrightarrow & P_{2h+i}^*\otimes L & \xrightarrow[(-1)^i\partial_{2h+i}^{\mathsf{T}}]{} & P_{2h+1+i}^*\otimes L & \longrightarrow & \cdots \end{array}$$

Again we get new commuting diagrams $(D_i)'$. The middle part of $\mathcal{P}$ now gives the commuting diagram:

$$\begin{array}{ccccc} P_{2h+2} & \xrightarrow{\partial_{2h+1}} & P_{2h+1} & \xrightarrow{\partial_{2h}} & P_{2h} \\ \downarrow{\scriptstyle \varphi_{2h}^{\intercal}} & & \downarrow{\scriptstyle \varphi_{2h+1}} & & \downarrow{\scriptstyle \varphi_{2h}} \\ P_{2h}^*\otimes L & \xrightarrow[\partial_{2h}^{\intercal}]{} & P_{2h+1}^*\otimes L & \xrightarrow[-\partial_{2h+1}^{\intercal}]{} & P_{2h+2}^*\otimes L \end{array}$$

Transposing the rightmost square, and reading off the first square we get:

$$-\varphi_{2h+1}^{\intercal}\circ\partial_{2h+1}=\partial_{2h}^{\intercal}\circ\varphi_{2h}^{\intercal}=\varphi_{2h+1}\circ\partial_{2h+1}.$$

This means that we can replace φ_{2h+1} by $\psi=\frac{1}{2}(\varphi_{2h+1}-\varphi_{2h+1}^{\intercal})$ without spoiling the commutativity of our diagrams. Then ψ will be an isomorphism by the five lemma, thus equipping P_{2h+1} with a skew-symmetric duality. The last case is analogous to this one, so we omit it. □

Proof of Claim 3. We apply Lemma 3.5 to our setting. Then $X=\mathbb{P}^3$, $F=E(\frac{r}{4}-1)$ and $L=\mathcal{O}_{\mathbb{P}^3}(-3)$, so that $F^*\otimes L=E(-\frac{r}{4}-2)$. The complex $\mathcal{P}$ that we are interested in is of course the cone $\mathcal{C}$ corresponding to the distinguished element $\beta\in\mathrm{Ext}^2(E(\frac{r}{4}-1),E(-\frac{r}{2}-2))$, so in particular $k=2$, $P_2=\mathcal{O}_{\mathbb{P}^3}(-2)^{r+2}$ and $P_1=\mathcal{O}_{\mathbb{P}^3}(-1)^{r+2}$.

Since E has rank 2, $\mathrm{H}^2(\wedge^2E(-\frac{r}{4}+1)\otimes\mathcal{O}_{\mathbb{P}^3}(-3))=0$, meaning that all elements $\beta\in\mathrm{Ext}^2(E(\frac{r}{4}-1),E(-\frac{r}{2}-2))$ are symmetric. Moreover conditions (3.10) and (3.11) translate respectively in:

$$\mathrm{Ext}^{>0}(P_1,P_2^*\otimes L)\simeq\mathrm{Ext}^{>0}(P_2,P_1^*\otimes L)\simeq\mathrm{H}^{>0}(\mathcal{O}_{\mathbb{P}^3}^{r+2})=0$$

and in:

$$\mathrm{Hom}(P_1,P_1^*\otimes L)\simeq\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(-1)^{r+2})=0,$$

and are thus trivially satisfied. By part (iii), the matrix A that we have constructed is skew-symmetrizable. This concludes Claim 3, as well as the proof of Theorem 3.3. □

Remark 3.1. Theorem 3.3 is consistent with the results known for symmetric matrices. As we remarked in Section 2.1, in this case the same computation of invariants of the vector bundles involved holds. In [IL99] the authors prove that if $r\geq 2$ is even, then the maximal dimension of a linear space of symmetric $n\times n$ matrices of constant rank r is $n-r+1$. In other words, 4-dimensional spaces of symmetric matrices of constant co-rank 2 do not exist.

It is also worth pointing out that Claim 3 is false on the projective plane $\mathbb{P}^2$, simply because the group $\mathrm{H}^2(\mathcal{O}_{\mathbb{P}^2}(-\frac{r}{2}-1))$ is non-zero as soon as $r\geq 6$.

3.3. **Simpler conditions for bundles with natural cohomology.** Imposing one further condition on the bundle E, namely natural cohomology, will enable us to simplify the requirements of $(\star)$ and $(\star\star)$. We recall that a vector bundle E on $\mathbb{P}^3$ has *natural cohomology* if $\mathrm{H}^i(E(t))\neq 0$ for at most one i, any t. Remark that a rank 2 bundle on $\mathbb{P}^3$ with $c_1=0$, $c_2>0$ and natural cohomology is (Mumford-Takemoto) *stable*, which in this setting is equivalent to the vanishing $\mathrm{H}^0(E)=0$. Indeed, by Riemann-Roch we see that $\chi(E)\leq 0$, and this, combined with the natural cohomology hypothesis, implies that the bundle has no sections.

Theorem 3.6. *Let E be as in Theorem 3.3. If E has natural cohomology, $(\star)$ and $(\star\star)$ reduce respectively to an isomorphism:*

($\diamond$) $\mathrm{H}^0(E(\frac{r}{4}-1))\simeq\mathrm{H}^2(E(-\frac{r}{4}-2))$,

and a surjection:

($\diamond\diamond$) $\mathrm{H}^0(E(\frac{r}{4}))\twoheadrightarrow\mathrm{H}^2(E(-\frac{r}{4}-1))$.

Hence if there exists an element $\beta \in \mathrm{Ext}^2(E(\frac{r}{4}-1), E(-\frac{r}{4}-2))$ that induces ($\diamond$) and ($\diamond\diamond$), then there exists a skew-symmetric matrix of linear forms, having size $r+2$ and constant rank r, and whose kernel is $E(-\frac{r}{4}-2)$.

Proof. We start with ($\star$). As remarked above, the bundle E is stable. But then $\mathrm{h}^0(E(-\frac{r}{4}-2)) = 0$, and by Riemann-Roch we see that $\chi(E(-\frac{r}{4}-2)) > 0$. Hence our hypothesis of natural cohomology translates in the fact that $\mathrm{h}^p(E(-\frac{r}{4}-2)) = 0$ for $p \neq 2$. It follows that once we impose that $\mathrm{H}^0(E(\frac{r}{4}-1)) \simeq \mathrm{H}^2(E(-\frac{r}{4}-2))$, then natural cohomology will force $\mathrm{h}^p(E(\frac{r}{4}-1)) = 0$ for $p \neq 0$ and all the other requirements are trivially satisfied. The same reasoning works for ($\star\star$). ☐

4. Instanton bundles

Let us take a closer look at general instantons. We call E a (mathematical) *instanton bundle of charge k*, or simply a k-instanton, if E is a rank 2 stable vector bundle on $\mathbb{P}^3$ with Chern classes $c_1 = 0$ and $c_2 = k$, satisfying the vanishing $\mathrm{H}^1(E(-2)) = 0$. A general k-instanton E has natural cohomology [HH82, Thm. 0.1(i)], so Theorem 3.6 applies. Here comes the main result of this Section, namely that for general instantons the requirements needed by Theorem 3.3 reduce to a single non-degeneracy condition.

Theorem 4.1. *Let r be a fixed integer number of the form $12s$ or $12s-4$, $s \in \mathbb{N}$. Let E be a general k-instanton, with $k = \frac{r(r+4)}{48}$. If E satisfies condition ($\diamond$) of Theorem 3.6, it also satisfies condition ($\diamond\diamond$).*

Recall that in our setting the only allowed second Chern class is $c_2(E) = \frac{r(r+4)}{48}$. However, for consistency with instanton literature, we still denote by k the charge of E, keeping in mind that $k = k(r) = \frac{r(r+4)}{48}$.

Our argument involves the (sheafified) minimal graded free resolution of a general k-instanton E. Let v be the smallest integer such that $E(v)$ has non-zero global sections. Using Riemann-Roch, we compute that [Har78, Rem. 8.2.3] $v = \frac{r}{4}-1$, and we find $\mathrm{h}^0(E(\frac{r}{4}-1)) = k$ and $\mathrm{h}^1(E(\frac{r}{4}-2)) = k$. Moreover, from [Rah97] we learn what the minimal graded free resolution of E looks like. A direct computation, together with the assumption of natural cohomology, then shows the following:

Proposition 4.2. *Let r be a fixed integer number of the form $12s$ or $12s-4$, $s \in \mathbb{N}$. Let E be a general k-instanton, with $k = \frac{r(r+4)}{48}$. Then E admits the following resolution:*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-\tfrac{r}{4}-1)^k \longrightarrow \begin{array}{c}\mathcal{O}_{\mathbb{P}^3}(-\frac{r}{4})^b \\ \oplus \\ \mathcal{O}_{\mathbb{P}^3}(-\frac{r}{4}-1)^c\end{array} \longrightarrow \begin{array}{c}\mathcal{O}_{\mathbb{P}^3}(-\frac{r}{4}+1)^k \\ \oplus \\ \mathcal{O}_{\mathbb{P}^3}(-\frac{r}{4})^a\end{array} \longrightarrow E \longrightarrow 0, \quad \textit{where:}$$

i) if $r = 8$, then $a = 4$, $b = 0$, $c = 6$;
ii) if $r = 12$, then $a = 4$, $b = 0$, $c = 10$;
iii) if $r = 20$, then $a = 2$, $b = 0$, $c = 20$;
iv) if $r \geq 24$, then $a = 0$, $b = k - \frac{r}{2} - 2$, $c = k + \frac{r}{2}$.

We are now ready to prove the main result of this Section.

Proof of Theorem 4.1. We use the structure of the graded module $\mathrm{H}^2_*(E) := \bigoplus_{t\in\mathbb{Z}} \mathrm{H}^2(E(t))$ to prove that there is a surjection:

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes \mathrm{H}^2(E(-\tfrac{r}{4}-2)) \twoheadrightarrow \mathrm{H}^2(E(-\tfrac{r}{4}-1)).$$

Combined with condition $(\diamond)$ this surjection gives us the diagram:

$$\begin{array}{ccc} \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes \mathrm{H}^0(E(\frac{r}{4}-1)) & \xrightarrow{\simeq} & \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes \mathrm{H}^2(E(-\frac{r}{4}-2)) \\ \downarrow & & \downarrow\!\!\downarrow \\ \mathrm{H}^0(E(\frac{r}{4})) & \longrightarrow & \mathrm{H}^2(E(-\frac{r}{4}-1)) \end{array}$$

which implies $\mathrm{H}^0(E(\frac{r}{4})) \twoheadrightarrow \mathrm{H}^2(E(-\frac{r}{4}-1))$, that is, condition $(\diamond\diamond)$.

Let us see this in detail. We call $R := \mathbb{C}[x_0, x_1, x_2, x_3]$ the polynomial ring in 4 variables, and for any R-module $\boldsymbol{M}$ we denote by $\boldsymbol{M}^\vee$ its dual as R-module and by $\widetilde{\boldsymbol{M}}$ its sheafification. Combining the results [HR91, Prop. 3.2] and [Dec90, Prop. 1], we see that if F is a rank 2 vector bundle on $\mathbb{P}^3$, with first Chern class $c_1 = c_1(F)$, then the module $\boldsymbol{M} = \mathrm{H}^1_*(F)$ admits a minimal graded free resolution of the form:

$$0 \to L_4 \to L_3 \to L_2 \oplus L_0^\vee(c_1) \to L_1 \to L_0 \to \boldsymbol{M} \to 0,$$

where F is the cohomology of the monad $\widetilde{L_0^\vee}(c_1) \to \widetilde{L_1} \to \widetilde{L_0}$, and has a minimal graded free resolution of the form:

$$0 \to \widetilde{L_4} \to \widetilde{L_3} \to \widetilde{L_2} \to F \to 0.$$

Now recall that a k-instanton bundle is the cohomology of a monad:

$$\mathcal{O}_{\mathbb{P}^3}(-1)^k \to \mathcal{O}_{\mathbb{P}^3}^{2k+2} \to \mathcal{O}_{\mathbb{P}^3}(1)^k. \tag{4.1}$$

If $r \geq 24$, by Proposition 4.2(iv) E admits the following resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-\tfrac{r}{4}-2)^k \to \mathcal{O}_{\mathbb{P}^3}(-\tfrac{r}{4}-1)^{k+\frac{r}{2}} \oplus \mathcal{O}_{\mathbb{P}^3}(-\tfrac{r}{4})^{k-\frac{r}{2}-2} \to \mathcal{O}_{\mathbb{P}^3}(-\tfrac{r}{4}+1)^k \to E \to 0. \tag{4.2}$$

From (4.2) and (4.1) we obtain the associated sequences of free R-modules, and by juxtaposing them we resolve the first cohomology module $\boldsymbol{M}$.

$$0 \to R(-\tfrac{r}{4}-2)^k \to \begin{array}{c} R(-\frac{r}{4})^{k-\frac{r}{2}-2} \\ \oplus \\ R(-\frac{r}{4}-1)^{k+\frac{r}{2}} \end{array} \to \begin{array}{c} R(-\frac{r}{4}+1)^k \\ \oplus \\ R(-1)^k \end{array} \to R^{2k+2} \to R(1)^k \to \boldsymbol{M} \to 0.$$

We have that $\mathrm{Ext}^i_R(\boldsymbol{M}, R) = 0$ for $i \neq 4$, and $\mathrm{Ext}^4_R(\boldsymbol{M}, R)$ is identified via Serre duality with $\boldsymbol{M}^*(4) \simeq \mathrm{H}^2_*(E)$, where $\boldsymbol{M}^*$ is the dual of $\boldsymbol{M}$ as vector spaces. This gives the following graded resolution of the module $\mathrm{H}^2_*(E)$:

$$0 \to R(-1)^k \to R^{2k+2} \to \begin{array}{c} R(\frac{r}{4}-1)^k \\ \oplus \\ R(1)^k \end{array} \to \begin{array}{c} R(\frac{r}{4})^{k-\frac{r}{2}-2} \\ \oplus \\ R(\frac{r}{4}+1)^{k+\frac{r}{2}} \end{array} \to R(\tfrac{r}{4}+2)^k \to \mathrm{H}^2_*(E) \to 0.$$

In particular $\mathrm{H}^2_*(E)$ is generated as graded R-module by its elements of minimal degree $-\frac{r}{4}-2$. Hence all elements of $\mathrm{H}^2(E(-\frac{r}{4}-1))$ can be obtained as linear combination of elements of $\mathrm{H}^2(E(-\frac{r}{4}-2))$, with linear forms as coefficients. This concludes our proof for the case $r \geq 24$.

The cases $r = 8, 12$ and 20 are identical, once we substitute (4.2) with the resolutions entailed by Proposition 4.2 $(i), (ii)$ and (iii). We obtain the resolution of the second cohomology module and thus surjections $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes \mathrm{H}^2(E(-\tau)) \twoheadrightarrow \mathrm{H}^2(E(-\tau+1))$ with $\tau = 4, 5$ and 7, for $r = 8, 12$ and 20 respectively. □

As a consequence of Theorem 4.1, in order for a general instanton E to produce new examples of $(r+2) \times (r+2)$ skew-symmetric matrices of constant co-rank 2, we only need to find an element of $\mathrm{Ext}^2(E(\frac{r}{4}-1), E(-\frac{r}{4}-2))$ satisfying condition $(\diamond)$ required by Theorem 3.6. Doing this is far from being easy. In the next Sections 5 and 6 we show how when $r = 8$ and $r = 12$ this result can be achieved.

It is worth underlining that the difficulty of finding examples increases significantly as r grows. Already for the next two cases $r = 20$ and $r = 24$ the space $\mathrm{Ext}^2(E(\frac{r}{4}-1), E(-\frac{r}{4}-2)) \simeq \mathrm{H}^2(E \otimes E(-\frac{r}{2}-1))$ is expected to be zero. Indeed, the two would correspond to instantons of charge 10 and 14 respectively: in the first case we have $\chi(S^2E(-11)) = 0$, whereas for the latter we are in the even worse situation where $\chi(S^2E(-13)) < 0$.

5. Instantons of charge two, matrices of rank eight and Westwick's example

Here we analyse in detail the case of skew-symmetric matrices A of linear forms of size 10, having constant rank 8, and their relation with 2-instantons.

5.1. **A new point of view on Westwick's example.** When we began our study, the only known example of a 4-dimensional linear space of skew-symmetric constant co-rank 2 matrices was due to Westwick. It appeared with almost no explanation in [Wes96, page 168], where the author simply exhibited the following matrix:

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & -x_0 & x_1 & 0 & x_2 & x_3 \\ 0 & 0 & 0 & 0 & x_0 & x_1 & 0 & x_2 & x_3 & 0 \\ 0 & 0 & 0 & -x_0 & 0 & 0 & x_2 & -x_3 & 0 & 0 \\ 0 & 0 & x_0 & -x_1 & 0 & 0 & x_3 & 0 & 0 & 0 \\ 0 & -x_0 & -x_1 & 0 & -x_2 & -x_3 & 0 & 0 & 0 & 0 \\ -x_0 & -x_1 & 0 & -x_2 & x_3 & 0 & 0 & 0 & 0 & 0 \\ -x_1 & 0 & -x_2 & -x_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_2 & -x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.1}$$

where x_0, x_1, x_2, x_3 are independent variables. The exact sequence (2.3) here reads:

$$0 \to E(-2) \to \mathcal{O}_{\mathbb{P}^3}^{10} \xrightarrow{W} \mathcal{O}_{\mathbb{P}^3}^{10}(1) \to E(3) \to 0,$$

with $c_1(E) = 0$ and $c_2(E) = 2$. The bundle $E(2)$ is globally generated, so a general global section s of $E(2)$ vanishes along a smooth irreducible curve Y of degree $6 = c_2(E(2))$ having canonical sheaf $\omega_Y = \mathcal{O}_Y(c_1(E(2)) - 4) = \mathcal{O}_Y$ and genus $g = 1$, i.e. Y is an elliptic sextic. We have the standard exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^3} \xrightarrow{s} E(2) \to \mathcal{I}_Y(4) \to 0. \tag{5.2}$$

Moreover, since Y is not contained in a quadric, computing cohomology from (5.2) we get that $\mathrm{H}^0(E) = 0$, so E is stable. Tensoring (5.2) by $\mathcal{O}_{\mathbb{P}^3}(-4)$ and computing cohomology again we see that $\mathrm{H}^1(E(-2)) = 0$, which means that E is an instanton bundle of charge 2. We refer to it as *Westwick instanton* E_W.

We recall that all 2-instantons are special 't Hooft instantons [Har78, Coroll. 9.6], where the terminology goes as follows: a k-instanton E is *'t Hooft* if it comes via Hartshorne-Serre correspondence from the union of $k+1$ disjoint lines, while E is called *special* if $\mathrm{h}^0(E(1))$ attains the maximum possible value, namely $\mathrm{h}^0(E(1)) = 2$. Any 't Hooft of charge 2 is special, because 3 skew lines in $\mathbb{P}^3$ are always contained in a quadric.

In the case of Westwick's example, we can recover this directly. One has $\mathrm{h}^0(E_W(1)) = \mathrm{h}^0(\mathcal{I}_Y(3)) > 0$, so a non-zero global section of E gives:

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to E_W(1) \to \mathcal{I}_Z(2) \to 0, \tag{5.3}$$

where Z is the union of three skew lines in $\mathbb{P}^3$, or a flat degeneration of it, that is, the union of a double structure on a line ℓ and a second line ℓ', or a triple structure on a line. The curves Z and Y are connected by a liaison of type $(3,3)$.

Notice that from the cohomology sequence associated to (5.3), it follows that $\mathrm{H}^1(E(-2)) \simeq \mathrm{Ext}^1(E(2), \mathcal{O}_{\mathbb{P}^3}) = 0$, so $E(2)$ cannot be obtained as a quotient of a vector bundle of rank bigger than 2. (At least not in the sense of [SU09, Def. 1(iii)].)

Theorem 3.3 gives a new interpretation of the matrix (5.1): it can be seen as the explicit decomposition of the cone of the morphism corresponding to the extension (2.1). We would like to achieve a better understanding of the instanton E_W. We are especially interested in finding out its orbit under the natural action of $\mathrm{SL}(4)$.

Let us recall some more facts about 2-instantons and the moduli space $M_{\mathbb{P}^3}(2;0,2)$ of stable rank 2 vector bundles E on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 2$. Our main references are [Har78] and [CO02]. Since every section of $E(1)$ vanishes on a curve Z of degree 3 and genus -2, E determines, and is uniquely determined by, the following data:

(1) a smooth quadric $Q \subset \mathbb{P}^3$;
(2) one of the two rulings of Q, that we call the *first family*;
(3) a linear system g^1_3 without base points on the first family.

The curve Z generates the quadric Q, the lines in the support vary in the rulings of the first family, and the curves Z describe a base-point-free g^1_3 in it. It follows that the moduli space $M_{\mathbb{P}^3}(2;0,2)$ is fibred over $\mathbb{P}^9 \setminus \Delta$, the space of smooth quadrics, by two copies of a variety $\mathcal{V} \subset \mathbb{G}(1,3)$. $\mathcal{V}$ is the open set formed by vector 2-planes in $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(3))$ corresponding to the base point free g^1_3's. In particular $M_{\mathbb{P}^3}(2;0,2)$ is smooth of dimension 13. There is a natural action of $\mathrm{SL}(4)$ on it induced by automorphisms of $\mathbb{P}^3$. Up to this action there is a 1-dimensional family of non-equivalent bundles: $M_{\mathbb{P}^3}(2;0,2) /\!\!/ \mathrm{SL}(4) \simeq \mathcal{V} /\!\!/ \mathrm{SL}(2)$, and the latter is a good quotient isomorphic to $\mathbb{A}^1$. All fibres are orbits except for one, which is a union of two orbits, corresponding respectively to pencils with one triple point (dimension 3), and two triple points (dimension 2).

Moreover, E is completely determined by the set of its jumping lines in the Grassmannian $\mathbb{G}(1,3)$, which is as follows:

- ℓ is 2-jumping, i.e. $E|_\ell \simeq \mathcal{O}_\ell(-2) \oplus \mathcal{O}_\ell(2)$, if and only if ℓ is a line of the second family on Q;
- ℓ is 1-jumping, i.e. $E|_\ell \simeq \mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell(1)$, if and only if either $\ell \subset Q$ belongs to the first family and is double or triple for the curve Z that contains it, or ℓ is not contained in Q but meets it in two points of a divisor Z of the g^1_3;
- ℓ cannot be k-jumping for $k \geq 3$.

For the Westwick instanton E_W, a direct computation shows that the 2-jumping lines can be parametrised in the form $(x_0, x_1, \alpha x_0, \alpha x_1)$, for $\alpha \in \mathbb{C}$. These lines form the second ruling in the quadric Q, that has therefore equation $x_0x_3 - x_1x_2 = 0$. The two lines parametrised by $(0, x_1, 0, x_3)$ and $(x_0, 0, x_2, 0)$ are the only 1-jumping lines. This proves the following:

Theorem 5.1. *The Westwick instanton belongs to the most special orbit of* $M_{\mathbb{P}^3}(2;0,2)$ *under the natural action of* $\mathrm{SL}(4)$, *corresponding to the* g^1_3*'s with two triple points.*

In Remark 2.1 we saw that if two matrices are equivalent under the action of $\mathrm{SL}(r+2) \times \mathrm{SL}(4)$, their associated bundles will not necessarily be isomorphic as vector bundles, but will belong to the same orbit under the natural action of $\mathrm{SL}(4)$. (We have seen this issue with some detail for 2-instantons, and we refer to [CO02, Lemma 4.10 and Theorem 4.13] for the general case.) Hence from Theorem 5.1 we can deduce that any instanton bundle E of charge 2 that does not lie in the most special orbit can potentially give examples of 10×10 skew-symmetric matrices of co-rank 2 different from Westwick's. We will now see that this is indeed the case.

5.2. **Instanton bundles of charge two and matrices of rank eight.** We show that the construction of Theorem 3.6 works for all charge 2 instanton bundles (not only Westwick's one, and not only general ones). In light of the results we saw on $M_{\mathbb{P}^3}(2;0,2) /\!\!/ \mathrm{SL}(4)$ and the previous remarks, this means that our method shows the existence of a continuous family (at least of dimension $1 = \dim(M_{\mathbb{P}^3}(2;0,2) /\!\!/ \mathrm{SL}(4)))$ of examples of 4-dimensional linear spaces of skew-symmetric matrices of size 10 and constant rank 8.

Theorem 5.2. *Any 2-instanton on $\mathbb{P}^3$ induces a skew-symmetric matrix of linear forms in 4 variables having size 10 and constant rank 8.*

Proof. Let E be a 2-instanton, and let $U = \mathrm{H}^0(E(1))$. We prove that the following natural map is surjective:

$$\mathrm{H}^2(E \otimes E(-5)) \simeq \mathrm{Ext}^2(E(1), E(-4)) \to U^* \otimes \mathrm{H}^2(E(-4)).$$

Then condition $(\diamond)$ follows, and Theorem 3.6 applies. Recall that $\mathrm{H}^2(E\otimes E(-5)) \cong \mathrm{H}^2(S^2E(-5))$, and note that by Serre duality we are reduced to show that the following natural map, corresponding to Yoneda product, is injective:

$$U \otimes \mathrm{H}^1(E) \to \mathrm{H}^1(S^2E(1)). \tag{5.4}$$

Computing cohomology, from the resolution in Proposition 4.2(i)—or from (5.3) if one prefers—we see that U has dimension 2. Moreover the natural evaluation of global sections of $E(1)$ gives an exact sequence:

$$0 \to U \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to E \to \mathcal{O}_Q(-3,0) \to 0,$$

where Q is a smooth quadric in $\mathbb{P}^3$, and we let the zero-locus of a global sections of E be a divisor of class $|\mathcal{O}_Q(3,0)|$. (See the previous Subsection 5.1 for more details.) The symmetric square S^2 of this exact sequence gives:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to U \otimes E(-1) \to S^2E \to \mathcal{O}_Q(-6,0) \to 0.$$

Tensoring by $\mathcal{O}_{\mathbb{P}^3}(1)$ and taking cohomology we see that (5.4) is injective, because the cohomology of $\mathcal{O}_{\mathbb{P}^3}(-1)$ vanishes and $\mathcal{O}_Q(-5,1)$ has no global sections. This concludes the proof. ☐

6. Instantons of charge four and matrices of rank twelve

According to Westwick's computation of invariants [Wes96], the next possible value that the rank r can attain after 8 is 12, so we are now looking at 14×14 skew-symmetric matrices of linear forms in 4 variables, having constant rank 12.

In this Section we show from a theoretical point of view that general 4-instantons provide examples of these matrices. More in detail, in Theorem 6.1 we prove the existence of this kind of matrices starting from 4-instantons with certain cohomological conditions. We stress that general 4-instantons satisfy the requirements of the Theorem.

Then, in the Appendix we give an explicit matrix, with a short outline of the strategy to produce such examples.

Theorem 6.1. *Let E be a 4-instanton on $\mathbb{P}^3$ with natural cohomology, such that $E(2)$ is globally generated. Then $\mathrm{Ext}^2(E(2), E(-5))) \neq 0$, and a general element β of this extension group induces a skew-symmetric matrix of linear forms in 4 variables, having size 14 and constant rank 12.*

Proof. Analogously to what we did in the case of 2-instantons, we show that there is a surjection:

$$\mathrm{H}^2(S^2E(-7)) \twoheadrightarrow \mathrm{H}^0(E(2))^* \otimes \mathrm{H}^2(E(-5)).$$

Set $U = \mathrm{H}^0(E(2)$. From the assumption that E has natural cohomology, and applying Riemann-Roch to $E(2)$, we get $\dim(U) = 4$. Evaluation of sections provides a natural map

$g : U \otimes \mathcal{O}_{\mathbb{P}^3} \to E(2)$, which is surjective by assumption. Set $E^t(-2) := \operatorname{Ker}(g)$. We have a short exact sequence:

$$0 \to E^t(-2) \to U \otimes \mathcal{O}_{\mathbb{P}^3} \to E(2) \to 0. \tag{6.1}$$

In [D'A00, Thm. 1] it is shown that E^t is an instanton bundle of charge 4, and that in fact on an open subset of the moduli space of 4-instantons, the map $E \mapsto E^t$ is an involution with no fixed points.

Dualise sequence (6.1), take its second symmetric power S^2, and tensor it with $\mathcal{O}_{\mathbb{P}^3}(-3)$.

We get the following 4-term exact sequence:

$$0 \to S^2E(-5) \xrightarrow{h_1} U^* \otimes E(-5) \xrightarrow{h_2} \wedge^2 U^* \otimes \mathcal{O}_{\mathbb{P}^3}(-3) \xrightarrow{h_3} \mathcal{O}_{\mathbb{P}^3}(1) \to 0.$$

Let $\mathcal{F}$ be the image of the map h_2 above. Looking at the short exact sequence:

$$0 \to \mathcal{F} \to \wedge^2 U^* \otimes \mathcal{O}_{\mathbb{P}^3}(-3) \xrightarrow{h_3} \mathcal{O}_{\mathbb{P}^3}(1) \to 0,$$

and taking cohomology, we see that $\mathrm{H}^2(\mathcal{F}) = 0$. Then, taking cohomology of the short exact sequence:

$$0 \to S^2E(-5) \xrightarrow{h_1} U^* \otimes E(-5) \to \mathcal{F} \to 0,$$

we get the required natural surjective map $\mathrm{H}^2(S^2E(-7)) \twoheadrightarrow \mathrm{H}^0(E(2))^* \otimes \mathrm{H}^2(E(-5))$. The theorem is thus proved. □

Appendix: construction of explicit examples

We outline here an algorithmic approach to the construction of skew-symmetric matrices in 4 variables, having size 14 and constant rank 12. The algorithm is based on the commutative algebra system `Macaulay2` [GS]. Since the system runs better over finite fields, we fix a finite field $\boldsymbol{k}$ of characteristic different from 2 and we work over the polynomial ring $R = \boldsymbol{k}[x_0 \dots, x_3]$. Conceptually, the construction holds without modification in characteristic zero.

The algorithm goes as follows.

Step 1. *Construct a general* 4*-instanton from an elliptic curve of degree* 8 *in* $\mathbb{P}^3$*.*

Blow up a point p in $\mathbb{P}^2$ and embed the blown-up plane by the system of cubics through p as a Del Pezzo surface of degree 8 in $\mathbb{P}^8$. A hyperplane section C_0 of this surface is an elliptic curve of degree 8 in $\mathbb{P}^7$. By a general projection into $\mathbb{P}^3$, we thus get a smooth elliptic curve C of degree 8 in $\mathbb{P}^3$. Let I_C be the ideal of C in R. Then I_C has the following minimal graded free resolution:

$$0 \to R(-7)^4 \to R(-6)^{10} \to R(-4)^3 \oplus R(-5)^4 \to I_C \to 0.$$

Let $\boldsymbol{E}$ be the kernel of the induced map $R(2)^3 \to I_C(6)$. It turns out that $\boldsymbol{E}$ is $\mathrm{H}^0_*(E^t)$ (see the previous subsection), where E is the instanton associated via the Hartshorne-Serre correspondence to the curve C. We denote by $\boldsymbol{M}$ the module $\mathrm{H}^2_*(E)$; it has Hilbert function $4, 6, 4$ in degrees $-5, -4, -3$.

Step 2. *Use the* R*-module* $\boldsymbol{E}$*, together with a surjective map of* R*-modules* $k : \boldsymbol{E} \to \boldsymbol{M}(-7)$ *to write a* 14×14 *matrix* B *of linear forms.*

To explain this step, we remark that given an element $\beta \in \operatorname{Ext}^2(E(2), E(-5))$, combining the maps μ^2_t (cf. Section 3) for all $t \in \mathbb{Z}$ we get a map of R-modules:

$$\mu^2_* : \boldsymbol{E} = \mathrm{H}^0_*(E) \to \mathrm{H}^2_*(E(-7)) = \boldsymbol{M}(-7).$$

According to our construction, μ^2_* has to be an isomorphism in degree 2, and an epimorphism in higher degree. This epimorphism is obtained by linearity from the isomorphism in degree 2, since both $\boldsymbol{E}$ and $\boldsymbol{M}(-7)$ are generated in degree 2. We have seen in the proof of Theorem

6.1 that all isomorphisms in degree 2 come from an element $\beta \in \operatorname{Ext}^2(E(2), E(-5))$. Hence we only need a general epimorphism $k : \boldsymbol{E} \to \boldsymbol{M}(-7)$ and in fact a general morphism will do. The system `Macaulay2` is capable of providing such morphism explicitly, and expresses k as a map between the generators of $\boldsymbol{E}$ and those of $\boldsymbol{M}$.

To complete the argument, we resolve the truncations $\boldsymbol{E}_{\geq 3}$ and $\boldsymbol{M}_{\geq 3}$. We get presentations $u : R(-1)^{34} \to R^{20} \to \boldsymbol{E}_{\geq 3}(3)$ and $v : R(-1)^{20} \to R^6 \to \boldsymbol{M}_{\geq 3}(3)$. Using the map k_3 from the expression of $\boldsymbol{E}_{\geq 3}$ to those of $\boldsymbol{M}_{\geq 3}$ induced by k, we get the following commutative exact diagram:

$$\begin{array}{ccccccc}
R(-1)^{34} & \xrightarrow{u} & R^{20} & \longrightarrow & \boldsymbol{E}_{\geq 3}(3) & \longrightarrow & 0 \\
\vdots\, h & & \downarrow k_3 & & \downarrow k & & \\
R(-1)^{20} & \xrightarrow{v} & R^{6} & \longrightarrow & \boldsymbol{M}_{\geq 3}(3) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

The map h above is induced by the diagram. Taking sygygies of the maps above, we can complete the diagram to the following:

$$\begin{array}{ccccccc}
R(-1)^{14} & \xrightarrow{B} & R^{14} & \longrightarrow & \boldsymbol{F} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
R(-1)^{34} & \xrightarrow{u} & R^{20} & \longrightarrow & \boldsymbol{E}_{\geq 3}(3) & \longrightarrow & 0 \\
\downarrow h & & \downarrow k_3 & & \downarrow k & & \\
R(-1)^{20} & \xrightarrow{v} & R^{6} & \longrightarrow & \boldsymbol{M}_{\geq 3}(3) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

The module $\boldsymbol{F}$ induced above is resolved by the matrix B. Since $\boldsymbol{E}$ and $\boldsymbol{F}$ differ only by the Artinian module $\boldsymbol{M}$, the induced coherent sheaves on $\mathbb{P}^3$ are isomorphic, hence B has constant rank 2.

Step 3. *Skew-symmetrise B to obtain the required matrix A.*

To perform this step, we note that B and the opposite transpose matrix $-B^{\intercal}$ give rise to two modules $\boldsymbol{F}$ and $\boldsymbol{F}'$ as their cokernels, and the associated cokernel sheaves are isomorphic. These sheaves are stable, and thus simple, so any endomorphism of each one of them is a multiple of the identity. Therefore, we can consider a random morphism δ from $\boldsymbol{F}$ to $\boldsymbol{F}'$ to build an isomorphism from the resolution of $\boldsymbol{F}$ to that of $\boldsymbol{F}'$. Then we get an exact commutative diagram:

$$\begin{array}{ccccccc}
R(-1)^{14} & \xrightarrow{B} & R^{14} & \longrightarrow & \boldsymbol{F} & \longrightarrow & 0 \\
\downarrow \Delta^{\intercal} & & \downarrow \Delta & & \downarrow \delta & & \\
R(-1)^{14} & \xrightarrow{-B^{\intercal}} & R^{14} & \longrightarrow & \boldsymbol{F}' & \longrightarrow & 0,
\end{array}$$

where the matrix (of scalars) Δ is invertible. Therefore $A = \Delta B$ is skew-symmetric, and its cokernel is $\boldsymbol{F}'$, so again we deduce that A has constant rank 12.

We conclude by exhibiting such a matrix A. To avoid cumbersome coefficients and make the matrix more readable, let us work on the field $\boldsymbol{k} = \mathbb{Z}/7\mathbb{Z}$. Then $A = x_0A_0 + x_1A_1 + x_2A_2 + x_3A_3$, where the A_i's are the following skew-symmetric matrices:

$$A_0 = \begin{pmatrix}
0 & -2 & -1 & -3 & 3 & 3 & 3 & 3 & 1 & -3 & 1 & 1 & 1 & -3 \\
2 & 0 & -1 & -2 & 3 & 0 & -3 & -2 & 0 & 3 & 0 & 0 & -2 & 0 \\
1 & 1 & 0 & 3 & -3 & 2 & -3 & -2 & -3 & -3 & -2 & -2 & -3 & -3 \\
3 & 2 & -3 & 0 & 2 & -3 & 1 & 0 & 2 & 1 & 3 & -1 & 0 & 1 \\
-3 & -3 & 3 & -2 & 0 & -2 & -3 & 3 & -3 & -2 & -1 & 2 & 2 & 1 \\
-3 & 0 & -2 & 3 & 2 & 0 & 3 & -3 & -3 & 1 & 3 & -1 & -3 & 1 \\
-3 & 3 & 3 & -1 & 3 & -3 & 0 & 1 & 3 & -1 & 3 & -2 & -1 & -2 \\
-3 & 2 & 2 & 0 & -3 & 3 & -1 & 0 & 3 & 3 & 2 & -3 & -1 & 0 \\
-1 & 0 & 3 & -2 & 3 & 3 & -3 & -3 & 0 & -1 & 3 & -2 & -3 & -3 \\
3 & -3 & 3 & -1 & 2 & -1 & 1 & -3 & 1 & 0 & -1 & -2 & 2 & -1 \\
-1 & 0 & 2 & -3 & 1 & -3 & -3 & -2 & -3 & 1 & 0 & 1 & -3 & -1 \\
-1 & 0 & 2 & 1 & -2 & 1 & 2 & 3 & 2 & 2 & -1 & 0 & 3 & -2 \\
-1 & 2 & 3 & 0 & -2 & 3 & 1 & 1 & 3 & -2 & 3 & -3 & 0 & 1 \\
3 & 0 & 3 & -1 & -1 & -1 & 2 & 0 & 3 & 1 & 1 & 2 & -1 & 0
\end{pmatrix}$$

$$A_1 = \begin{pmatrix}
0 & -2 & -2 & 3 & -3 & 0 & -2 & -3 & 3 & -1 & 2 & 0 & 2 & -3 \\
2 & 0 & 3 & -1 & 1 & 2 & -3 & -1 & -2 & -1 & -1 & -3 & 1 & 2 \\
2 & -3 & 0 & -2 & 1 & 1 & 1 & -1 & 2 & -3 & 0 & -3 & 2 & -3 \\
-3 & 1 & 2 & 0 & 2 & -1 & 1 & -2 & -1 & -2 & 1 & 2 & 2 & -3 \\
3 & -1 & -1 & -2 & 0 & 1 & -1 & 1 & -2 & 0 & -1 & 2 & 0 & 0 \\
0 & -2 & -1 & 1 & -1 & 0 & 3 & 0 & -2 & 2 & 2 & -3 & -3 & 1 \\
2 & 3 & -1 & -1 & 1 & -3 & 0 & -3 & 2 & 3 & -1 & -2 & -2 & 3 \\
3 & 1 & 1 & 2 & -1 & 0 & 3 & 0 & 1 & 1 & 3 & 0 & 3 & -1 \\
-3 & 2 & -2 & 1 & 2 & 2 & -2 & -1 & 0 & 2 & -1 & -3 & 1 & 2 \\
1 & 1 & 3 & 2 & 0 & -2 & -3 & -1 & -2 & 0 & -1 & 1 & 3 & -1 \\
-2 & 1 & 0 & -1 & 1 & -2 & 1 & -3 & 1 & 1 & 0 & -3 & 2 & -3 \\
0 & 3 & 3 & -2 & -2 & 3 & 2 & 0 & 3 & -1 & 3 & 0 & 3 & -1 \\
-2 & -1 & -2 & -2 & 0 & 3 & 2 & -3 & -1 & -3 & -2 & -3 & 0 & 0 \\
3 & -2 & 3 & 3 & 0 & -1 & -3 & 1 & -2 & 1 & 3 & 1 & 0 & 0
\end{pmatrix}$$

$$A_2 = \begin{pmatrix}
0 & 2 & 2 & -3 & 3 & 2 & -1 & -1 & 1 & 1 & 0 & 2 & -3 & -2 \\
-2 & 0 & -2 & 3 & 3 & -1 & 1 & -1 & -3 & -2 & 1 & -3 & -2 & -2 \\
-2 & 2 & 0 & 1 & 3 & 1 & 3 & 2 & 2 & 3 & 2 & 1 & 0 & -3 \\
3 & -3 & -1 & 0 & -3 & -1 & 1 & -3 & 3 & -1 & -3 & 2 & -3 & 1 \\
-3 & -3 & -3 & 3 & 0 & 3 & -2 & -3 & 3 & 1 & -3 & -1 & 0 & 2 \\
-2 & 1 & -1 & 1 & -3 & 0 & 3 & -2 & 0 & -2 & 0 & -2 & -2 & -2 \\
1 & -1 & -3 & -1 & 2 & -3 & 0 & 0 & 2 & -1 & -2 & -3 & 2 & -2 \\
1 & 1 & -2 & 3 & 3 & 2 & 0 & 0 & -2 & 0 & 2 & -2 & 0 & 3 \\
-1 & 3 & -2 & -3 & -3 & 0 & -2 & 2 & 0 & -1 & -1 & -1 & 0 & -1 \\
-1 & 2 & -3 & 1 & -1 & 2 & 1 & 0 & 1 & 0 & -2 & 3 & -2 & 3 \\
0 & -1 & -2 & 3 & 3 & 0 & 2 & -2 & 1 & 2 & 0 & -3 & 3 & -1 \\
-2 & 3 & -1 & -2 & 1 & 2 & 3 & 2 & 1 & -3 & 3 & 0 & 0 & -1 \\
3 & 2 & 0 & 3 & 0 & 2 & -2 & 0 & 0 & 2 & -3 & 0 & 0 & 3 \\
2 & 2 & 3 & -1 & -2 & 2 & 2 & -3 & 1 & -3 & 1 & 1 & -3 & 0
\end{pmatrix}$$

$$A_3 = \begin{pmatrix}
0 & -3 & 2 & -3 & -1 & -1 & 3 & -2 & 3 & 3 & 3 & 0 & -3 & -3 \\
3 & 0 & -3 & 1 & 1 & 2 & -1 & -3 & -1 & 0 & 3 & -3 & 0 & -1 \\
-2 & 3 & 0 & 1 & -1 & 0 & -1 & -2 & 3 & 0 & -1 & -2 & 1 & -2 \\
3 & -1 & -1 & 0 & 3 & 2 & -1 & 0 & 1 & -3 & -3 & -1 & -1 & 3 \\
1 & -1 & 1 & -3 & 0 & 3 & 3 & 0 & 0 & -3 & -3 & 3 & 2 & 3 \\
1 & -2 & 0 & -2 & -3 & 0 & -3 & 3 & 3 & -3 & -3 & -2 & 1 & 1 \\
-3 & 1 & 1 & 1 & -3 & 3 & 0 & 0 & 3 & 2 & 0 & -3 & 2 & 0 \\
2 & 3 & 2 & 0 & 0 & -3 & 0 & 0 & 0 & 1 & -1 & -2 & 1 & 1 \\
-3 & 1 & -3 & -1 & 0 & -3 & -3 & 0 & 0 & 3 & 2 & -3 & -1 & -1 \\
-3 & 0 & 0 & 3 & 3 & 3 & -2 & -1 & -3 & 0 & -2 & -2 & -3 & -2 \\
-3 & -3 & 1 & 3 & 3 & 3 & 0 & 1 & -2 & 2 & 0 & 1 & 2 & -3 \\
0 & 3 & 2 & 1 & -3 & 2 & 3 & 2 & 3 & 2 & -1 & 0 & 2 & -2 \\
3 & 0 & -1 & 1 & -2 & -1 & -2 & -1 & 1 & 3 & -2 & -2 & 0 & -3 \\
3 & 1 & 2 & -3 & -3 & -1 & 0 & -1 & 1 & 2 & 3 & 2 & 3 & 0
\end{pmatrix}$$

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