RANK 2 STABLE SHEAVES WITH ODD DETERMINANT ON FANO THREEFOLDS OF GENUS 9.

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Abstract. According to Mukai and Iliev, a smooth prime Fano threefold X of genus 9 is associated with a surface $\mathbb{P}(\mathcal{V})$, ruled over a smooth plane quartic Γ, and the derived category of Γ embeds into that of X by a theorem of Kuznetsov.

We use this setup to study the moduli spaces of rank-2 stable sheaves on X with odd determinant. For each $c_2 \geq 7$, we prove that a component of their moduli space $\mathsf{M}_X(2,1,c_2)$ is birational to a Brill-Noether locus of vector bundles with fixed rank and degree on Γ, having enough sections when twisted by $\mathcal{V}$.

For $c_2 = 7$, we prove that $\mathsf{M}_X(2,1,7)$ is isomorphic to the blow-up of the Picard variety $\mathrm{Pic}^2(\Gamma)$ along the curve parametrizing lines contained in X.

1. Introduction

Let X be a smooth complex projective threefold, whose Picard group is generated by an ample divisor H_X. We consider Maruyama's coarse moduli scheme $\mathsf{M}_X(r, c_1, c_2)$ of H_X-semistable rank r sheaves F on X with $c_i(F) = c_i$ and $c_3(F) = 0$. This moduli space is a projective variety, and not much more is known in general; still many results are available in special cases. For instance, rank 2 bundles on $\mathbb{P}^3$ have been intensively studied since [Bar77].

Since [AHDM78] and [AW77], the case which has attracted most attention is that of *instanton bundles* on $\mathbb{P}^3$, i.e. stable rank 2 bundles F with $c_1(F) = 0$ and $\mathrm{H}^1(\mathbb{P}^3, F(-2)) = 0$. The starting points to analyze these bundles are given by Beilinson's theorem describing the derived category of $\mathbb{P}^n$, and by the related notion of monad, see [BH78]. Smoothness of the moduli space of these bundles has recently been announced for any c_2 (see [JV11]) together with irreducibility for odd c_2 (see [Tik11]).

If one desires to set up a similar analysis over a threefold X other than $\mathbb{P}^3$, one direction is to look at Fano threefolds. Recall that if the anticanonical divisor $-K_X$ is linearly equivalent to $i_X H_X$, for some positive integer i_X and ample H_X, then the variety X is called a *Fano threefold of index* i_X. A natural way to generalize instanton bundles to X would appear to consider stable rank 2 bundles $\mathcal{E}$ on X with:

$$\mathrm{H}^1(X, \mathcal{E}) = 0, \qquad \mathcal{E}^* \otimes \omega_X \cong \mathcal{E}. \tag{1.1}$$

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In the case $X = \mathbb{P}^3$, the bundle $\mathcal{E}(2)$ would thus be an instanton bundle.

In this paper we look at Fano threefolds X of index 1, and study the moduli space $\mathsf{M}_X(2, c_1, c_2)$, and its subspace consisting of bundles $\mathcal{E}$ with the above property. The genus of a Fano threefold X of index 1 is defined as $g = H_X^3/2 + 1$. Note that one can assume $c_1 \in \{0, 1\}$, so we speak of bundles with odd or even determinant. The bundles satisfying (1.1) have odd determinant for $i_X = 1$, in fact $c_1(\mathcal{E}(1)) = 1$, and one can dare call them odd instantons. See [Fae11] for more on these notions.

To describe our point of view, let us start with low c_2. One sees that $\mathsf{M}_X(2, 1, c_2)$ is empty for $c_2 < m_g = \lceil g/2 \rceil + 1$. The case of minimal $c_2 = m_g$ is well understood, and we refer to the survey we gave in [BF11]. For higher c_2, we constructed in [BF08] a component $\mathsf{M}(d)$ of $\mathsf{M}_X(2, 1, d)$, whose general element is a vector bundle $F(-1) = \mathcal{E}$ such that $\mathcal{E}$ is an odd instanton, i.e. it satisfies (1.1). In order to study the component $\mathsf{M}(d)$ and, as far as possible, the whole space $\mathsf{M}_X(2, 1, d)$, our idea is to make use of Kuznetsov's semiorthogonal decomposition of the derived category of X (see [Kuz06]). This provides a suitable homological method, rephrasing the language of monads and Beilinson's theorem.

To state our results, we focus on Fano threefolds X of genus 9. Recall that, by a result of Mukai, [Muk88,Muk89], the variety X is a linear section of the Lagrangian Grassmannian sixfold $\Sigma = \mathbb{LG}(3, 6)$, that is $X = \Sigma \cap \mathbb{P}^{11}$. We consider the projectively dual plane quartic $\Gamma = \Sigma^\vee \cap \mathbb{P}^2$, obtained cutting the dual variety of Σ with the linear section orthogonal to X, and the integral functor $\mathbf{\Phi}^! : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma)$, according to Kuznetsov's theorem, [Kuz06]. This functor is right adjoint to the fully faithful functor $\mathbf{\Phi}$, provided by the universal sheaf $\mathscr{E}$ on $X \times \Gamma$ for the fine moduli space:

$$\Gamma \cong \mathsf{M}_X(2, 1, 6).$$

Recall that the threefold X is associated with a rank 2 stable bundle $\mathcal{V}$ on Γ, in such a way that $\mathbb{P}(\mathcal{V})$ is isomorphic to the Hilbert scheme $\mathscr{H}_2^0(X)$ of conics contained in X, see [Ili03]. The main result of this paper is the following.

Theorem. *The map $\varphi : F \mapsto \mathbf{\Phi}^!(F)$ gives:*

A) for any $d \geq 8$, a birational map of $\mathsf{M}(d)$ to a generically smooth $(2d-11)$-dimensional component of the Brill-Noether locus:

$$\{\mathcal{F} \in \mathsf{M}_\Gamma(d-6, d-5) \,|\, \mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{F}) \geq d-6\};$$

B) an isomorphism of $\mathsf{M}_X(2, 1, 7)$ with the blow-up of $\mathrm{Pic}^2(\Gamma)$ along a curve isomorphic to the Hilbert scheme $\mathscr{H}_1^0(X)$ of lines contained in X. The exceptional divisor consists of the sheaves in $\mathsf{M}_X(2, 1, 7)$ which are not globally generated.

In particular we prove that $\mathsf{M}_X(2, 1, 7)$ is an irreducible threefold which is smooth as soon as $\mathscr{H}_1^0(X)$ is smooth. Note that this result closely resembles those of [Dru00], [IM00], [MT01], regarding rank 2 sheaves on a smooth cubic threefold in $\mathbb{P}^4$, and relying on the Abel-Jacobi mapping.

The paper is organized as follows. In the next section we set up some notation. Then, in Section 3, we review the geometry of prime Fano threefolds X of genus 9, and we interpret some well-known facts concerning lines and conics contained in X in the language of vector bundles and derived categories. In Section 4, we prove part (A) of the theorem above. Section 5 is devoted to part (B).

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2. Definitions and preliminary results

In this section we will collect some preliminary material and set up some notation.

2.1. **Basic material.** Let n be a positive integer. We consider a smooth connected complex projective n-dimensional variety X, equipped with an ample divisor class H_X. The canonical bundle of X is denoted by ω_X.

Given a subscheme Z of X, we write F_Z for $F \otimes \mathscr{O}_Z$ and we denote by $\mathcal{I}_{Z,X}$ the ideal sheaf of Z in X, and by $N_{Z,X}$ its normal sheaf. We will frequently drop the second subscript. The degree $\deg(L)$ of a divisor class L over X is defined as the degree of $L \cdot H_X^{n-1}$.

2.1.1. *Cohomology and derived categories.* Given a pair of coherent sheaves (F, E) on X, we will write $\mathrm{ext}^k_X(F, E)$ for the dimension of the vector space $\mathrm{Ext}^k_X(F, E)$, and similarly $\mathrm{h}^k(X, F) = \dim \mathrm{H}^k(X, F)$. The Euler characteristic of (F, E) is defined as $\chi(F, E) = \sum_k (-1)^k \mathrm{ext}^k_X(F, E)$ and $\chi(F)$ is defined as $\chi(\mathscr{O}_X, F)$.

Given two sheaves E, F, we denote by $e_{E,F}$ the natural evaluation map:

$$e_{E,F} : \mathrm{Hom}_X(E, F) \otimes E \to F.$$

A sheaf F on X is called *simple* if $\hom_X(F, F) = 1$, and *exceptional* if it is simple and moreover $\mathrm{Ext}^k_X(F, F) = 0$ for all $k \geq 1$.

As a basic tool, we will use the derived category $\mathbf{D^b}(X)$ of complexes of coherent sheaves on X with bounded cohomology. For definitions and notation we refer to [GM96]. In particular we write $[j]$ for the j-th shift to the left in the derived category, and by $D(-)$ the anti-auto-equivalence of $\mathbf{D^b}(X)$ defined by $D(-) = \mathbf{R}\mathcal{H}om_X(-, \mathscr{O}_X)$.

Recall that a subcategory $\mathcal{A}$ of $\mathbf{D^b}(X)$ is called left or right admissible if the inclusion $i_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathbf{D^b}(X)$ has a left or right adjoint, which will be denoted as usual by $i^*_{\mathcal{A}}$ and $i^!_{\mathcal{A}}$ ($\mathcal{A}$ is called admissible if it is so in both ways). Assuming $\mathcal{A}$ admissible, we have $\mathbf{D^b}(X) = \langle \mathcal{A}, {}^\perp\mathcal{A} \rangle = \langle \mathcal{A}^\perp, \mathcal{A} \rangle$, $\mathcal{A}^\perp$ is left admissible and ${}^\perp\mathcal{A}$ is right admissible. In this situation, the left and right mutations through $\mathcal{A}$ are defined respectively as:

$$L_{\mathcal{A}} = i_{\mathcal{A}^\perp} i^*_{\mathcal{A}^\perp} \qquad \text{and} \qquad R_{\mathcal{A}} = i_{{}^\perp\mathcal{A}} i^!_{{}^\perp\mathcal{A}}.$$

We refer to [Gor90, Bon89] for more details.

If $\mathcal{A}$ is generated by an exceptional object A, and B is an object of $\mathbf{D^b}(X)$, the left and right mutations of B through A are defined, respectively, by the triangles:

$$\mathrm{L}_A B[-1] \to \mathrm{Hom}(A, B) \otimes A \to B \to \mathrm{L}_A B,$$
$$\mathrm{R}_A B \to A \to \mathrm{Hom}(A, B)^* \otimes B \to \mathrm{R}_A B[1].$$

2.1.2. *ACM varieties and sheaves.* Assume H_X very ample, so that X is a smooth n-dimensional subvariety of $\mathbb{P}^m$. Given a coherent sheaf F on X, we write $F(t)$ for $F \otimes \mathscr{O}_X(tH_X)$. If the coordinate ring of X is Cohen-Macaulay, then X is said to be arithmetically Cohen-Macaulay (ACM). If $n > 0$, then X is ACM if and only if $\mathrm{H}^k(\mathbb{P}^m, \mathcal{I}_{X,\mathbb{P}^m}(t)) = 0$ for any integer t and for any $0 < k \leq n$. A locally free sheaf F on an ACM variety X is called *ACM* (*arithmetically Cohen-Macaulay*) if it has no intermediate cohomology, i.e. if $\mathrm{H}^k(X, F(t)) = 0$ for all integer t and for any $0 < k < n$. The corresponding module over the coordinate ring of X is thus a maximal Cohen-Macaulay module. For more details see e.g. [CDH05].

2.1.3. *Hilbert schemes.* Given a coherent sheaf F on X, we write $F(t)$ for $F \otimes \mathscr{O}_X(tH_X)$, we denote by $p(F,t)$ the Hilbert polynomial $\chi(F(t))$ of the sheaf F. Given a numerical polynomial $p(t)$, we let $\mathrm{Hilb}_{p(t)}(X)$ be the *Hilbert scheme* of closed subschemes of X with Hilbert polynomial $p(t)$. In case $p(t)$ has degree one, we let $\mathscr{H}^g_d(X)$ be the union of irreducible components of $\mathrm{Hilb}_{1-g+dt}(X)$ containing integral curves of degree d and arithmetic genus g. Both $\mathscr{H}^g_d(X)$ and $\mathrm{Hilb}_{p(t)}(X)$ are projective schemes.

2.1.4. *Chern classes.* The Chern classes $c_k(F)$ are defined for any coherent sheaf F on X, and actually for any object of $\mathbf{D^b}(X)$, and take values in $\mathrm{H}^{k,k}(X)$ (see for instance [Căl05]). The degree of a coherent sheaf F is defined as $\deg(c_1(F))$. In the sequel, the Chern classes will be denoted by integers as soon as $\mathrm{H}^{k,k}(X)$ has dimension 1 and the choice of a generator is clear. The Chern polynomial of a coherent sheaf F on X is defined as $c_F(t) = 1 + c_1(F)t + \ldots + c_n(F)t^n$. Let Z be an integral subscheme of X, of codimension $m \geq 1$, denote by $[Z]$ its fundamental class in $\mathrm{H}^{m,m}(X)$. We recall that when a sheaf T is supported at Z, and has rank r at a generic point of Z, then we have $c_k(T) = 0$ for $1 \leq k \leq m-1$ and:

$$c_m(T) = (-1)^{m-1} r[Z]. \tag{2.1}$$

If Z is not integral, a similar formula holds by taking the sum over all integral components of minimal codimension appearing in the support of T, weighted by their multiplicity.

2.2. **Torsion-free and reflexive sheaves.** Given a coherent sheaf F on X, we denote by $F^* = \mathcal{H}om_X(F, \mathscr{O}_X)$ the dual of F. Recall that a coherent sheaf F on X is *reflexive* if the natural map $F \to F^{**}$ of F to its double dual is an isomorphism. Any locally free sheaf is reflexive, and any reflexive sheaf is torsion-free. Recall that a coherent sheaf F on X is reflexive if and only if it can be included into a locally free sheaf E with E/F torsion-free, see [Har80, Proposition 1.1]. Moreover, by [Har80, Proposition 1.9], any reflexive rank-1 sheaf is invertible (recall that X is smooth and irreducible). Finally, we will use a straightforward generalization of [Har80, Proposition 2.6] which implies that the third Chern class $c_3(F)$ of a rank 2 reflexive sheaf F on a smooth projective threefold satisfies $c_3(F) \geq 0$, with equality attained iff F is locally free. We will need the following simple lemma.

Lemma 2.1. *Let F be a vector bundle on X and $\mathcal{F}$ be a torsion-free sheaf such that $c_1(F) = c_1(\mathcal{F})$ and $\mathrm{rk}(F) = \mathrm{rk}(\mathcal{F})$. Then, any injective map $F \to \mathcal{F}$ is an isomorphism.*

Proof. We have the exact sequence:

$$0 \to F \to \mathcal{F} \to T \to 0$$

where the quotient T is a torsion sheaf with $c_1(T) = 0$. Assume $T \neq 0$. Note that $\mathscr{E}xt^k_X(T, F) \cong \mathscr{E}xt^k_X(T, \mathscr{O}_X) \otimes F = 0$ for $k = 0, 1$, because T is supported in codimension at least 2, since $c_1(T) = 0$. Then from the spectral sequence:

$$E_2^{p,q} = \mathrm{H}^p(X, \mathscr{E}xt^q_X(T, \mathscr{O}_X)) \Longrightarrow \mathrm{Ext}^{p+q}_X(T, \mathscr{O}_X),$$

we get $\mathrm{Ext}^1_X(T, F) = 0$. This implies $\mathcal{F} \cong F \oplus T$, which is a contradiction because $\mathcal{F}$ is torsion-free. □

2.3. **Summary on semistable vector bundles and sheaves.** Let us now recall a few well-known facts about semi-stable sheaves on projective varieties. We refer to the book [HL97] for a more detailed account of these notions. Stability depends on the choice of an ample divisor H_X, so we will deal with polarized manifolds from now on. We recall that a torsion-free coherent sheaf F on X is (Gieseker) *semi-stable* if for any coherent subsheaf E, with $\mathrm{rk}(E) < \mathrm{rk}(F)$, one has $p(E, t)/\mathrm{rk}(E) \leq p(F, t)/\mathrm{rk}(F)$ for $t \gg 0$. The sheaf F is called *stable* if the inequality above is strict for all E and $t \gg 0$. A semi-stable sheaf is called *poly-stable* if it is the direct sum of stable sheaves of the same slope.

The *slope* of a torsion-free sheaf $F \neq 0$ is defined as $\mu(F) = \deg(F)/\mathrm{rk}(F)$. The *normalized* twist F_{norm} of F is set to be the unique sheaf $F(t)$ with $-1 < \mu(F(t)) \leq 0$. We recall that a torsion-free coherent sheaf F is *μ-semi-stable* if for any coherent subsheaf E, with $\mathrm{rk}(E) < \mathrm{rk}(F)$, one has $\mu(E) \leq \mu(F)$. The sheaf F is called *μ-stable* if the above inequality is strict for all E. We recall that the *discriminant* of a sheaf F is:

$$\Delta(F) = 2rc_2(F) - (r-1)c_1(F)^2.$$

Bogomolov's inequality, see for instance [HL97, Theorem 3.4.1], states that if F is μ-semi-stable, then we have:

$$\Delta(F) \cdot H_X^{n-2} \geq 0.$$

Another useful tool is Hoppe's criterion, see [Hop84, Lemma 2.6], or [AO94, Theorem 1.2]. It says that, if the line bundle H_X is very ample and generates $\mathrm{Pic}(X)$, and F is a vector bundle on X of rank r, we have:

(2.2)

$$\text{if } \mathrm{H}^0(X, (\wedge^p F)_{\mathrm{norm}}) = 0, \text{ for all } 0 < p < r, \text{ then the bundle } F \text{ is } \mu\text{-stable.}$$

We introduce here some notation concerning moduli spaces. Recall that two semi-stable sheaves are *S-equivalent* if the direct sum of all successive quotients associated with their Jordan-Hölder filtrations are isomorphic. We denote by $\mathsf{M}_X(r, c_1, \ldots, c_n)$ the moduli space of S-equivalence classes of rank r torsion-free semi-stable sheaves on X with Chern classes $c_1, \ldots, c_n$. We will drop the values of the classes c_k from k_0 on when they are zero from k_0 on. The class in $\mathsf{M}_X(r, c_1, \ldots, c_n)$ of a given sheaf F will be denoted again by F.

2.4. **Basic notions on prime Fano threefolds and K3 surfaces.** Let now X be a smooth projective variety of dimension 3. Recall that X is called *Fano* if its anticanonical divisor class $-K_X$ is ample. A Fano threefold X is *prime* if its Picard group is generated by the class of K_X. These varieties are classified up to deformation, see for instance [IP99, Chapter IV]. The number of deformation classes is 10. Each class is characterized by the *genus*, which is the integer g such that $\deg(X) = -K_X^3 = 2g - 2$. Recall that the genus g of a prime Fano threefold satisfies $2 \leq g \leq 10$ or $g = 12$.

If X is a prime Fano threefold of genus g, the Hilbert scheme $\mathscr{H}_1^0(X)$ of lines contained in X is a scheme of pure dimension 1. The threefold X is said to be *exotic* if the Hilbert scheme $\mathscr{H}_1^0(X)$ contains a component which is non-reduced at any point. It turns out that no threefold of genus 9 is exotic, see [GLN06]. In particular the normal bundle of a general line $L \subset X$ splits as $\mathscr{O}_L \oplus \mathscr{O}_L(-1)$. It is well-known that, if X is general, then the scheme $\mathscr{H}_1^0(X)$ is a smooth irreducible curve.

Remark that the cohomology groups $\mathrm{H}^{k,k}(X)$ of a prime Fano threefold X of genus g are generated by the divisor class H_X (for $k = 1$), the class L_X of a line contained in X (for $k = 2$), the class P_X of a closed point of X (for $k = 3$). Hence we will denote the Chern classes of a sheaf on X by the integral multiple of the corresponding generator. Recall that $H_X^2 = (2g - 2)L_X$.

We will use the geometry of lines and conics contained in X. So let us note that, given a line L and a conic C contained in X, the Chern classes of $\mathscr{O}_L$ and $\mathscr{O}_C$ satisfy:

$$
\begin{aligned}
&c_1(\mathscr{O}_L(t)) = 0, && c_2(\mathscr{O}_L(t)) = -1, && c_3(\mathscr{O}_L(t)) = 1 + 2t,\\
&c_1(\mathscr{O}_C(t)) = 0, && c_2(\mathscr{O}_C(t)) = -2, && c_3(\mathscr{O}_C(t)) = 4t,
\end{aligned}
$$

where the first and the second Chern classes are given by (2.1) and the third Chern classes can be computed by Hirzebruch-Riemann-Roch (we will write down the formula explicitly for $g = 9$, see (3.11) next; for the formula for any genus see [BF08, (2.11)]).

A smooth projective surface S is a *K3 surface* if it has trivial canonical bundle and irregularity zero. We recall by [HL97, Part II, Chapter 6] that, given a stable sheaf F of rank r on a K3 surface S polarized by H_S, the dimension at F of the moduli space $\mathsf{M}_S(r, c_1, c_2)$ is:

$$\Delta(F) - 2\,(r^2 - 1).$$

3. Geometry of prime Fano threefolds of genus 9

From now on we will denote by X a smooth prime Fano threefold of genus 9. We will collect in this section some fundamental remarks on these threefolds.

3.1. **Tautological bundle of rank 3.** Most of the basic features of a Fano threefolds X of genus 9 come from $\mathsf{Sp}(3)$-geometry, and can be encoded in the properties of a tautological bundle of rank 3. For a detailed account on these varieties, we refer to the papers [Muk88, Muk89, Ili03, IR05].

By a result of Mukai, the threefold X is isomorphic to a 3-codimensional linear section of the Lagrangian Grassmannian Σ of 3-dimensional subspaces of a 6-dimensional vector space V which are isotropic with respect to a skew-symmetric 2-form ω. The manifold Σ is homogeneous for the complex Lie group $\mathsf{Sp}(3)$, which acts on V preserving ω, and is equivariantly embedded in $\mathbb{P}^{13} = \mathbb{P}(\ker(\wedge^2 V \xrightarrow{\omega} \mathbb{C}))$. In other words, the manifold Σ, which is a Hermitian symmetric space, can be written as $\mathsf{Sp}(3)/\mathsf{P}(\alpha_3)$, where $\mathsf{P}(\alpha_3)$ is the parabolic subgroup associated with the longest root of the Lie algebra of $\mathsf{Sp}(3)$. The Lie algebra of this group has dimension 21 and its Dynkin diagram is of type C_3.

The divisor class H_X embeds X in $\mathbb{P}^{10}$ as an ACM variety. Indeed, by [IR05, Section 2.3], the variety Σ is ACM in $\mathbb{P}^{13}$, and this implies that the linear section X is ACM too. A very general hyperplane section S of X is a smooth K3 surface polarized by the restriction H_S of H_X to S, with Picard number 1 and sectional genus 9, and S is an ACM subvariety of $\mathbb{P}^9$.

The manifold Σ is equipped with a tautological homogeneous rank 3 subbundle $\mathcal{U}$, and we still denote by $\mathcal{U}$ its restriction to X. The tautological exact sequence on X, obtained as restriction from Σ, reads:

$$0 \to \mathcal{U} \to V \otimes \mathscr{O}_X \to \mathcal{U}^* \to 0. \tag{3.1}$$

Let us review the properties of the vector bundle $\mathcal{U}$. Its Chern classes satisfy $c_1(\mathcal{U}) = -1$, $c_2(\mathcal{U}) = 8$, $c_3(\mathcal{U}) = -2$. The bundle $\mathcal{U}$ is exceptional (see e.g. [Kuz06, Section 6.3]). We have the following lemma.

Lemma 3.1. *The bundle $\mathcal{U}$ is stable and ACM. The same is true for its restriction $\mathcal{U}_S$ to a smooth hyperplane section surface S with $\mathrm{Pic}(S) = \langle H_S \rangle$.*

Proof. Applying Borel-Bott-Weil's theorem (see e.g. [Wey03]) on Σ, we obtain:

$$\mathrm{H}^k(\Sigma, \mathcal{U}(-t)) = 0, \qquad \text{for} \qquad \begin{cases} \text{all } k \text{ and } 0 \le t \le 3, \\ k \neq 0 \text{ and } t \le -1, \\ k \neq 6 \text{ and } t \ge 4. \end{cases} \tag{3.2}$$

Consider the Koszul complex of X in Σ:

$$0 \to \mathscr{O}_\Sigma(-3) \to \mathscr{O}_\Sigma(-2)^3 \to \mathscr{O}_\Sigma(-1)^3 \to \mathscr{O}_\Sigma \to \mathscr{O}_X \to 0.$$

Tensoring it with $\mathcal{U}$, and using (3.2), it easily follows that $\mathcal{U}$ is ACM on X. Since $\wedge^2 \mathcal{U} \cong \mathcal{U}^*(-1)$, by Serre duality we get $\mathrm{H}^0(X, \wedge^2 \mathcal{U}) = 0$, so $\mathcal{U}$ is stable by Hoppe's criterion, see (2.2).

To check the statement on S, consider the defining exact sequence:

$$0 \to \mathscr{O}_X(-1) \to \mathscr{O}_X \to \mathscr{O}_S \to 0. \tag{3.3}$$

Since $\mathcal{U}$ is ACM on X, tensoring (3.3) by $\mathcal{U}(-t)$, and using $\mathrm{H}^0(X, \mathcal{U}) = 0$, we get:

$$\mathrm{H}^1(S, \mathcal{U}_S(-t)) = 0, \qquad \text{for } t \ge 0, \qquad \text{and} \quad \mathrm{H}^0(S, \mathcal{U}_S) = 0.$$

Tensoring (3.3) by $\mathcal{U}^*(-t)$, recalling that we have proved $\mathrm{H}^0(X, \mathcal{U}^*(-1)) = 0$, and that $\mathcal{U}$ is ACM on X, making use of Serre duality we obtain:

$$\mathrm{H}^1(S, \mathcal{U}_S(t)) = 0, \qquad \text{for } t \ge 1, \qquad \text{and} \quad \mathrm{H}^0(S, \mathcal{U}_S^*(-1)) = 0.$$

This proves that the bundle $\mathcal{U}_S$ is ACM and that it is stable again by Hoppe's criterion. $\square$

Remark 3.2. Given a line L contained in X, the bundle $\mathcal{U}^*$ splits over L as $\mathscr{O}_L^2 \oplus \mathscr{O}_L(1)$. Indeed, it must split (by Grothendieck's theorem) as $\oplus_{i=1,2,3}\mathscr{O}_L(a_i)$ where we order the a_i's to that $a_1 \leq a_2 \leq a_3$. But $\mathcal{U}^*$ is globally generated, so $a_i \geq 0$; and $c_1(\mathcal{U}^*) = 1$ implies $a_1 + a_2 + a_3 = 1$, so $a_1 = a_2 = 0$ and $a_3 = 1$.

3.2. **Universal bundles and the decomposition of the derived category.** Here we review the structure of the derived category of a smooth prime Fano threefold X of genus 9, in terms of the semiorthogonal decomposition provided by [Kuz06]. We will interpret this decomposition in terms of the universal vector bundle of the moduli space $\mathsf{M}_X(2,1,6)$.

3.2.1. *Semiorthogonal decomposition in terms of moduli space of bundles.* In view of the results of [IR05], and recalling [BF11, Lemma 3.4], the moduli space $\mathsf{M}_X(2,1,6)$ is fine and isomorphic to a smooth plane quartic curve Γ. Further, the universal sheaf $\mathscr{E}$ for this moduli space is locally free by [BF08, Proposition 3.5]. It is defined on $X \times \Gamma$, and we denote by p and q respectively the projections to X and Γ. The curve Γ can be obtained as the intersection of the $\mathsf{Sp}(3)$-invariant quartic in $\check{\mathbb{P}}^{13}$ projectively dual to Σ, with the $\mathbb{P}^2$ spanned by the web of linear forms vanishing on X. This curve is also called the homologically projectively dual curve to X. Indeed, by Kuznetsov's theorem, see [Kuz06, Section 6.3]), we have a semiorthogonal decomposition:

$$\mathbf{D^b}(X) = \langle \mathscr{O}_X, \mathcal{U}^*, \mathbf{\Phi}(\mathbf{D^b}(\Gamma)) \rangle, \tag{3.4}$$

where $\mathbf{\Phi}$ is the integral functor associated with a sheaf $\mathscr{F}$ on $X \times \Gamma$, flat over Γ, and the symbol $\langle - \rangle$ denotes "generated" i.e. the minimal full triangulated subcategory containing a given family of $\mathbf{D^b}(X)$. We show now that the sheaf $\mathscr{F}$ is given by the moduli functor on $\Gamma = \mathsf{M}_X(2,1,6)$.

Lemma 3.3. *The sheaf $\mathscr{F}$ is isomorphic to $\mathscr{E}$, up to twist by an invertible sheaf.*

Proof. Given a point $y \in \Gamma$, we denote by $\mathscr{F}_y$ the sheaf $\mathscr{F}$ restricted to $X \times \{y\}$. We claim that, by [Kuz06, Appendix A], $\mathscr{F}_y$ fits into a long exact sequence:

$$0 \to \mathscr{O}_X \to \mathcal{U}^* \to \mathscr{F}_y \to \mathscr{O}_Z \to 0, \tag{3.5}$$

where Z is a conic contained in X. In order to see this, we first inspect [Kuz06, Proposition A.9] (observe that $\mathscr{F}_y$ is denoted by C in that paper). We observe that Z is obtained as the intersection of a sub-Grassmannian $\mathrm{LGr}(2,4) \subset \Sigma$ of 2-dimensional Lagrangian subspaces in a 4-subspace of V, with a codimension-2 linear section of Σ containing X. Then, we note that $\mathrm{LGr}(2,4)$ is a smooth quadric threefold, so that, since X does not contain planes or 2-dimensional quadrics, Z must be a conic.

We prove now that $\mathscr{F}_y$ is torsion-free. Note first that, for all $y \in \Gamma$, the sheaf $\mathscr{F}_y$ is simple by Bondal-Orlov's criterion, [BO95], since the functor $\mathbf{\Phi}$ is fully faithful. Hence $\mathscr{F}_y$ is indecomposable. Let now I be the image of the middle map in (3.5), and recall from [Kuz06, Proposition A.9] that I is torsion-free. Let T be the torsion part of $\mathscr{F}_y$ and assume by contradiction that $T \neq 0$. Then (3.5) induces:

$$0 \to I \to \mathscr{F}_y/T \to \mathscr{O}_D \to 0, \tag{3.6}$$

where $D \subset C$ is defined by $\mathscr{O}_D = \mathscr{O}_C/T$. So D is either a line in X or a finite set of points. In both cases, applying $\mathrm{Hom}_X(-, \mathscr{O}_D(-1))$ to

$$0 \to \mathscr{O}_X \to \mathcal{U}^* \to I \to 0,$$

we easily compute $\mathrm{Ext}^2_X(I, \mathscr{O}_D(-1)) = 0$. Hence, by Serre duality we obtain $\mathrm{Ext}^1_X(\mathscr{O}_D, I) = 0$, so by (3.6) we get $\mathscr{F}_y/T \cong I \oplus \mathscr{O}_D$. We deduce that I is a direct summand of $\mathscr{F}_y$, contradicting that $\mathscr{F}_y$ is simple.

Now, we compute $c_1(\mathscr{O}_Z) = 0$, $c_2(\mathscr{O}_Z) = -2$, $c_3(\mathscr{O}_Z) = 0$, and so $c_1(\mathscr{F}_y) = 1$, $c_2(\mathscr{F}_y) = 6$, $c_3(\mathscr{F}_y) = 0$. Further, we easily check that $\mathscr{F}_y$ is a stable sheaf, i.e. $\mathscr{F}_y$ sits in $\mathsf{M}_X(2, 1, 6)$. Note that, by [BF08, Proposition 3.5], $\mathscr{F}_y$ must be a vector bundle.

Since $\mathscr{E}$ is a universal vector bundle for the fine moduli space $\Gamma = \mathsf{M}_X(2, 1, 6)$, we have thus that $\mathscr{F}$ is the twist by a line bundle on Γ of a pull-back of $\mathscr{E}$ via a map $f : \Gamma \to \Gamma$. Finally, we show that f is not constant (hence f is an isomorphism and we are done). Indeed, again by Bondal-Orlov's criterion we have $\mathrm{Ext}^k_X(\mathscr{F}_y, \mathscr{F}_z) = 0$, for all k if $y \neq z \in \Gamma$. But if f was constant, we would have $\hom_X(\mathscr{F}_y, \mathscr{F}_z) = 1$, for any $y, z \in \Gamma$. □

Since the universal sheaf $\mathscr{E}$ is defined up to a line bundle, we may now assume $\mathscr{F} \cong \mathscr{E}$.

3.2.2. *An explicit formulation of the semiorthogonal decomposition.* Let us write down explicitly Kuznetsov's semiorthogonal decomposition. The functor $\mathbf{\Phi}$ is defined as follows:

$$\mathbf{\Phi} : \mathbf{D^b}(\Gamma) \to \mathbf{D^b}(X), \qquad \mathbf{\Phi}(-) = \mathbf{R}p_*(q^*(-) \otimes \mathscr{E}).$$

Recall that $\mathbf{\Phi}$ is fully faithful and the corresponding right and left adjoint functors $\mathbf{\Phi}^!$ and $\mathbf{\Phi}^*$ are defined by the formulas:

$$\mathbf{\Phi}^! : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma), \qquad \mathbf{\Phi}^!(-) = \mathbf{R}q_*(p^*(-) \otimes \mathscr{E}^*(\omega_\Gamma))[1], \tag{3.7}$$

$$\mathbf{\Phi}^* : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma), \qquad \mathbf{\Phi}^*(-) = \mathbf{R}q_*(p^*(-) \otimes \mathscr{E}^*(-H_X))[3]. \tag{3.8}$$

Given a sheaf F over X, in view of [Gor90], the semiorthogonal decomposition (3.4) gives a functorial exact triangle:

$$\mathbf{\Phi}(\mathbf{\Phi}^!(F)) \to F \to \mathbf{\Psi}(\mathbf{\Psi}^*(F)), \tag{3.9}$$

where $\mathbf{\Psi}$ is the inclusion of the subcategory $\langle \mathscr{O}_X, \mathcal{U}^* \rangle$ in $\mathbf{D^b}(X)$ and $\mathbf{\Psi}^*$ is the left adjoint functor to $\mathbf{\Psi}$. The k-th term of the complex $\mathbf{\Psi}(\mathbf{\Psi}^*(F))$ can be written as follows:

$$(\mathbf{\Psi}(\mathbf{\Psi}^*(F)))^k \cong \mathrm{Ext}^{-k}_X(F, \mathscr{O}_X)^* \otimes \mathscr{O}_X \oplus \mathrm{Ext}^{1-k}_X(F, \mathcal{U})^* \otimes \mathcal{U}^*. \tag{3.10}$$

3.2.3. *Grothendieck Riemann-Roch's formula.* Consider a coherent sheaf F of (generic) rank r on $X \times \Gamma$ and with Chern classes c_i, d_j lying in:

$$c_i \in \mathrm{H}^{i,i}(X) \otimes \mathrm{H}^{0,0}(\Gamma), \qquad d_j \in \mathrm{H}^{j-1,j-1}(X) \otimes \mathrm{H}^{1,1}(\Gamma).$$

We let $c_i, d_j \in \mathbb{Z}$ by considering the integral multiples of the natural positive generator of the trace over $\mathbb{Z}$ of each of the cohomology group above, and:

$$\xi \in \mathrm{H}^{2,1}(X) \otimes \mathrm{H}^{0,1}(\Gamma) \oplus \mathrm{H}^{1,2}(X) \otimes \mathrm{H}^{1,0}(\Gamma),$$

with $\xi^2 = \alpha \in \mathbb{Z}$. Then Hirzebruch-Riemann-Roch applied to the sheaf F on $X \times \Gamma$ reads:

$$\begin{aligned}\chi(F) =& \tfrac{8}{3}c_1^3 d_1 - \tfrac{8}{3}c_1^2 d_2 - \tfrac{1}{3}c_1 c_2 d_1 + \tfrac{1}{6}c_1 d_3 + \tfrac{1}{6}c_2 d_2 + \tfrac{1}{24}c_3 d_1 - \tfrac{1}{6}d_4 - \\ &- \tfrac{16}{3}c_1^3 + 4c_1^2 d_1 + c_1 c_2 - 4c_1 d_2 - \tfrac{1}{4}c_2 d_1 - \tfrac{1}{4}c_3 + \tfrac{1}{4}d_3 - \\ &-8c_1^2 + \tfrac{10}{3}c_1 d_1 + c_2 - \tfrac{10}{3}d_2 - \tfrac{20}{3}c_1 + d_1 + \tfrac{1}{12}\alpha - 2r.\end{aligned}$$

By Grothendieck-Riemann-Roch we also have:

$$\chi(F) = \sum_{i=0}^{3}(-1)^i \chi(\mathbf{R}^i p_*(F)) = \sum_{j=0}^{1}(-1)^j \chi(\mathbf{R}^j q_*(F)).$$

It will be handy to specialize this formula to a coherent sheaf F on X of rank r, with Chern classes c_1, c_2, c_3 (this is actually the Hirzebruch-Riemann-Roch formula for a sheaf on X):

$$\chi(F) = r + \tfrac{10}{3}\, c_1 + 4\, c_1^2 - \tfrac{1}{2} c_2 + \tfrac{8}{3}\, c_1^3 - \tfrac{1}{2}\, c_1\, c_2 + \tfrac{1}{2}\, c_3. \tag{3.11}$$

Let us now go back to our universal bundle $\mathscr{E}$. We have the following:

$$c_1(\mathscr{E}) = H_X + N, \qquad c_2(\mathscr{E}) = 6L_X + H_X M + \eta,$$

where N and M are divisor classes on Γ, and η sits in $\mathrm{H}^3(X, \mathbb{C}) \otimes \mathrm{H}^1(\Gamma, \mathbb{C})$. The formulas above help obtaining the following lemma:

Lemma 3.4. *We have* $\eta^2 = 6$ *and* $\deg(N) = 2\deg(M) - 1$. *Since the universal bundle* $\mathscr{E}$ *is determined up to twisting by the pull-back of a line bundle on* Γ, *we may adopt the convention:*

$$\deg(N) = \deg(\mathscr{E}_x) = 5. \tag{3.12}$$

Proof. From the definition of $\mathbf{\Phi}$, it follows that $\mathbf{\Phi}(\mathscr{O}_y) \cong \mathscr{E}_y$. Since the functor $\mathbf{\Phi}$ is fully faithful, we easily obtain also the isomorphism $\mathbf{\Phi}^!(\mathscr{E}_y) \cong \mathscr{O}_y$. Since $\mathscr{O}_X$ lies in $\mathbf{\Phi}(\mathbf{D^b}(\Gamma))^\perp$, we have:

$$\mathbf{\Phi}^!(\mathscr{O}_X) = 0.$$

Plugging the equations $\chi(\mathbf{\Phi}^!(\mathscr{O}_X)) = 0$ and $\chi(\mathbf{\Phi}^!(\mathscr{E}_y)) = 1$ into Grothendieck-Riemann-Roch's formula, we get our claim. □

3.2.4. *Mutation of the tautological bundle.* One can play with mutations to obtain different semiorthogonal decompositions. For instance the vector bundle $\mathcal{U}$ is given by the left mutation of $\mathcal{U}^*$ through $\mathscr{O}_X$, in view of (3.1), more precisely $L_{\mathscr{O}_X}\mathcal{U}^* \cong \mathcal{U}[1]$.

Lemma 3.5. *We have the natural isomorphisms:*

$$\mathcal{H}^0(\mathbf{\Phi}(\mathbf{\Phi}^*(\mathcal{U}^*))) \cong \mathcal{U}^*, \qquad \mathcal{H}^1(\mathbf{\Phi}(\mathbf{\Phi}^*(\mathcal{U}^*))) \cong \mathcal{U}(1).$$

Proof. We first replace the semiorthogonal decomposition (3.4) by $\langle \mathcal{U}, \mathscr{O}_X, \mathbf{\Phi}(\mathbf{D^b}(\Gamma)\rangle$ by left-mutating $\mathcal{U}^*$ through $\mathscr{O}_X$. Let $S : F \mapsto F \otimes \mathscr{O}_X(-1)[3]$ be the Serre functor of $\mathbf{D^b}(X)$. It is well-known that, right-mutating $\mathcal{U}$ through $\mathscr{O}_X$ and $\mathbf{\Phi}(\mathbf{D^b}(\Gamma))$, we must obtain $S^{-1}(\mathcal{U})$ i.e. $\mathcal{U}(1)[-3]$ (see [Bon89]). This gives the semiorthogonal decomposition:

$$\mathbf{D^b}(X) = \langle \mathscr{O}_X, \mathbf{\Phi}(\mathbf{D^b}(\Gamma), \mathcal{U}(1))\rangle,$$

Now, we have $\mathrm{R}_{\mathscr{O}_X}\mathcal{U} \cong \mathcal{U}^*[-1]$, so $\mathrm{R}_{\mathbf{\Phi}(\mathbf{D^b}(\Gamma))}\mathcal{U}^*[-1] \cong \mathcal{U}(1)[-3]$. On the other hand, we have the mutation triangle:

$$\mathrm{R}_{\mathbf{\Phi}(\mathbf{D^b}(\Gamma))}\mathcal{U}^* \to \mathcal{U}^* \to \mathbf{\Phi}(\mathbf{\Phi}^*(\mathcal{U}^*)). \tag{3.13}$$

Taking cohomology of the above triangle proves the lemma, and (3.13) becomes:

$$\mathcal{U}(1)[-2] \to \mathcal{U}^* \to \mathbf{\Phi}(\mathbf{\Phi}^*(\mathcal{U}^*)). \tag{3.14}$$

□

3.3. **Conics contained in** X**.** In this section we review some facts concerning the geometry of conics contained in X. In Proposition 3.10 we recover Iliev's description of their Hilbert scheme, see [Ili03]. We outline a different proof, which holds for any smooth prime Fano threefolds of genus 9. We first look at the vanishing locus of sections of $\mathcal{U}^*$.

Lemma 3.6. *Let $\sigma \in \mathrm{H}^0(X,\mathcal{U}^*)$ be a non-zero global section of $\mathcal{U}^*$. Then σ vanishes either along a conic, or a line, or a pair of points. Further, for $\sigma' \in \mathrm{H}^0(X,\mathcal{U}^*)$, not proportional to σ, the locus where σ and σ' vanish simultaneously is a line, or a point, or the empty set.*

Proof. Recall that X is cut in $\Sigma \subset \mathbb{P}^{13}$ by a 3-codimensional linear section $\mathbb{P}^{10} \subset \mathbb{P}^{13}$. From Lemma 3.1 we see that $\mathrm{H}^0(X,\mathcal{U}^*)$ is isomorphic to $\mathrm{H}^0(\Sigma,\mathcal{U}^*)$, so σ and σ' lift to sections $\tilde{\sigma}$ and $\tilde{\sigma}'$ on Σ. The vanishing locus of σ and of (σ,σ') in X are given as the intersection of $\mathbb{P}^{10}$ with the vanishing locus of $\tilde{\sigma}$ and of $(\tilde{\sigma},\tilde{\sigma}')$ in Σ.

Now, the vanishing locus of $\tilde{\sigma}$ is a sub-Grassmannian of 2-dimensional Lagrangian subspaces in a 4-subspace of V, i.e. a quadric threefold Q in Σ. We have already observed that X contains no 2-dimensional quadrics or planes, so the locus in X obtained cutting Q with $\mathbb{P}^{10}$ must be a pair of points (if the dimension is the expected one) or a conic, or a line.

Further, the locus in Σ where $\tilde{\sigma}$ and $\tilde{\sigma}'$ vanish together is a sub-Grassmannian of 1-dimensional Lagrangian subspaces in a 2-subspace of V, i.e. a $\mathbb{P}^1 \subset \Sigma$. Cutting this $\mathbb{P}^1$ with the $\mathbb{P}^{10}$ defining X we obtain either a line, or a point, or the empty set. □

Lemma 3.7. *Let C be any conic contained in X. Then we have:*

$$\mathrm{h}^0(X,\mathcal{U}\otimes\mathscr{O}_C) = 1, \qquad \mathrm{h}^1(X,\mathcal{U}\otimes\mathscr{O}_C) = 0, \tag{3.15}$$

$$\hom_X(\mathcal{U},\mathcal{I}_C) = 1, \qquad \mathrm{ext}^k_X(\mathcal{U},\mathcal{I}_C) = 0, \qquad \textit{for } k \neq 0 \tag{3.16}$$

Proof. By Riemann-Roch we have $\chi(\mathcal{U}^*\otimes\mathcal{I}_C) = 1$ and one can easily prove $\mathrm{Ext}^k_X(\mathcal{U},\mathcal{I}_C) = 0$, for $k \geq 2$. So, to prove (3.16), it only remains to show that $\hom_X(\mathcal{U}^*\otimes\mathcal{I}_C) \leq 1$. But, assuming $\hom_X(\mathcal{U}^*\otimes\mathcal{I}_C) \geq 2$, we have that C is (scheme theoretically) contained in the simultaneous zero locus of a pencil of global sections of $\mathcal{U}^*$, which is, by Lemma 3.6, either a line or a point or the empty set. This is nonsense, and (3.16) is proved.

Now, in order to prove (3.15), we first apply $\mathrm{Hom}_X(-,\mathcal{I}_C)$ to (3.1) and we find $\mathrm{ext}^1_X(\mathcal{U}^*,\mathcal{I}_C) = 1$ and $\mathrm{ext}^k_X(\mathcal{U}^*,\mathcal{I}_C) = 0$ for all $k \neq 0$. Since $\mathrm{H}^k(X,\mathcal{U}) = 0$ for all k, (3.15) follows. □

Lemma 3.8. *Let E be a sheaf in $\mathsf{M}_X(2,1,6)$. Let α be any non-zero element in $\mathrm{Hom}_X(\mathcal{U}^*,E)$. Then α gives the long exact sequence:*

$$0 \to \mathscr{O}_X \xrightarrow{\sigma} \mathcal{U}^* \xrightarrow{\alpha} E \to \mathscr{O}_C \to 0, \tag{3.17}$$

where C is a conic contained in X and σ is a global section of $\mathcal{U}^*$.

Proof. Let I be the image of the non-zero map $\alpha : \mathcal{U}^* \to E$. Recall by Lemma 3.1 that $\mathcal{U}$ is stable. Thus, by stability of E we get $\mathrm{rk}(\ker(\alpha)) = 1$ and $c_1(\ker(\alpha)) = 0$. Since $\ker(\alpha)$ is reflexive, it must be invertible and we get an exact sequence of the form:

$$0 \to \mathscr{O}_X \to \mathcal{U}^* \to E \to T \to 0, \tag{3.18}$$

where $T = \mathrm{cok}(\alpha)$ is a torsion sheaf.

We have $c_1(T) = 0$, $c_2(T) = -2$, $c_3(T) = 0$, so the support of T is a non-empty subscheme of codimension 2 in X. Recall that E is locally free (see [BF08, Proposition 3.5]), so that dualizing (3.18) we get:

$$0 \to E^* \to \mathcal{U} \to \mathscr{O}_X \to \mathscr{E}xt^2_X(T,\mathscr{O}_X) \to 0, \tag{3.19}$$

and the vanishing $\mathscr{E}xt^k_X(T,\mathscr{O}_X) = 0$ for $k \neq 2$. This says that $\mathscr{E}xt^2_X(T,\mathscr{O}_X) \cong \mathscr{O}_C$, where C is the zero-locus of $\sigma \in \mathrm{H}^0(X,\mathcal{U}^*)$. So, by Lemma 3.6, we have that C is either a conic in X or a line in X or a pair of points of X. But the last two cases are impossible, since we may compute from (3.19) the value $c_2(\mathscr{E}xt^2_X(T,\mathscr{O}_X)) = 2$. Note that, if $y \in \Gamma$ is the point corresponding to the sheaf $E = \mathscr{E}_y$, we can write (3.19) as:

$$0 \to \mathscr{E}_y^* \xrightarrow{\alpha^\top} \mathcal{U} \xrightarrow{\sigma^\top} \mathcal{I}_C \to 0. \tag{3.20}$$

Dualizing again (3.19) we must get back (3.18), so that $T = \mathscr{E}xt^2_X(\mathscr{O}_C,\mathscr{O}_X) \cong \mathscr{O}_C$, and the lemma is proved. Note that the exact sequence (3.17) is precisely (3.5). □

Lemma 3.9. *Let $E = \mathscr{E}_y$ be a sheaf in $\mathsf{M}_X(2,1,6)$. Then we have:*

$$\begin{aligned} &\mathrm{hom}_X(\mathcal{U}^*,E) = 2, && \\ &\mathrm{Ext}^k_X(\mathcal{U}^*,E) = 0, && \text{for all } k \geq 1, \\ &\mathrm{Ext}^k_X(\mathcal{U},E^*) = \mathrm{Ext}^k_X(E,\mathcal{U}^*) = 0, && \text{for all } k. \end{aligned}$$

Proof. Tensor (3.17) by $\mathcal{U}$ and recall that $\mathrm{H}^k(X,\mathcal{U}) = 0$ for all k. The bundle $\mathcal{U}$ is exceptional, so $\mathrm{h}^0(X,\mathcal{U}\otimes\mathcal{U}^*) = 1$ and $\mathrm{H}^k(X,\mathcal{U}\otimes\mathcal{U}^*) = 0$ for all $k \geq 1$. By (3.15), we conclude that $\mathrm{h}^0(X,\mathcal{U}\otimes E) = \mathrm{hom}_X(\mathcal{U}^*,E) = 2$ and $\mathrm{H}^k(X,\mathcal{U}\otimes E) = \mathrm{ext}^k_X(\mathcal{U}^*,E) = 0$ for all $k \geq 1$.

Since $\Gamma = \mathsf{M}_X(2,1,6)$, we have $E \cong \mathscr{E}_y \cong \mathbf{\Phi}(\mathscr{O}_y)$ for some $y \in \Gamma$. Then we have:

$$\mathrm{Ext}^k_X(\mathcal{U},\mathscr{E}_y^*) \cong \mathrm{Ext}^k_X(\mathscr{E}_y,\mathcal{U}^*) \cong \mathrm{Ext}^k_X(\mathbf{\Phi}(\mathscr{O}_y),\mathcal{U}^*),$$

and the last term vanishes, for all k, since $\mathbf{\Phi}(\mathbf{D^b}(\Gamma))$ is left-orthogonal to $\mathcal{U}^*$. ☐

Proposition 3.10 (Iliev)**.** *Let X be a smooth prime Fano threefold of genus 9. Then the sheaf $\mathcal{V} = q_*(p^*(\mathcal{U}) \otimes \mathscr{E})$ is a rank 2 vector bundle on Γ, and we have a natural isomorphism:*

$$\mathcal{V}^* \cong \mathbf{\Phi}^*(\mathcal{U}^*). \tag{3.21}$$

The Hilbert scheme $\mathscr{H}_2^0(X)$ is isomorphic to the projective bundle $\mathbb{P}(\mathcal{V})$ over Γ. In particular, $\mathscr{H}_2^0(X)$ is a smooth irreducible surface.

Proof. In view of Lemma 3.9, we have $\mathbf{R}^k q_*(p^*(\mathcal{U}) \otimes \mathscr{E}) = 0$, for $k \geq 1$, and $\mathcal{V}$ is a locally free sheaf on Γ of rank $\mathrm{h}^0(X, \mathcal{U} \otimes \mathscr{E}_y) = 2$.

By an instance of Grothendieck duality, see [Har66, Chapter III], given a sheaf $\mathscr{P}$ on $X \times \Gamma$, we have:

$$D(\mathbf{R}q_*(\mathscr{P})) \cong \mathbf{R}q_*(\mathscr{O}_X(-1) \otimes D(\mathscr{P}))[3], \tag{3.22}$$

and the isomorphism is functorial. Setting $\mathscr{P} = p^*(\mathcal{U}) \otimes \mathscr{E}$ in (3.22), we get (3.21).

Consider now an element ξ of the projective bundle $\mathbb{P}(\mathcal{V})$. It is uniquely represented by a pair $([\alpha], y)$, where y is a point of Γ, and $[\alpha]$ is an element of $\mathbb{P}(\mathrm{H}^0(X, \mathcal{U} \otimes \mathscr{E}_y))$. Setting $E = \mathscr{E}_y$ in Lemma 3.8, the morphism α gives (3.20), thus defining a conic C associated with α. This defines an algebraic map $\vartheta : \mathbb{P}(\mathcal{V}) \to \mathscr{H}_2^0(X)$.

To conclude the proof, let us provide an inverse to ϑ. Let C be a conic contained in X. By (3.16), there exists a unique (up to scalar) morphism $\delta : \mathcal{U} \to \mathcal{I}_C$. The composition of δ with the embedding of $\mathcal{I}_C \to \mathscr{O}_X$ provides a non-zero global section σ of $\mathcal{U}^*$ whose zero locus contains C. But in view of Lemma 3.6 this zero locus is thus precisely C, so that δ is surjective. Then, setting $K = \ker(\delta)$, it is easy to see that $K(1)$ is a sheaf in $\mathsf{M}_X(2, 1, 6)$, so that $K(1) \cong \mathscr{E}_y$ for some $y \in \Gamma$. We then associate with the conic C the point $\xi = ([\alpha], y) \in \mathbb{P}(\mathcal{V})$, where α is the transpose of the inclusion of K in $\mathcal{U}$. This provides an algebraic inverse to ϑ and completes the proof. ☐

Lemma 3.11. *We have a natural isomorphism $\mathbf{\Phi}^!(\mathcal{U}(1))[-1] \cong \mathbf{\Phi}^*(\mathcal{U}^*)$. In particular, we get $\det(\mathcal{V}^*) \cong \omega_\Gamma(-N)$, where $c_1(\mathscr{E}) = H_X + N$.*

Proof. We apply the functor $\mathbf{\Phi}^!$ to the triangle (3.14). Since $\mathbf{\Phi}^!(\mathcal{U}^*) = 0$ by semi-orthogonality of (3.4), we obtain:

$$\mathbf{\Phi}^!(\mathcal{U}(1))[-1] \cong \mathbf{\Phi}^!(\mathbf{\Phi}(\mathbf{\Phi}^*(\mathcal{U}^*))) \cong \mathbf{\Phi}^*(\mathcal{U}^*),$$

where the last equivalence follows from the fully faithfulness of $\mathbf{\Phi}$. The first statement is thus proved.

In view of the previous proposition, we have $\mathcal{V} \cong \mathbf{\Phi}^*(\mathcal{U}^*)^*$. Since the bundle $\mathscr{E}$ has rank two and $c_1(\mathscr{E}) = H_X + N$ we have $\mathscr{E}^* \cong \mathscr{E}(-H_X - N)$. Plugging this in the definition of $\mathbf{\Phi}^!(\mathcal{U}(1))$ and recalling the definition of $\mathcal{V} = q_*(p^*(\mathcal{U}) \otimes \mathscr{E})$, we have:

$$\mathcal{V} \cong \mathbf{\Phi}^!(\mathcal{U}(1)) \otimes \omega_\Gamma^*(N)[-1] \cong \mathcal{V}^* \otimes \omega_\Gamma^*(N).$$

Since $\mathcal{V}$ has rank 2, we have $\mathcal{V} \cong \mathcal{V}^* \otimes \det(\mathcal{V})$, and the second statement of the lemma follows. ☐

Remark 3.12. Recall that $\deg(N) = 5$ by assumption (3.12). Hence from Lemma 3.11 it follows that $\deg(\mathcal{V}) = 1$. In view of the previous results, we can identify $\mathcal{V}$ with a twist of the stable bundle of rank 2 of degree 3, defined by Iliev in [Ili03, Section 5]. Let $K_\Gamma = c_1(\omega_\Gamma)$ and recall that by Mukai's theorem, [Muk01], X is isomorphic to the *type II Brill-Noether locus*:

$$\mathsf{M}_\Gamma(2, K_\Gamma, 3\mathcal{V}) = \{\mathcal{F} \in \mathsf{M}_\Gamma(2, c_1(\mathcal{V}) + K_\Gamma) \,|\, \mathrm{h}^0(\Gamma, \mathcal{F} \otimes \mathcal{V}^*) \geq 3\}.$$

Therefore, the bundle $\mathscr{E}$ is universal also for the moduli space $X \cong \mathsf{M}_\Gamma(2, K_\Gamma, 3\mathcal{V})$.

In fact, the moduli space of smooth prime Fano threefolds of genus 9 can be identified with the space of pairs $(\Gamma, \mathcal{V})$ where Γ is a smooth plane quartic and $\mathcal{V}$ is a vector bundle of rank 2 of odd degree and such that any section of $\mathbb{P}(\mathcal{V})$ has self-intersection at least 3.

3.4. **Lines contained in X.** Here we focus on lines contained in X. We will describe their Hilbert scheme as a certain Brill-Noether locus of the Picard variety $\mathrm{Pic}^2(\Gamma)$ of line bundles of degree 2 on Γ. We start with the following preliminary lemma:

Lemma 3.13. *Let $\mathcal{L}$ be a line bundle in $\mathrm{Pic}^2(\Gamma)$ such that $\mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{L}) \geq 2$. Then $\mathbf{\Phi}(\mathcal{L})$ is a stable vector bundle of rank 5 and degree 2.*

Proof. Given $\mathcal{L}$ as above, we consider a 2-dimensional subspace $\Lambda \subset \mathrm{H}^0(\Gamma, \mathcal{V} \otimes \mathcal{L})$, and the natural evaluation of sections:

$$\beta : \mathcal{V}^* \to \mathcal{L} \otimes \Lambda^*.$$

First of all we prove that $\det(\beta) \neq 0$. Indeed, assume $\det(\beta) = 0$, and note that the image $\mathrm{Im}(\beta)$ would be a line bundle admitting 2 linearly independent morphisms to $\mathcal{L}$, by definition of the evaluation map. Then, since Γ is not rational, we have $\deg(\mathcal{L}) - \deg(\mathrm{Im}(\beta)) \geq 2$, so $\deg(\mathrm{Im}(\beta)) \leq 0$. But, by a result of Mukai (see [Muk01, Theorem 9.1]), we know that any section of the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{V})$ has self-intersection number at least 3. This implies that the degree of any line bundle which is quotient of $\mathcal{V}^*$ is at least 1, so $\deg(\mathrm{Im}(\beta)) \geq 1$, a contradiction.

Now since $\det(\beta) \neq 0$, we have the following exact sequence:

$$0 \to \mathcal{V}^* \xrightarrow{j} \mathcal{L} \otimes \Lambda^* \xrightarrow{h} \mathscr{O}_{y_1} \oplus \ldots \oplus \mathscr{O}_{y_5} \to 0, \tag{3.23}$$

where $y_1, \ldots, y_5$ are points of Γ. Applying the functor $\mathbf{\Phi}$ to (3.23) and using (3.21) and Lemma 3.5, we obtain:

$$0 \to \mathcal{U}^* \xrightarrow{i} \mathbf{\Phi}(\mathcal{L}) \otimes \Lambda^* \xrightarrow{g} \mathscr{E}_{y_1} \oplus \ldots \oplus \mathscr{E}_{y_5} \to \mathcal{U}(1) \to 0. \tag{3.24}$$

It is easy to compute $\deg(\mathbf{\Phi}(\mathcal{L})) = 2$ and $\mathrm{rk}(\mathbf{\Phi}(\mathcal{L})) = 5$. Note also that $\mathbf{\Phi}(\mathcal{L})$ is indecomposable. Indeed, it is simple by fully faithfulness of $\mathbf{\Phi}$, because $\mathrm{Hom}_X(\mathbf{\Phi}(\mathcal{L}), \mathbf{\Phi}(\mathcal{L})) \cong \mathrm{Hom}_\Gamma(\mathcal{L}, \mathcal{L}) \cong \mathbb{C}$. Applying $\mathbf{\Phi}^*$ to (3.24) we get:

$$\mathbf{\Phi}^*(g) = h, \qquad \mathbf{\Phi}^*(i) = j.$$

Now we want to prove that $\mathbf{\Phi}(\mathcal{L}) \otimes \Lambda^* \cong \mathbf{\Phi}(\mathcal{L})^2$ is semi-stable and this will imply that $\mathbf{\Phi}(\mathcal{L})$ is stable, because its degree and rank are co-prime. Assume the contrary, and let Q be a torsion-free quotient of $\mathbf{\Phi}(\mathcal{L})^2$ with $\mathrm{rk}(Q) \leq 9$

and $\mu(Q) < \frac{2}{5}$. Let K be the kernel of the projection $\mathbf{\Phi}(\mathcal{L})^2 \to Q$. Clearly we have $\mu(K) > \frac{2}{5}$ and $\mathrm{rk}(K) \leq 9$. We have thus the diagram:

(3.25)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{U}^* & \longrightarrow & \mathbf{\Phi}(\mathcal{L})^2 & \longrightarrow & \mathrm{Im}(g) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q' & \longrightarrow & Q & \longrightarrow & Q'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Now since $\mathcal{U}^*$ is stable we have $\mu(K') \leq \frac{1}{3}$ and since $\mathrm{Im}(g)$ is included in $\mathscr{E}_{y_1} \oplus \ldots \oplus \mathscr{E}_{y_5}$, which is poly-stable, we have either $\mu(K'') < \frac{1}{2}$, or $\mu(K'') = \frac{1}{2}$ and (up to reordering of the y_i's) $K'' \cong \mathscr{E}_{y_1} \oplus \ldots \oplus \mathscr{E}_{y_k}$ for some $1 \leq k \leq 4$. Imposing all these numerical conditions we easily exclude the cases $\mathrm{rk}(K) = 3, 5, 9$. Hence we are lead to deal with the following cases:

i) $\mu(K) = \frac{1}{2}$. In this case $K' = 0$, $\mu(K'') = \frac{1}{2}$, and so, up to reordering of the y_i's, we have $K \cong K'' \cong \mathscr{E}_{y_1} \oplus \ldots \oplus \mathscr{E}_{y_k}$ for some $1 \leq k \leq 3$. Then we have a non-zero map $\mathscr{E}_{y_1} \to \mathbf{\Phi}(\mathcal{L})^2$, and since $\mathbf{\Phi}$ is fully faithful and $\mathscr{E}_y \cong \mathbf{\Phi}(\mathscr{O}_y)$, this amounts to a non-zero map $\mathscr{O}_{y_1} \to \mathcal{L}^2$, which is absurd since $\mathcal{L}$ is torsion-free.

ii) $\mu(K) = \frac{3}{7}$ and $K' = 0$. In this case $K \cong K''$, so the third line in the diagram is: $0 \to \mathcal{U}^* \to Q \to Q'' \to 0$, where $c_1(Q) = c_1(\mathcal{U}^*) = 1$ and $\mathrm{rk}(Q) = \mathrm{rk}(\mathcal{U}^*) = 3$. Recall that Q is torsion-free by assumption. Hence we can apply Lemma 2.1 and we get $Q'' = 0$. Then i splits, hence so does j, for $\mathbf{\Phi}^*(i) = j$. This is absurd.

iii) $\mu(K) = \frac{3}{7}$ and $\mathrm{rk}(K') = 1$. This time $\mathrm{rk}(K'') = 6$, $c_1(K'') = 3$ and, up to reordering of the y_i's, we have $K'' \cong \mathscr{E}_{y_1} \oplus \mathscr{E}_{y_2} \oplus \mathscr{E}_{y_3}$ and $c_1(K') = 0$. Looking at the first column in the above diagram, we see that K' is reflexive of rank 1, so $K' \cong \mathscr{O}_X$. But for all $i = 1, 2, 3$, we have $\mathrm{Ext}^1_X(\mathscr{E}_{y_i}, \mathscr{O}_X) = 0$, because each $\mathscr{E}_{y_i}$ is ACM (or because $\mathbf{\Phi}(\mathbf{D^b}(\Gamma))$ is left-orthogonal to $\mathscr{O}_X$). So the first row of (3.25) splits. As in (i), we would then get a non-zero map $\mathscr{E}_{y_i} \to \mathbf{\Phi}(\mathcal{L})^2$, which is absurd.

iv) $\mu(K) = \frac{3}{7}$ and $\mathrm{rk}(K') = 3$. In this case we must have $c_1(K') = 1$ and $\mu(K'') = \frac{1}{2}$. Then it follows again that K'' is a direct sum of some of the $\mathscr{E}_{y_i}$'s. On the other hand, since $\mu(K') = \frac{1}{3}$, we get $K' \cong \mathcal{U}^*$ by stability of $\mathcal{U}^*$. By Lemma 3.9 we know that $\mathrm{Ext}^1_X(\mathscr{E}_{y_i}, \mathcal{U}^*) = 0$, so again the first row of (3.25) splits, and we are lead to a contradiction.

□

Proposition 3.14. *Let L be a line contained in X. Then $\mathbf{\Phi}^!(\mathscr{O}_L(-1))$ is a line bundle of degree 2 and we have a functorial exact sequence:*

(3.26)
$$0 \to \mathscr{O}_X \to A_L \otimes \mathcal{U}^* \xrightarrow{\zeta_L} \mathbf{\Phi}(\mathbf{\Phi}^!(\mathscr{O}_L(-1))) \to \mathscr{O}_L(-1) \to 0,$$

where $A_L = \mathrm{H}^1(L, \mathcal{U}_L^*(-2)) \cong \mathbb{C}^2$. *Moreover, the map:*

$$\psi : L \mapsto \mathbf{\Phi}^!(\mathscr{O}_L(-1))$$

gives an isomorphism of the Hilbert scheme $\mathscr{H}_1^0(X)$ *with the Brill-Noether locus:*

$$W = \{\mathcal{L} \in \mathrm{Pic}^2(\Gamma) \mid \mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{L}) \geq 2\}. \tag{3.27}$$

In fact, the last inequality must be an equality.

Proof. Recall that, for each $y \in \Gamma$, the sheaf $\mathscr{E}_y$ is a globally generated bundle (by [BF11, Theorem 3.2]) with $c_1(\mathscr{E}_y) = 1$. Thus, it splits over L as $\mathscr{O}_L \oplus \mathscr{O}_L(1)$. It easily follows that $\mathbf{\Phi}^!(\mathscr{O}_L(-1))$ is a sheaf concentrated in degree 0, having rank equal to $\mathrm{h}^0(L, \mathscr{E}_y^*) = 1$. By Grothendieck-Riemann-Roch formula we compute $\deg(\mathbf{\Phi}^!(\mathscr{O}_L(-1))) = 2$.

To get (3.26), we use (3.9) and (3.10). We have thus to compute the cohomology groups $\mathrm{Ext}_X^k(\mathscr{O}_L(-1), \mathscr{O}_X)$ and $\mathrm{Ext}_X^k(\mathscr{O}_L(-1), \mathcal{U})$. Note that $\mathcal{U}^*$ splits over L as $\mathscr{O}_L^2 \oplus \mathscr{O}_L(1)$ by Remark 3.2. So, using Serre duality, we see that $\mathrm{Ext}_X^k(\mathscr{O}_L(-1), \mathscr{O}_X) = \mathrm{Ext}_X^k(\mathscr{O}_L(-1), \mathcal{U}) = 0$ for $k \neq 2$, while for $k = 2$ we have $\mathrm{ext}_X^2(\mathscr{O}_L(-1), \mathscr{O}_X) = 1$ and $\mathrm{ext}_X^2(\mathscr{O}_L(-1), \mathcal{U}) = 2$. Setting $A_L = \mathrm{H}^1(L, \mathcal{U}_L^*(-2)) \cong \mathrm{Ext}_X^2(\mathscr{O}_L(-1), \mathcal{U})^*$, we obtain the functorial resolution (3.26) and $\dim(A_L) = 2$.

Set $\mathcal{L} = \mathbf{\Phi}^!(\mathscr{O}_L(-1))$, and recall the isomorphism (3.21). Applying the functor $\mathrm{Hom}_X(\mathcal{U}^*, -)$ to the long exact sequence (3.26), since $\mathcal{U}$ is exceptional, and both $\mathrm{Hom}_X(\mathcal{U}^*, \mathscr{O}_X)$ and $\mathrm{Hom}_X(\mathcal{U}^*, \mathscr{O}_L(-1))$ vanish, we get the natural isomorphisms:

$$\begin{gathered} \mathrm{Hom}_\Gamma(\mathcal{V}^*, \mathcal{L}) \cong \mathrm{Hom}_\Gamma(\mathbf{\Phi}^*(\mathcal{U}^*), \mathcal{L}) \cong \mathrm{Hom}_X(\mathcal{U}^*, \mathbf{\Phi}(\mathcal{L})) \cong A_L, \\ \mathrm{Ext}_\Gamma^1(\mathcal{V}^*, \mathcal{L}) \cong \mathrm{Ext}_\Gamma^1(\mathbf{\Phi}^*(\mathcal{U}^*), \mathcal{L}) \cong \mathrm{Ext}_X^1(\mathcal{U}^*, \mathbf{\Phi}(\mathcal{L})) \cong \mathrm{H}^1(L, \mathcal{U}(-1)). \end{gathered} \tag{3.28}$$

Therefore, the line bundle $\mathcal{L}$ lies in the locus defined by (3.27), and actually we have $\mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{L}) = 2$. Moreover, up to multiplication by a non-zero scalar, the morphism ζ_L coincides with the natural evaluation map $e_{\mathcal{U}^*, \mathbf{\Phi}(\mathcal{L})}$. Thus, the mapping $L \mapsto \mathbf{\Phi}^!(\mathscr{O}_L(-1))$ is injective, since $\mathscr{O}_L(-1)$ can be recovered as $\mathrm{cok}(\zeta_L)$.

Now in order to prove that ψ is an isomorphism, we will build the inverse $\psi^{-1} : W \to \mathscr{H}_1^0(X)$. Given a line bundle $\mathcal{L}$ in W we know by Lemma 3.13 that $\mathbf{\Phi}(\mathcal{L})$ is a stable vector bundle of rank 5 and degree 2. Choose now a 2-dimensional subspace $\Lambda \subset \mathrm{H}^0(\Gamma, \mathcal{V} \otimes \mathcal{L})$, and let e be the evaluation map $e : \Lambda \otimes \mathcal{V}^* = \Lambda \otimes \mathbf{\Phi}^*(\mathcal{U}^*) \to \mathcal{L}$. By adjunction, from this map we obtain $\mathcal{H}^0(\mathbf{\Phi}(e)) : \Lambda \otimes \mathcal{U}^* \to \mathbf{\Phi}(\mathcal{L})$, hence we deduce that $\mathcal{H}^0(\mathbf{\Phi}(e)) \neq 0$. Let K and T be the kernel and the image of $\mathcal{H}^0(\mathbf{\Phi}(e))$, and write the exact sequence:

$$0 \to K \to \Lambda \otimes \mathcal{U}^* \xrightarrow{\mathcal{H}^0(\mathbf{\Phi}(e))} \mathbf{\Phi}(\mathcal{L}) \to T \to 0. \tag{3.29}$$

By poly-stability of $\Lambda \otimes \mathcal{U}^*$ and stability of $\mathbf{\Phi}(\mathcal{L})$ (see Lemma 3.13), it easily follows that $\mathrm{rk}(K) = 1$ and $c_1(K) = 0$. Moreover K is reflexive and hence invertible. This means that $K \cong \mathscr{O}_X$ and T is a torsion sheaf with $c_1(T) = 0$,

$c_2(T) = -1$, $c_3(T) = -1$. Dualizing (3.29) we get

$$0 \to \mathbf{\Phi}(\mathcal{L})^* \to \Lambda^* \otimes \mathcal{U} \to \mathscr{O}_X \to \mathscr{E}xt^2_X(T, \mathscr{O}_X) \to 0.$$

and $\mathscr{E}xt^k_X(T, \mathscr{O}_X) = 0$ for $k \neq 2$. This says that $\mathscr{E}xt^2_X(T, \mathscr{O}_X) \cong \mathscr{O}_C$, where C is the zero-locus of two independent global sections of $\mathcal{U}^*$. So, by Lemma 3.6 and since $c_2(\mathscr{E}xt^2_X(T, \mathscr{O}_X)) = -1$, it follows that $C = L$ must be a line contained in X. Dualizing again we easily get $T \cong \mathscr{O}_L(-1)$.

This shows that (3.29) has the same form of (3.26), with $\mathcal{L} \cong \mathbf{\Phi}^!(\mathscr{O}_L(-1))$. It follows that $\mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{L}) = \dim(A_L) = 2$. This proves the last assertion, and also that Λ above is all of $\mathrm{H}^0(\Gamma, \mathcal{V} \otimes \mathcal{L})$. We define $\psi^{-1}(\mathcal{L}) = [L] \in \mathscr{H}^0_1$ and it is clear by the construction that ψ^{-1} is the inverse of ψ. □

Lemma 3.15. *Given a line L contained in X, we have the triangle*

$$\mathcal{U}^* \to \mathbf{\Phi}(\mathbf{\Phi}^!(\mathscr{O}_L[-1])) \to \mathscr{O}_L[-1] \tag{3.30}$$

and the map $\mathscr{O}_L \mapsto \mathbf{\Phi}^!(\mathscr{O}_L)[-1]$ is an isomorphism of $\mathscr{H}^0_1(X)$ onto the Brill-Noether locus:

$$\widetilde{W} = \{\mathcal{M} \in \mathrm{Pic}^1(\Gamma) \mid \mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{M}) \geq 1\}$$

which is isomorphic to the locus W defined in (3.27). *Further, for any $\mathcal{M} \in \widetilde{W}$ we have:*

$$\mathcal{H}^0(\mathbf{\Phi}(\mathcal{M})) \cong \mathcal{U}^*, \qquad \mathcal{H}^1(\mathbf{\Phi}(\mathcal{M})) \cong \mathscr{O}_L, \tag{3.31}$$

for the unique line $L \subset X$ such that $\mathbf{\Phi}^!(\mathscr{O}_L)[-1] \cong \mathcal{M}$.

Proof. By Serre duality we have $\mathrm{Ext}^k_X(\mathscr{O}_L, \mathscr{O}_X) = 0$ for all k, $\mathrm{ext}^3_X(\mathscr{O}_L, \mathcal{U}) = 1$ and $\mathrm{Ext}^k_X(\mathscr{O}_L, \mathcal{U}) = 0$ for all $k \neq 3$, since by Remark 3.2 we know that $\mathcal{U}^*$ splits over L as $\mathscr{O}_L^2 \oplus \mathscr{O}_L(1)$. Then by using (3.9) and (3.10), we get $\mathbf{\Psi}(\mathbf{\Psi}^*(\mathscr{O}_L)) \cong \mathcal{U}^*[2]$ and the triangle (3.30). In turn, this gives:

$$\mathcal{H}^k(\mathbf{\Phi}(\mathbf{\Phi}^!(\mathscr{O}_L))) \cong \begin{cases} \mathcal{U}^* & \text{if } k = -1; \\ \mathscr{O}_L & \text{if } k = 0; \\ 0 & \text{if } k \neq -1, 0. \end{cases} \tag{3.32}$$

Now, with any line bundle $\mathcal{L} \in W$ we associate the line bundle $\tau(\mathcal{L}) := \mathcal{L}^* \otimes \omega_\Gamma \otimes \det \mathcal{V}^*$. By Serre duality we have $\mathrm{H}^1(\Gamma, \tau(\mathcal{L}) \otimes \mathcal{V}) \cong \mathrm{H}^0(\Gamma, \mathcal{L} \otimes \mathcal{V})^*$, since $\mathcal{V}^* \otimes \det \mathcal{V} \cong \mathcal{V}$. Hence $\mathrm{h}^1(\Gamma, \tau(\mathcal{L}) \otimes \mathcal{V}) = 2$, because $\mathcal{L} \in W$. By Riemann-Roch we know that $\chi(\tau(\mathcal{L}) \otimes \mathcal{V}) = -1$, hence it follows that $\mathrm{h}^0(\Gamma, \tau(\mathcal{L}) \otimes \mathcal{V}) = 1$. Hence τ is a map from W to $\widetilde{W}$ which is clearly an isomorphism. Since we proved in Proposition 3.14 that $\mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{L}) = 2$, we obtain $\mathrm{h}^0(\Gamma, \mathcal{V} \otimes \tau(\mathcal{L})) = 1$.

Recall that in Proposition 3.14 we have established an isomorphism $\psi : L \mapsto \mathbf{\Phi}^!(\mathscr{O}_L(-1))$ from the Hilbert scheme $\mathscr{H}^0_1(X)$ to the locus W. Now $\tau \circ \psi : \mathscr{H}^0_1(X) \to \widetilde{W}$ is again an isomorphism and we want to show that it coincides with $\mathbf{\Phi}^!(-)[-1]$. Setting $\mathscr{P} = p^*(\mathscr{O}_L(-1)) \otimes \mathscr{E}^* \otimes \omega_\Gamma$ in Grothendieck duality formula (3.22), and recalling the definition of $\mathbf{\Phi}^!$, we have:

$$D(\mathbf{\Phi}^!(\mathscr{O}_L(-1))) \cong \mathbf{R}q_*(\mathscr{O}_X(-1) \otimes D(p^*(\mathscr{O}_L(-1)) \otimes \mathscr{E}^* \otimes \omega_\Gamma))[2],$$

Recalling that $\mathscr{E}^* \cong \mathscr{E}(-H_X - N)$ and $\det(\mathcal{V}^*) \cong \omega_\Gamma(-N)$, we have $\mathscr{E}^* \otimes \omega_\Gamma \cong \mathscr{E}(-H_X) \otimes \det \mathcal{V}^*$. Since $D(\mathscr{O}_L(-1))[2] \cong \mathscr{O}_L$ we can conclude that:

$$D(\mathbf{\Phi}^!(\mathscr{O}_L(-1)) \cong \mathbf{R}q_*(p^*(\mathscr{O}_L) \otimes \mathscr{E}^* \otimes \det \mathcal{V}) \cong \mathbf{\Phi}^!(\mathscr{O}_L)[-1] \otimes \omega_\Gamma^* \otimes \det \mathcal{V}^*.$$

This proves that $\tau \circ \psi = \mathbf{\Phi}^!(-)[-1]$ and so $\mathbf{\Phi}^!(-)[-1]$ is an isomorphism of $\mathscr{H}_1^0(X)$ onto $\widetilde{W}$. To check the last statement, we use that $\mathbf{\Phi}^!(-)[-1]$ is an isomorphism, so for any $\mathcal{M} \in \widetilde{W}$, there is a unique $L \in \mathscr{H}_1^0(X)$ such that $\mathcal{M} \cong \mathbf{\Phi}^!(\mathscr{O}_L)[-1]$. For this line L, we already proved the desired statement in (3.32). □

4. Stable sheaves of rank 2 with odd determinant

Recall from [BF08, Theorem 3.9] that, for each $c_2 \geq 6$, there exists an irreducible component $\mathsf{M}(c_2)$ of dimension $2c_2 - 11$ of $\mathsf{M}_X(2, 1, c_2)$ containing a locally free sheaf F which satisfies:

$$\mathrm{H}^1(X, F(-1)) = 0. \tag{4.1}$$

$$\mathrm{Ext}^2_X(F, F) = 0, \tag{4.2}$$

and the extra assumption $\mathrm{H}^0(X, F \otimes \mathscr{O}_L(-1)) = 0$, for some line $L \subset X$ having normal bundle $\mathscr{O}_L \oplus \mathscr{O}_L(-1)$. For $c_2 = 6$, we have $\mathsf{M}(6) = \mathsf{M}_X(2, 1, 6) \cong \Gamma$. For $c_2 \geq 7$, $\mathsf{M}(c_2)$ is defined recursively as the unique component of $\mathsf{M}_X(2, 1, c_2)$ which contains a sheaf F fitting into:

$$0 \to F \to G \to \mathscr{O}_L \to 0, \tag{4.3}$$

where G is a general sheaf lying in $\mathsf{M}(c_2 - 1)$. Here we are going to prove the following result, which amounts to Part A of our main theorem.

Theorem 4.1. *For any integer $c_2 \geq 8$, there is a birational map φ, generically defined by $F \mapsto \mathbf{\Phi}^!(F)$, from $\mathsf{M}(c_2)$ to a generically smooth $(2c_2 - 11)$-dimensional component $\mathsf{B}(c_2)$ of the locus:*

$$\{\mathcal{F} \in \mathsf{M}_\Gamma(c_2 - 6, c_2 - 5) \,|\, \mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{F}) \geq c_2 - 6\}. \tag{4.4}$$

We begin with a series of lemmas.

Lemma 4.2. *Let $c_2 \geq 7$, and let F be a sheaf in $\mathsf{M}_X(2, 1, c_2)$, satisfying (4.1). Then F belongs to the subcategory $\langle \mathcal{U}^*, \mathbf{\Phi}(\mathbf{D^b}(\Gamma)) \rangle$ of $\mathbf{D^b}(X)$, and $\mathbf{\Phi}^!(F)$ is a vector bundle on Γ, of rank $c_2 - 6$ and degree $c_2 - 5$.*

Proof. By stability of F and Serre duality, we have $\mathrm{H}^k(X, F(-1)) = 0$ for $k = 0, 3$. Using (4.1) and Riemann-Roch's formula, we conclude:

$$\mathrm{H}^k(X, F(-1)) = 0, \qquad \text{for all } k.$$

By Serre duality, this is equivalent to:

$$\mathrm{Ext}^k_X(F, \mathscr{O}_X) = 0, \qquad \text{for all } k. \tag{4.5}$$

This implies that the bundle F belongs to $\langle \mathcal{U}^*, \mathbf{\Phi}(\mathbf{D^b}(\Gamma)) \rangle \subset \langle \mathscr{O}_X, \mathcal{U}^*, \mathbf{\Phi}(\mathbf{D^b}(\Gamma)) \rangle$.

By the definition (3.7) of $\mathbf{\Phi}^!$, the stalk of $\mathcal{H}^k(\mathbf{\Phi}^!(F))$ over the point $y \in \Gamma$ is governed by:

$$\mathrm{H}^{k+1}(X, \mathscr{E}_y^* \otimes F) \otimes \omega_{\Gamma, y}. \tag{4.6}$$

Let us check that (4.6) vanishes for all $y \in \Gamma$ and for $k \neq 0$. For $k = -1$, the statement is clear. Indeed, by stability, any non-zero morphism $f : \mathscr{E}_y \to F$ would be injective. Then f should be an isomorphism by Lemma 2.1, and this is impossible.

To check the case $k = 1$, by Serre duality we can show $\mathrm{Ext}^1_X(F, \mathscr{E}_y^*) = 0$. Recall by [BF08, Proposition 3.5] that $\mathscr{E}_y$ is globally generated, and we have thus an exact sequence:

$$0 \to K \to \mathscr{O}_X^6 \to \mathscr{E}_y \to 0.$$

Dualizing it, we obtain:

$$0 \to \mathscr{E}_y^* \to \mathscr{O}_X^6 \to K^* \to 0,$$

where K is a stable vector bundle of rank 4. Applying $\mathrm{Hom}_X(F, -)$, to this sequence, in view of (4.5) we get:

$$\mathrm{Ext}^1_X(F, \mathscr{E}_y^*) = \mathrm{Hom}_X(F, K^*) = 0,$$

where the last equality holds by stability (note that F has slope $1/2$ while K^* has slope $1/4$).

Finally, (4.6) vanishes for $k = 2$ again by stability. Indeed, $\mathrm{H}^3(X, \mathscr{E}_y^* \otimes F)$ is dual to $\mathrm{Hom}_X(F, \mathscr{E}_y^*)$, and the slopes here are $1/2$ and $-1/2$ so that this homomorphism group is zero. We have thus proved that $\mathbf{\Phi}^!(F)$ is a vector bundle on Γ, for the dimension of $\mathrm{H}^1(X, \mathscr{E}_y^* \otimes F)$ does not depend on y. By Riemann-Roch we compute its rank as $\mathrm{rk}(\mathbf{\Phi}^!(F)) = \chi(F \otimes \mathscr{E}_y) = c_2 - 6$.

Using Grothendieck-Riemann-Roch's formula, one can easily compute the degree of $\mathbf{\Phi}^!(F)$. □

Lemma 4.3. *Let $c_2 \geq 7$, and let F be a sheaf in $\mathsf{M}_X(2, 1, c_2)$, satisfying (4.1). Then we have a functorial resolution of the form:*

$$0 \to A_F \otimes \mathcal{U}^* \xrightarrow{\zeta_F} \mathbf{\Phi}(\mathbf{\Phi}^!(F)) \to F \to 0, \tag{4.7}$$

where $A_F = \mathrm{Ext}^2_X(F, \mathcal{U})^*$ has dimension $c_2 - 6$.

Proof. To write down (4.7), we use the exact triangle (3.9). Note that by Lemma 4.2 we know that $F \in \langle \mathcal{U}^*, \mathbf{\Phi}(\mathbf{D^b}(\Gamma)) \rangle$, and we need only to calculate the groups $\mathrm{Ext}^k_X(F, \mathcal{U})$ for all k.

If $k = 0, 3$, we easily get $\mathrm{Ext}^k_X(F, \mathcal{U}) = 0$ by stability of the sheaves $\mathcal{U}$ and F. Applying the functor $\mathrm{Hom}_X(F, -)$ to (3.1) we get $\mathrm{Ext}^1_X(F, \mathcal{U}) \cong \mathrm{Hom}_X(F, \mathcal{U}^*) = 0$, where the vanishing follows from the stability of F and $\mathcal{U}$. By Riemann-Roch we get $\mathrm{ext}^2_X(F, \mathcal{U}) = c_2 - 6$. □

Lemma 4.4. *Let $c_2 \geq 8$, and let F be a sheaf in $\mathsf{M}_X(2, 1, c_2)$, satisfying (4.1). Then:*

$$A_F \cong \mathrm{Hom}_X(\mathcal{U}^*, \mathbf{\Phi}(\mathbf{\Phi}^!(F))), \tag{4.8}$$

$$\mathrm{Ext}^1_X(\mathcal{U}^*, F) \cong \mathrm{Ext}^1_X(\mathcal{U}^*, \mathbf{\Phi}(\mathbf{\Phi}^!(F))), \tag{4.9}$$

$$\mathrm{Ext}^k_X(\mathcal{U}^*, F) = 0, \qquad \textit{for } k \neq 1. \tag{4.10}$$

The map ζ_F in (4.7) coincides, up to conjugacy, with the natural evaluation map.

Proof. Let us first show $\mathrm{Hom}_X(\mathcal{U}^*, F) = 0$. By contradiction, we consider a non-zero map $\gamma : \mathcal{U}^* \to F$. By the argument of Lemma 3.8 we have $\ker(\gamma) \cong \mathscr{O}_X$. Then the cokernel of γ is a torsion sheaf with Chern classes $c_1(\mathrm{cok}(\gamma)) = 0$, $c_2(\mathrm{cok}(\gamma)) = c_2 - 8 \geq 0$ and $c_3(\mathrm{cok}(\gamma)) = 6 - c_2 < 0$, and this is impossible by formula (2.1).

Now, in view of Lemma 4.3, we have the resolution (4.7). We apply to it the functor $\mathrm{Hom}_X(\mathcal{U}^*, -)$. Since $\mathrm{Hom}_X(\mathcal{U}^*, F) = 0$, and $\mathcal{U}^*$ is exceptional, we get (4.8) and (4.9). It is now easy to show (4.10) also for $k = 2, 3$. Indeed, for $k \geq 2$, we have:

$$\mathrm{Ext}^k_X(\mathcal{U}^*, F) \cong \mathrm{Ext}^k_X(\mathcal{U}^*, \mathbf{\Phi}(\mathbf{\Phi}^!(F))) \cong \mathrm{Ext}^k_\Gamma(\mathbf{\Phi}^*(\mathcal{U}^*), \mathbf{\Phi}^!(F)) = 0,$$

since $\mathbf{\Phi}^*(\mathcal{U}^*)$ and $\mathbf{\Phi}^!(F)$ are both sheaves on a curve, the first one by (3.21), the second one by Lemma 4.2.

The last statement follows from the diagram:

$$\begin{array}{ccc} \mathrm{Hom}_X(\mathcal{U}^*, A_F \otimes \mathcal{U}^*) \otimes \mathcal{U}^* & \xrightarrow{\cong} & \mathrm{Hom}_X(\mathcal{U}^*, \mathbf{\Phi}(\mathbf{\Phi}^!(F))) \otimes \mathcal{U}^* \\ \downarrow{\cong} & & \downarrow \\ A_F \otimes \mathcal{U}^* & \xrightarrow{\zeta_F} & \mathbf{\Phi}(\mathbf{\Phi}^!(F)) \end{array}$$

where the left arrow is given by the isomorphism $\mathrm{Hom}_X(\mathcal{U}^*, A_F \otimes \mathcal{U}^*) \cong A_F$, the top one is obtained by (4.8), and the right arrow is the natural evaluation map $e_{\mathcal{U}^*, \mathbf{\Phi}(\mathbf{\Phi}^!(F))}$. □

Lemma 4.5. *Let $c_2 \geq 8$, let F be a sheaf in $\mathsf{M}_X(2, 1, c_2)$ satisfying (4.1), and set $\mathcal{F} = \mathbf{\Phi}^!(F)$.*

i) The bundle $\mathcal{F}$ is simple, and satisfies $\mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{F}) = c_2 - 6$;
ii) setting e for the natural evaluation map:

$$e := e_{\mathcal{V}^*, \mathcal{F}} : \mathrm{Hom}_\Gamma(\mathcal{V}^*, \mathcal{F}) \otimes \mathcal{V}^* \to \mathcal{F},$$

we get that $\mathcal{H}^0(\mathbf{\Phi}(e))$ agrees with ζ_F up to conjugacy;
iii) if F satisfies (4.2), then the natural map:

$$\mathrm{H}^0(\Gamma, \mathcal{V} \otimes \mathcal{F}) \otimes \mathrm{H}^0(\Gamma, \mathcal{V}^* \otimes \mathcal{F}^* \otimes \omega_\Gamma) \to \mathrm{H}^0(\Gamma, \mathcal{F} \otimes \mathcal{F}^* \otimes \omega_\Gamma) \tag{4.11}$$

is injective.

Proof. Recall the notation $A_F = \mathrm{Ext}^2_X(F, \mathcal{U})^*$, and the isomorphism $\mathbf{\Phi}^*(\mathcal{U}^*) \cong \mathcal{V}^*$ (cf. (3.21)). Since $c_2 \geq 8$ we can invoke Lemma 4.4. So, we can use the natural isomorphisms (4.8), (4.9). We have thus:

$$A_F \cong \mathrm{Hom}_\Gamma(\mathcal{V}^*, \mathcal{F}), \qquad \mathrm{Ext}^1_X(\mathcal{U}^*, F) \cong \mathrm{Ext}^1_\Gamma(\mathcal{V}^*, \mathcal{F}),$$

and we have seen that A_F has dimension $c_2 - 6$. This proves the second statement of (i).

Next, we apply the functor $\mathrm{Hom}_X(-, F)$ to the fundamental exact sequence (4.7), and we use the fact that $\mathbf{\Phi}^!$ is right adjoint to $\mathbf{\Phi}$. The vanishing (4.10) at $k = 0$ thus gives:

$$\mathrm{Hom}_\Gamma(\mathcal{F}, \mathcal{F}) \cong \mathrm{Hom}_\Gamma(\mathbf{\Phi}^!(F), \mathbf{\Phi}^!(F)) \cong \mathrm{Hom}_X(\mathbf{\Phi}(\mathbf{\Phi}^!(F)), F) \cong \mathrm{Hom}_X(F, F), \tag{4.12}$$

because F is stable, hence simple. So $\mathcal{F}$ is simple and we have proved (i).

In order to show (ii), we prove that $\mathcal{H}^0(\mathbf{\Phi}(e))$ agrees with the natural evaluation map $e_{\mathcal{U}^*,\mathbf{\Phi}(\mathcal{F})}$, and we use Lemma 4.4. Since we have $\mathrm{Hom}_\Gamma(\mathcal{V}^*,\mathcal{F}) \cong \mathrm{Hom}_X(\mathcal{U}^*,\mathbf{\Phi}(\mathcal{F})) \cong A_F$, the map $e : \mathrm{Hom}_\Gamma(\mathcal{V}^*,\mathcal{F}) \otimes \mathcal{V}^* \cong A_F \otimes \mathbf{\Phi}^*(\mathcal{U}^*) \to \mathcal{F}$ by adjunction gives the map $\mathcal{H}^0(\mathbf{\Phi}(e)) : A_F \otimes \mathcal{U}^* \to \mathbf{\Phi}(\mathcal{F})$. So, by functoriality of the evaluation map, $e_{\mathcal{U}^*,\mathbf{\Phi}(\mathcal{F})}$ coincides with $\mathcal{H}^0(\mathbf{\Phi}(e))$, and we are done. So (ii) is proved.

To prove (iii), we note that the map (4.11) is the transpose of the Petri map $\mathrm{Ext}^1_\Gamma(e,\mathcal{F})$, and we have to check that $\mathrm{Ext}^1_\Gamma(e,\mathcal{F})$ is surjective as soon as F satisfies (4.2). To see this, we write the long exact sequence obtained applying $\mathrm{Hom}_X(-,F)$ to (4.7) and we use (4.12), to obtain:

$$\mathrm{Ext}^1_X(F,F) \to \mathrm{Ext}^1_\Gamma(\mathcal{F},\mathcal{F}) \xrightarrow{\mathrm{Ext}^1_X(\zeta_F,F)} \mathrm{Ext}^1_X(\mathcal{U}^*,F) \otimes A_F^* \to \mathrm{Ext}^2_X(F,F) \to 0,$$

where the last zero is due to $\mathrm{Ext}^2_\Gamma(\mathcal{F},\mathcal{F}) = 0$, since $\mathcal{F}$ is a sheaf on a curve. Therefore, in order to prove our claim, we only have to look at the map $\mathrm{Ext}^1_X(\zeta_F,F)$ and to check that it agrees with $\mathbf{\Phi}(\mathrm{Ext}^1_\Gamma(e,\mathcal{F}))$. But this is clear, since:

$$\mathrm{Ext}^1_X(\zeta_F,F) = \mathrm{Ext}^1_X(\mathbf{\Phi}(\mathbf{\Phi}^*(\zeta_F)),F) = \mathbf{\Phi}(\mathrm{Ext}^1_\Gamma(\mathbf{\Phi}^*(\zeta_F),\mathcal{F})) = \mathbf{\Phi}(\mathrm{Ext}^1_\Gamma(e,\mathcal{F})).$$

□

Lemma 4.6. *Given a non-split exact sequence of vector bundles on* Γ*:*

$$0 \to F' \to F \to F'' \to 0, \tag{4.13}$$

assume $\mathrm{rk}(F') = \deg(F') = 1$*,* $\deg(F) = \mathrm{rk}(F) + 1$ *and* F'' *μ-stable. Then F is μ-stable.*

Proof. Set $r = \mathrm{rk}(F)$. We have $\deg(F) = r + 1$, $\mathrm{rk}(F'') = r - 1$ and $\deg(F'') = r$. Assume by contradiction that F is not μ-stable, so it contains a subsheaf K with $\mu(K) \geq \mu(F)$ and $\mathrm{rk}(K) < r$. Set $s = \mathrm{rk}(K)$ and $c = \deg(K)$. The sequence (4.13) induces an exact sequence of torsion-free sheaves:

$$0 \to K' \to K \to K'' \to 0,$$

with $K'' \subset F''$ and $K' \subset F'$.

If $K' = 0$, then $K \cong K''$ and so $\mu(K) < \mu(F'')$ for F'' is μ-stable and (4.13) is non-split. Now $\mu(K) = c/s \geq (r+1)/r = \mu(F)$ gives $c > r + 1$ because $s < r$, and similarly $\mu(K) = c/s < \mu(F'') = r/(r-1)$ gives $c(r-1) < r^2$. So $r^2 - 1 < c(r-1) < r^2$, which is impossible.

If $\mathrm{rk}(K') = 1$, we have either $\deg(K') \leq 0$, or $K' \cong F'$. In the first case we have $\deg(K'') \geq \deg(K)$ and so $\mu(F) \leq \mu(K) \leq \frac{\deg(K'')}{s-1} = \mu(K'') < \mu(F'')$ and again $(r+1)/r \leq c/s < r/(r-1)$ is impossible if $s < r$. In the second case, we get $(c-1)/(s-1) = \mu(K'') < \mu(F'') = r/(r-1)$ and $c/s = \mu(K) \geq \mu(F) = (r+1)/r$. Using $s < r$, from the first inequality we get $(c-1)(r-1) < r(r-1)$, so $r + 1 > c$, and plugging into the second inequality we get $cr > cs$, which is absurd. □

We are now in position to prove the main result of this section.

Proof of Theorem 4.1. The proof goes by induction on $c_2 \geq 8$, but we first need some results on the case $c_2 = 7$. According to [BF08, Proposition 3.5], any sheaf F of $\mathsf{M}_X(2,1,7)$ satisfies (4.1), so we may apply Lemma

4.2 to F and we obtain that $\mathcal{F} = \mathbf{\Phi}^!(F)$ is a line bundle (hence stable) of degree 2 on Γ. Since $\mathcal{V}$ has rank 2 and degree 1 (see Remark 3.12), applying Riemann-Roch on Γ we get:

$$\mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{F}) \geq \chi(\Gamma, \mathcal{V} \otimes \mathcal{F}) = 1.$$

Now let us look at $c_2 \geq 8$. Recall from [BF08, Theorem 3.9] that there exists an irreducible component $\mathsf{M}(c_2)$ of dimension $2c_2 - 11$ of $\mathsf{M}_X(2,1,c_2)$ containing a vector bundle F satisfying (4.1), hence by semicontinuity there exists an open dense subset of $\mathsf{M}(c_2)$ where any vector bundle satisfies (4.1). Thus, for any sheaf F in this open set, by Lemma 4.2 it follows that $\mathcal{F} = \mathbf{\Phi}^!(F)$ is a vector bundle on Γ of rank $c_2 - 6$ and degree $c_2 - 5$, and it satisfies $\mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{F}) = c_2 - 6$ by Lemma 4.5. The mapping that sends $\mathcal{F}$ to F is injective for $c_2 \geq 8$, because, in view of Lemma 4.4, the sheaf F can be recovered from $\mathcal{F}$ as the cokernel of the natural evaluation map:

$$\mathrm{Hom}_X(\mathcal{U}^*, \mathbf{\Phi}(\mathcal{F})) \otimes \mathcal{U}^* \to \mathbf{\Phi}(\mathcal{F}).$$

Let us now prove that, if F is general in $\mathsf{M}(c_2)$, then the vector bundle $\mathbf{\Phi}^!(F)$ is stable over Γ. In fact we prove that, if F is a sheaf fitting into (4.3), and G is general in $\mathsf{M}(c_2 - 1)$, then $\mathcal{F} = \mathbf{\Phi}^!(F)$ is stable over Γ. Since stability is an open property by [Mar76], this will imply that $\mathbf{\Phi}^!(F)$ is stable for F general in $\mathsf{M}(c_2)$. By induction, we may assume that $\mathbf{\Phi}^!(G)$ is a stable vector bundle of rank $c_2 - 7$ and degree $c_2 - 6$.

Applying $\mathbf{\Phi}^!$ to (4.3), we get an exact sequence of bundles on Γ:

$$0 \to \mathbf{\Phi}^!(\mathscr{O}_L)[-1] \to \mathcal{F} \to \mathbf{\Phi}^!(G) \to 0,$$

where $\mathbf{\Phi}^!(\mathscr{O}_L)[-1]$ is a line bundle of degree 1 by Lemma 3.15. Note that this extension must be non-split. Indeed, $\mathcal{F}$ is simple in view of (4.12), hence indecomposable. Moreover, $\deg(\mathcal{F}) = \mathrm{rk}(\mathcal{F}) + 1$, so we may apply Lemma 4.6 and conclude that $\mathcal{F}$ is stable.

We have thus proved that an open dense subset of $\mathsf{M}(c_2)$ maps into the locus defined by (4.4). This locus is equipped with a natural structure of a subvariety of the moduli space $\mathsf{M}_\Gamma(c_2 - 6, c_2 - 5)$. Its tangent space at the point $[\mathcal{F}]$ is identified with $\ker(\mathrm{Ext}^1_\Gamma(e, \mathcal{F}))$, while the obstruction sits in $\mathrm{cok}(\mathrm{Ext}^1_\Gamma(e, \mathcal{F}))$, where again $e = e_{\mathcal{V}^*, \mathcal{F}}$. Notice that, by Lemma 4.5, the latter space vanishes if F satisfies (4.2).

We have thus proved that the mapping $F \mapsto \mathcal{F}$ provides an open immersion of a Zariski open dense subset of the $(2d-11)$-dimensional variety $\mathsf{M}(c_2)$, into a Zariski open dense piece of $\mathsf{B}(c_2)$. This finishes the proof. □

5. The moduli space $\mathsf{M}_X(2,1,7)$ as a blow-up of the Picard variety

In this section, we set up a more detailed study of the moduli space $\mathsf{M}_X(2,1,7)$, of which we give a biregular (rather than birational) description. In fact, the map $\varphi : F \mapsto \mathbf{\Phi}^!(F)$ sends the whole space $\mathsf{M}_X(2,1,7)$ to the abelian variety $\mathrm{Pic}^2(\Gamma)$. In turn, $\mathrm{Pic}^2(\Gamma)$ contains a copy of the Hilbert scheme $\mathscr{H}^0_1(X)$, via the map ψ (see Proposition 3.14), as a subvariety of codimension 2. The relation between these varieties is given by the main result of this section, which provides Part B of our main theorem.

Theorem 5.1. *The mapping* $\varphi : F \mapsto \mathbf{\Phi}^!(F)$ *gives an isomorphism of the moduli space* $\mathsf{M}_X(2,1,7)$ *to the blow-up of* $\mathrm{Pic}^2(\Gamma)$ *along the subvariety* $W = \psi(\mathscr{H}_1^0(X))$. *The exceptional divisor consists of the sheaves in* $\mathsf{M}_X(2,1,7)$ *which are not globally generated.*

The proof of this result occupies the rest of the paper. The strategy goes as follows. First, we show that $\mathsf{M}_X(2,1,7)$ is isomorphic to the the Quot scheme of quotients of $\mathcal{V}$ with Hilbert polynomial $t+1$. Then, we prove that this Quot scheme is isomorphic to the blow-up of $\mathrm{Pic}^2(\Gamma)$ along the subvariety W. In doing so, we will first need a detailed analysis of the locus in $\mathsf{M}_X(2,1,7)$ of sheaves that are not globally generated.

5.1. **Characterization and properties of globally generated sheaves.** We first study in detail the condition for a sheaf in $\mathsf{M}_X(2,1,7)$ to be globally generated. In particular, we need to show that a sheaf in $\mathsf{M}_X(2,1,7)$ is not globally generated iff it fails to be so on a line $L \subset X$, see the next lemma.

Lemma 5.2. *Let* F *be a sheaf in* $\mathsf{M}_X(2,1,7)$. *Then, we have:*

$$\mathrm{H}^k(X, F(-1)) = \mathrm{H}^k(X, F) = 0, \qquad \textit{for } k = 1, 2. \tag{5.1}$$

Moreover, either F *is locally free, or there exist a bundle* E *in* $\mathsf{M}_X(2,1,6)$, *and a line* $L \subset X$, *and an exact sequence:*

$$0 \to F \to E \to \mathscr{O}_L \to 0.$$

Furthermore, the following statements are equivalent:

i) the sheaf F *is not globally generated;*
ii) the vector space $\mathrm{Hom}_X(\mathcal{U}^*, F)$ *is non-zero;*
iii) there exists a line $L \subset X$, *and a long exact sequence:*

$$0 \to \mathscr{O}_X \to \mathcal{U}^* \to F \to \mathscr{O}_L(-1) \to 0. \tag{5.2}$$

Proof. The first two statements are taken from [BF08, Proposition 3.5], and we only have to prove the equivalence of (i), (ii) and (iii). Clearly condition (iii) implies both conditions (i) and (ii).

Let us prove (ii) $\Rightarrow$ (iii). Consider a non-zero map $\gamma : \mathcal{U}^* \to F$. The argument of Lemma 3.8 implies $\ker(\gamma) \cong \mathscr{O}_X$ and the cokernel T of γ has $c_1(T) = 0$, $c_2(T) = -1$, $c_3(T) = -1$. So we have $T \cong \mathscr{O}_L(-1)$, for some line $L \subset X$, if T is supported on a Cohen-Macaulay curve. In turn, this holds if the support of T has no isolated or embedded points, which follows once we prove $\mathrm{H}^0(X, T(-1)) = 0$. But, if I if the image of the middle map in (5.2), we have $\mathrm{H}^1(X, I(-1)) = 0$, so by $\mathrm{H}^0(X, F(-1)) = 0$, we have $\mathrm{H}^0(X, T(-1)) = 0$, and we are done.

It remains to show (i) $\Rightarrow$ (ii). We follow here an argument suggested by the referee, that simplifies our previous proof. Let F be a sheaf in $\mathsf{M}_X(2,1,7)$. We assume $\mathrm{Hom}_X(\mathcal{U}^*, F) = 0$, and we want to prove that F is globally generated.

First, we observe that $\mathrm{Hom}_X(\mathcal{U}^*, F) = 0$ implies that $F(1)$ lies in the subcategory ${}^{\perp}(\mathcal{U}^*)$ of $\mathbf{D^b}(X)$, and that this subcategory is $\langle \mathbf{\Phi}(\mathbf{D^b}(\Gamma)), \mathscr{O}_X(1) \rangle$. The second assertion is clear from $\mathbf{D^b}(X) = \langle \mathscr{O}_X, \mathcal{U}^*, \mathbf{\Phi}(\mathbf{D^b}(\Gamma)) \rangle$, since mutating $\mathscr{O}_X$ through $\langle \mathcal{U}^*, \mathbf{\Phi}(\mathbf{D^b}(\Gamma)) \rangle$ we get $\mathscr{O}_X(1)$. The relation $F \in {}^{\perp}(\mathcal{U}^*)$

amounts to show $\mathrm{Ext}^{3-k}_X(F(1),\mathcal{U}^*) = 0$ for $0 \le k \le 3$, or by Serre duality $\mathrm{Ext}^k_X(\mathcal{U}^*, F) = 0$ for $0 \le k \le 3$. This is clear for $k = 2, 3$ by the argument proving (4.10) in Lemma 4.4. Our assumption $\mathrm{Hom}_X(\mathcal{U}^*, F) = 0$ gives this vanishing for $k = 0$, so we can deduce the vanishing for $k = 1$ by Hirzebruch-Riemann-Roch formula (3.11).

Now, we know that $\mathrm{H}^k(X, F) = 0$ for $k > 0$, so the decomposition $\langle \mathbf{\Phi}(\mathbf{D^b}(\Gamma)), \mathscr{O}_X(1)\rangle$ gives an exact triangle:

$$\mathrm{H}^0(X, F) \otimes \mathscr{O}_X(1) \to F(1) \to \mathbf{\Phi}(\mathbf{\Phi}^*(F(1)).$$

We will be done if we show $\mathcal{H}^0(\mathbf{\Phi}(\mathbf{\Phi}^*(F(1))) = 0$, for in this case the above triangle show that F is globally generated. Note that, by definition of $\mathbf{\Phi}^*$ and $\mathbf{\Phi}^!$ (see (3.8) and (3.7)), we have $\mathbf{\Phi}^*(F(1)) \cong \mathbf{\Phi}^!(F) \otimes \omega^*_\Gamma[2]$. But we have proved that $\mathbf{\Phi}^!(F)$ is a sheaf, so $\mathbf{\Phi}^*(F(1))$ is a sheaf concentrated in degree -2, supported on the curve Γ. This clearly implies $\mathcal{H}^0(\mathbf{\Phi}(\mathbf{\Phi}^*(F(1))) = 0$, and we are done. □

5.2. **Sheaves on the Fano threefold and Quot scheme on the dual curve.** We refer to [HL97] for the description and properties of Grothendieck's Quot scheme. We let $\mathbf{P} = \mathrm{Quot}_\Gamma(\mathcal{V}, t + 1)$ be the Quot scheme of quotients of $\mathcal{V}$ having Hilbert polynomial $t + 1$. An element of $\mathbf{P}$ is a quotient Q of $\mathcal{V}$ of rank 1 and degree 3. The dual of the kernel of the map $\mathcal{V} \to Q$ is a line bundle of degree 2 on Γ, so we have a map $v : \mathbf{P} \to \mathrm{Pic}^2(\Gamma)$.

Proposition 5.3. *The moduli space $\mathsf{M}_X(2, 1, 7)$ is isomorphic to the Quot scheme $\mathbf{P}$.*

Proof. We would like to find two maps between $\mathsf{M}_X(2, 1, 7)$ and $\mathbf{P}$ that are mutually inverse. We now define a map $k : \mathsf{M}_X(2, 1, 7) \to \mathbf{P}$. To do so, let F be any sheaf in $\mathsf{M}_X(2, 1, 7)$ set $\mathcal{L} = \mathbf{\Phi}(F)$ and recall that, by (4.7), F is the cokernel of a canonically defined map $\zeta_F : \mathcal{U}^* \to \mathbf{\Phi}(\mathcal{L})$. Apply $\mathbf{\Phi}^*$ and recall that $\mathbf{\Phi}^*(\mathcal{U}^*) \cong \mathcal{V}^*$ (Proposition 3.10) and that $\mathbf{\Phi}^*(\mathbf{\Phi}(\mathcal{L})) \cong \mathcal{L}$ lies in $\mathrm{Pic}^2(\Gamma)$ (Lemma 4.2). Then, with F we associate a non-zero map $\mathcal{V}^* \to \mathcal{L}$ and its transpose gives an injective map $\mathcal{L}^* \to \mathcal{V}$, whose cokernel we define as $k(F) \in \mathbf{P}$.

Now we define a map $h : \mathbf{P} \to \mathsf{M}_X(2, 1, 7)$, inverse of k. To do so, consider a quotient Q of $\mathcal{V}$ and let $\mathcal{L}^*$ be the kernel of the projection $\mathcal{V} \to Q$. The sheaf $\mathcal{L}$ is a line bundle of degree 2 on Γ. Transposing the injection $\mathcal{L}^* \to \mathcal{V}$ we get a non-zero map $a_Q : \mathcal{V}^* \to \mathcal{L}$. We can now take, by adjunction, the map:

$$\varepsilon_Q = \mathcal{H}^0(\mathbf{\Phi}(a_Q)) : \mathcal{U}^* \to \mathbf{\Phi}(\mathcal{L}).$$

We shall prove that $\mathscr{F}_Q = \mathrm{cok}(\varepsilon_Q)$ is a sheaf in $\mathsf{M}_X(2, 1, 7)$. We distinguish two cases, according to whether a_Q is surjective or not.

i) If a_Q is surjective, then $K = \ker(a_Q)$ is a line bundle on Γ of degree -3, and we have:

$$0 \to K \to \mathcal{V}^* \to \mathcal{L} \to 0. \tag{5.3}$$

Note that, using stability of the $\mathscr{E}_y$'s, we can easily check that $\mathbf{\Phi}(\mathcal{L})$ and $\mathbf{\Phi}(K)$ are vector bundles, concentrated in degree 0 and 1 respectively, and the rank and Chern classes of these bundles are determined by Grothendieck-Riemann-Roch. Applying $\mathbf{\Phi}$ to (5.3) we get:

$$0 \to \mathcal{U}^* \xrightarrow{\varepsilon_Q} \mathbf{\Phi}(\mathcal{L}) \to \mathbf{\Phi}(K)[1] \to \mathcal{U}(1) \to 0.$$

Then one can easily see that $\mathscr{F}_Q$, being the image of the middle map above, is reflexive and lies in $\mathsf{M}_X(2,1,7)$.

ii) Assume now that a_Q is not surjective. We set $K = \ker(a_Q)$ and $\mathcal{M} = \operatorname{Im}(a_Q)$ and we have $\operatorname{rk}(\mathcal{M}) = 1$. Note that $\deg \mathcal{M} \geq 1$ by the same argument of Lemma 3.13, hence in fact $\deg(\mathcal{M}) = 1$. We have then the exact sequence:

$$0 \to \mathcal{M} \to \mathcal{L} \to \mathscr{O}_y \to 0. \tag{5.4}$$

The line bundle $\mathcal{M}$ obviously lies in the locus $\widetilde{W}$ defined by Lemma 3.15. Then we have, by (3.31), that $\mathcal{H}^0(\mathbf{\Phi}(\mathcal{M})) \cong \mathcal{U}^*$ and $\mathcal{H}^1(\mathbf{\Phi}(\mathcal{M})) \cong \mathscr{O}_M$, where $M \subset X$ is a line. So, applying $\mathbf{\Phi}$ to (5.4), we get:

$$0 \to \mathcal{U}^* \xrightarrow{\varepsilon_Q} \mathbf{\Phi}(\mathcal{L}) \to \mathscr{E}_y \to \mathscr{O}_M \to 0.$$

Therefore the sheaf $\mathscr{F}_Q = \operatorname{cok}(\varepsilon_Q)$ lies in $\mathsf{M}_X(2,1,7)$, and we know moreover that $\mathscr{F}_Q$ is not locally free over M and that $\mathscr{F}_Q^{**} \cong \mathscr{E}_y$.

We have thus completed the construction of h. It is now straightforward to check that h and k are mutually inverse. □

5.3. **Quot scheme and the blow-up of the Picard variety.** Recall that we proved in Proposition 3.14 that the Hilbert scheme of lines contained in X is isomorphic to a subvariety W of $\operatorname{Pic}^2(\Gamma)$, and denote by $\iota : W \hookrightarrow \operatorname{Pic}^2(\Gamma)$ the inclusion. We consider the blow-up $\mathbf{B}$ of $\operatorname{Pic}^2(\Gamma)$ at W, and we denote by $\mathbf{E}$ the exceptional divisor.

Proposition 5.4. *The Quot scheme* $\mathbf{P}$ *is isomorphic to the blown-up Picard variety* $\mathbf{B}$.

We need the following lemma. Let us denote by r and s the projections from $\Gamma \times \operatorname{Pic}^2(\Gamma)$ onto Γ and $\operatorname{Pic}^2(\Gamma)$, respectively:

$$\Gamma \xleftarrow{\;r\;} \Gamma \times \operatorname{Pic}^2(\Gamma) \xrightarrow{\;s\;} \operatorname{Pic}^2(\Gamma)$$

A Poincaré bundle $\mathscr{P}$ on $\Gamma \times \operatorname{Pic}^2(\Gamma)$ is a line bundle satisfying $\mathscr{P}_{|\Gamma \times \mathcal{L}} \cong \mathcal{L}$, for all $\mathcal{L} \in \operatorname{Pic}^2(\Gamma)$.

Lemma 5.5. *There is a choice of a Poincaré line bundle* $\mathscr{P}$ *on* $\Gamma \times \operatorname{Pic}^2(\Gamma)$ *that gives:*

$$\begin{aligned} & s_*(r^*(\mathcal{V}) \otimes \mathscr{P}) \cong \mathscr{O}_{\operatorname{Pic}^2(\Gamma)}, \\ & \mathbf{R}^1 s_*(r^*(\mathcal{V}) \otimes \mathscr{P}) \cong \iota_*(\omega_W), \\ & \mathbf{R}s_*(r^*(\mathcal{V}^* \otimes \omega_\Gamma) \otimes \mathscr{P}^*) \cong \mathcal{I}_W[-1]. \end{aligned}$$

Proof. We will first choose $\mathscr{P}$ to be any Poincaré line bundle on $\Gamma \times \operatorname{Pic}^2(\Gamma)$, and indicate at the end of the proof how to modify $\mathscr{P}$ so that it has the desired properties.

Recall that W is purely 1-dimensional subscheme of $\mathrm{Pic}^2(\Gamma)$, see Proposition 3.14. Moreover, by the same proposition, W is Cohen-Macaulay, as it can be seen using the formalism of determinantal subvarieties, as explained in [ACGH85].

As a first step, we compute $\mathbf{R}^k s_*(r^*(\mathcal{V}) \otimes \mathscr{P})$ and $\mathbf{R}^k s_*(r^*(\mathcal{V}^* \otimes \omega_\Gamma) \otimes \mathscr{P}^*)$. These sheaves are zero for $k \neq 0, 1$. Further, we have:

$$\mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathscr{P}_{\mathcal{L}}) = 2 \qquad \mathrm{h}^1(\Gamma, \mathcal{V} \otimes \mathscr{P}_{\mathcal{L}}) = 1 \qquad \text{iff } \mathcal{L} \in W,$$
$$\mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathscr{P}_{\mathcal{L}}) = 1 \qquad \mathrm{h}^1(\Gamma, \mathcal{V} \otimes \mathscr{P}_{\mathcal{L}}) = 0 \qquad \text{iff } \mathcal{L} \in \mathrm{Pic}^2(\Gamma) \setminus W.$$

This says that $s_*(r^*(\mathcal{V}) \otimes \mathscr{P})$ is a line bundle $\mathscr{M}$ on $\mathrm{Pic}^2(\Gamma)$, and that $\mathbf{R}^1 s_*(r^*(\mathcal{V}) \otimes \mathscr{P})$ is $\iota_*(\mathscr{N})$, where $\mathscr{N}$ is an invertible sheaf on W. Further, we have:

$$\mathrm{h}^0(\Gamma, \mathcal{V}^* \otimes \omega_\Gamma \otimes \mathscr{P}^*_{\mathcal{L}}) = 1 \quad \mathrm{h}^1(\Gamma, \mathcal{V}^* \otimes \omega_\Gamma \otimes \mathscr{P}^*_{\mathcal{L}}) = 2 \quad \text{iff } \mathcal{L} \in W,$$
$$\mathrm{h}^0(\Gamma, \mathcal{V}^* \otimes \omega_\Gamma \otimes \mathscr{P}^*_{\mathcal{L}}) = 0 \quad \mathrm{h}^1(\Gamma, \mathcal{V}^* \otimes \omega_\Gamma \otimes \mathscr{P}^*_{\mathcal{L}}) = 1 \quad \text{iff } \mathcal{L} \in \mathrm{Pic}^2(\Gamma) \setminus W.$$

So $s_*(r^*(V^* \otimes \omega_\Gamma) \otimes \mathscr{P}^*) = 0$, and $\mathbf{R}s_*(r^*(\mathcal{V}^* \otimes \omega_\Gamma) \otimes \mathscr{P}^*)$ is a sheaf on $\mathrm{Pic}^2(\Gamma)$, concentrated in degree 1, and having (generic) rank 1.

To relate these two calculations, we use Grothendieck duality, which gives:

$$D(\mathbf{R}s_*(r^*(\mathcal{V}) \otimes \mathscr{P})) \cong \mathbf{R}s_*(r^*(\mathcal{V}^* \otimes \omega_\Gamma) \otimes \mathscr{P}^*)[1]. \tag{5.5}$$

We have $D(\iota_*(\mathscr{N})) \cong \iota_*(\mathscr{N}^* \otimes \omega_W)[2]$, since W is Cohen-Macaulay of codimension 2 in $\mathrm{Pic}^2(\Gamma)$. Looking at the spectral sequence associated with (5.5), we get that $\mathbf{R}s_*(r^*(\mathcal{V}^* \otimes \omega_\Gamma) \otimes \mathscr{P}^*)[1]$ is the kernel of a surjective map:

$$\mathscr{M}^* \to \iota_*(\mathscr{N}^* \otimes \omega_W).$$

Hence we get the isomorphisms:

$$\mathbf{R}s_*(r^*(\mathcal{V}^* \otimes \omega_\Gamma) \otimes \mathscr{P}^*) \cong \mathcal{I}_W \otimes \mathscr{M}^*[-1], \qquad \mathscr{N} \cong \mathscr{M} \otimes \iota_*(\omega_W).$$

It is now clear that the desired Poincaré line bundle is obtained as $\mathscr{P} \otimes \mathscr{M}^*$. □

Proof of Proposition 5.4. Consider the blow-up $\mathbf{B}$ of $\mathrm{Pic}^2(\Gamma)$ along the subvariety W, and let $b : \mathbf{B} \to \mathrm{Pic}^2(\Gamma)$ denote the blow-up map. Recall that $\mathbf{B}$ is the projectivization of the sheaf $\mathcal{I}_W$. So, a point z of $\mathbf{B}$ is uniquely given by a surjection $\mathcal{I}_W \to \mathscr{O}_{b(z)}$. In fact, a non-zero map $\mathcal{I}_W \to \mathscr{O}_{b(z)}$ can only be a surjection, since $\mathscr{O}_{b(z)}$ has no non-trivial sub-sheaves. Summing up, points of $\mathbf{B}$ are just non-zero maps $\mathcal{I}_W \to \mathscr{O}_{b(z)}$.

We consider now the functor $\boldsymbol{\Upsilon} : \mathbf{D^b}(\mathrm{Pic}^2 \Gamma) \to \mathbf{D^b}(\Gamma)$ defined by $\boldsymbol{\Upsilon}(-) = \mathbf{R}r_*(s^*(-) \otimes \mathscr{P})$, and its left adjoint functor $\boldsymbol{\Upsilon}^*(-) = \mathbf{R}s_*(r^*(-) \otimes \mathscr{P}^* \otimes \omega_\Gamma)[1]$. Then the previous lemma says:

$$\boldsymbol{\Upsilon}^*(\mathcal{V}^*) \cong \mathcal{I}_W.$$

Recall now that $b(z)$ represents a line bundle of degree 2 on Γ, namely $\mathscr{P}_{b(z)}$. Let us abbreviate $\mathcal{L} = \mathscr{P}_{b(z)}$. We consider the following identifications:

$$\mathrm{Hom}_{\mathrm{Pic}^2(\Gamma)}(\mathcal{I}_W, \mathscr{O}_{b(z)}) \cong \mathrm{Hom}_{\mathrm{Pic}^2(\Gamma)}(\boldsymbol{\Upsilon}^*(\mathcal{V}^*), \mathscr{O}_{b(z)}) \cong \mathrm{Hom}_\Gamma(\mathcal{V}^*, \boldsymbol{\Upsilon}(\mathscr{O}_{b(z)})) \cong \mathrm{Hom}_\Gamma(\mathcal{V}^*, \mathcal{L}).$$

Recall that a point of $\mathbf{P}$ is a quotient Q of $\mathcal{V}$ with rank 1 and degree 3. We already pointed out that such a quotient $\mathcal{V} \to Q$ determines a kernel

$\mathcal{L}^*$ (with $\mathcal{L} \in \mathrm{Pic}^2(\Gamma)$) and so a non-zero map $a : \mathcal{V}^* \to \mathcal{L}$. Vice-versa, a non-zero map $a : \mathcal{V}^* \to \mathcal{L}$ determines Q as the cokernel of its transpose.

Now the correspondence between $\mathbf{B}$ and $\mathbf{P}$ is clear. Namely, a point z of $\mathbf{B}$ is a non-zero element of $\mathrm{Hom}_{\mathrm{Pic}^2(\Gamma)}(\mathcal{I}_W, \mathscr{O}_{b(z)}) \cong \mathrm{Hom}_\Gamma(\mathcal{V}^*, \mathcal{L})$, so z corresponds to a map $a : \mathcal{V}^* \to \mathcal{L}$, and we associate with z the cokernel of the map $\mathcal{L}^* \to \mathcal{V}$, obtained transposing a. This cokernel is an element of $\mathbf{Q}$. Clearly the construction is reversible, and the proposition is proved. □

The proof of Theorem 5.1 is now almost completed. The only statement that still needs to be checked is that the exceptional divisor $\mathbf{E}$ of the blow-up is mapped by our isomorphism onto the set of sheaves in $\mathsf{M}_X(2,1,7)$ that are not globally generated. To see this, recall that an element $w \in \mathbf{B}$ lies in $\mathbf{E}$ if and only if it is mapped by b to a point of W. In turn, a line bundle $\mathcal{L} = \mathbf{\Phi}^!(F)$ lies in W if and only if it satisfies $\mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{L}) = 2$. But this is equivalent to $\hom_X(\mathcal{U}^*, \mathbf{\Phi}(\mathcal{L})) = 2$, which in turn happens if and only if $\mathrm{Hom}_X(\mathcal{U}^*, F) \neq 0$, as one sees immediately applying $\mathrm{Hom}_X(\mathcal{U}^*, -)$ to (4.7). Therefore, by Lemma 5.2, we get that $w \in \mathbf{B}$ lies in $\mathbf{E}$ if and only if the corresponding sheaf $F_w \in \mathsf{M}_X(2,1,7)$ fails to be globally generated. This concludes the proof.

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