LOGARITHMIC BUNDLES OF DEFORMED WEYL ARRANGEMENTS OF TYPE A_2

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ABSTRACT. We consider deformations of the Weyl arrangement of type A_2, which include the extended Shi and Catalan arrangements. These last ones are well-known to be free. We study their sheaves of logarithmic vector fields in all other cases, and show that they are Steiner bundles. Also, we determine explicitly their unstable lines. As a corollary, some counter-examples to the shift isomorphism problem are given.

INTRODUCTION

Let Φ be an irreducible crystallographic root system in Euclidean space $V \simeq \mathbb{R}^m$, let $\Phi^+ \subset \Phi$ be the positive roots, and let η be the Coxeter number of Φ. Let $x_1, \ldots, x_m$ be coordinates of V, set $S = \mathbb{R}[x_0, \ldots, x_m]$, and denote by $\mathrm{Der}(S)$ the free S-module of derivations of S, generated by the partial derivatives $\frac{\partial}{\partial x_0}, \ldots, \frac{\partial}{\partial x_m}$. For $s \in \mathbb{Z}$ and $\alpha \in \Phi^+$, define the hyperplanes:

$$H_{\alpha,s} = \{x \in \mathbb{P}^m \mid \alpha(x_1, \ldots, x_m) = sx_0\} \subset \mathbb{P}^m.$$

Fix integers $k, j \geq 0$, and define the (cone over the) *deformation of the Weyl arrangement of type* Φ:

$$\mathcal{A}_\Phi^{[-j,k+j]} = \{x_0 = 0\} \cup \{H_{\alpha,s} \mid \alpha \in \Phi^+, -j \leq s \leq k+j\}.$$

The combinatorics, topology and algebra of $\mathcal{A} = \mathcal{A}_\Phi^{[-j,k+j]}$ have been studied by several authors, for instance by Postnikov and Stanley in [10], by Athanasiadis in [2], by Edelman and Reiner in [5], and by Yoshinaga in [14], especially when $k \in \{0, 1\}$. In particular, the freeness of $\mathcal{A}$ when $k = 0, 1$ was conjectured by Edelman and Reiner and proved in [14] by Yoshinaga. By *freeness* here we mean freeness of logarithmic derivation module of $\mathcal{A}$:

$$D_0(\mathcal{A}) := \{\theta \in \mathrm{Der}(S) \mid \theta(f_{j,k}) = 0\},$$

where $f_{j,k}$ is the form of degree $n = |\mathcal{A}|$ given as product of linear forms defining the hyperplanes of $\mathcal{A}$. Equivalently, freeness means splitting of the the sheafification $T_\mathcal{A}$ of $D_0(\mathcal{A})$. This is a reflexive sheaf of rank m called *logarithmic sheaf*. It can also be defined as the kernel of the Jacobian map:

$$\mathcal{O}_{\mathbb{P}^m}^{m+1} \xrightarrow{\nabla(f_{j,k})} \mathcal{O}_{\mathbb{P}^m}(n-1).$$

In spite of the good knowledge of $T_\mathcal{A}$ for $k \in \{0,1\}$, almost nothing is known about $T_\mathcal{A}$ for $k \geq 2$, not even for A_2. For example, setting $\mathcal{B} = \mathcal{A}_\Phi^{[-j-1,k+j+1]}$, the *shift isomorphism problem*, cf. [15, Remark 3.7] asks whether there is an isomorphism:

$$(1) \qquad\qquad T_\mathcal{A} \;\simeq\; T_\mathcal{B}(\eta).$$

Another question is the *shifted dual isomorphism problem*, to the effect that:

$$(2) \qquad\qquad T_\mathcal{A} \;\simeq\; T_\mathcal{A}^\vee(-\eta(k+2j+1)).$$

These isomorphisms hold when $k = 0,1$ by [14]. However, even equality of characteristic polynomials (i.e., of Chern classes) of these sheaves is unknown in general: this is the so-called "functional equation" conjecture of [10], cf. also [15, Conjecture 3.4 and 3.5]. However the roots of the characteristic polynomial should have real part $\eta(k+2j+1)/2$ by the "Riemann hypothesis" of [10], verified for Φ of type A, B, C, D in [3].

In this paper, we are most interested in the case $\Phi = A_2$. We switch to the notation (z, x, y) rather than (x_0, x_1, x_2), and we fix $\mathcal{A} = \mathcal{A}_{A_2}^{[-j,k+j]}$. We have $\eta = 3$. In this case $T_\mathcal{A}$ is locally free (a vector bundle) of rank 2, and the lines of $\mathcal{A}$ are defined by vanishing of the form:

$$f_{j,k} = z \prod_{-j \leq s \leq k+j} (x - sz)(y - sz)(y + x - sz).$$

Concerning resolutions, our main theorem is the following.

Theorem 1. *For any $k \geq 2$ and $j \geq 0$, there is a resolution:*

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^{k-1} \to \mathcal{O}_{\mathbb{P}^2}^{k+1} \to T_\mathcal{A}(2k+1+3j) \to 0.$$

In particular, $T_\mathcal{A}(2k+1+3j)$ is a Steiner bundle.

By *Steiner bundle* here we mean a vector bundle whose resolution is given by a matrix of linear forms. This agrees and gives a new interpretation of the following formulas, easily obtained for instance counting multiple points and using [7, Remark 2.2]:

$$c_1(T_\mathcal{A}(2k+3j+1)) = k-1, \qquad c_2(T_\mathcal{A}(2k+3j+1)) = \tfrac{k(k-1)}{2}.$$

Since $T_\mathcal{A}(2k+1+3j)$ is a Steiner bundle, for any line $L \subset \mathbb{P}^2$, by restriction onto L we get a surjective map:

$$\mathcal{O}_L^{k+1} \longrightarrow T_\mathcal{A}(2k+1+3j)|_L.$$

This implies that $T_\mathcal{A}(2k+1+3j)|_L = \mathcal{O}_L(a) \oplus \mathcal{O}_L(k-1-a)$ with $0 \leq a \leq k-1$. When $a = 0$ or $a = k-1$ the number $|k-1-2a|$ is as large as possible. This justifies the next definition, cf. [11, Page 508] or [6, Definition 2.1].

Definition 1. Let $k \geq 2$ and E be a Steiner bundle defined by:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{k-1} \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{k+1} \longrightarrow E \longrightarrow 0.$$

A line L such that $E|_L = \mathcal{O}_L \oplus \mathcal{O}_L(k-1)$ or equivalently $H^0(\mathbb{P}^2, E^\vee|_L) \neq 0$ or equivalently $H^1(L, E|_L(-2)) \neq 0$ is called *unstable*. The set of such lines is denoted by $W(E)$, it is naturally a subscheme of $\check{\mathbb{P}}^2$. These unstable lines were first called *superjumping lines* in [4].

Our next result, tightly related with Theorem 1, deals with the set of unstable lines of $T_{\mathcal{A}}$, which we can determine explicitly. The figure shows them in case $j = 0$ and $k = 3$ or $k = 4$, the thick orange lines being unstable (the solid ones lie in the arrangement, the dashed ones don't).
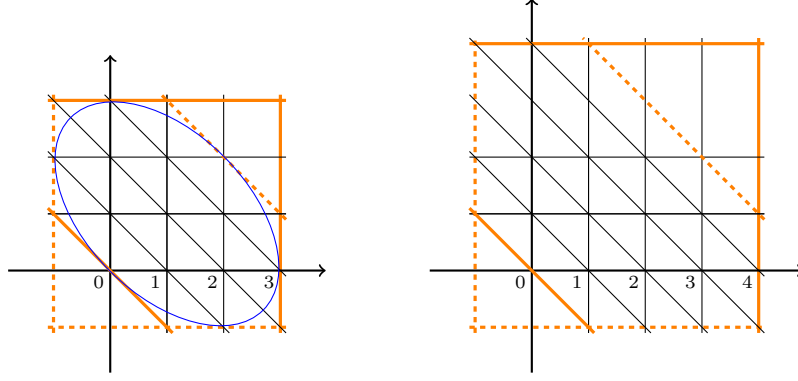
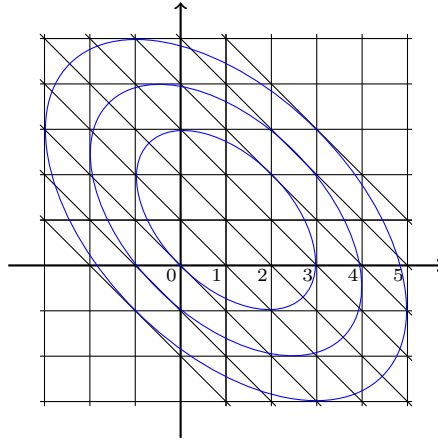Conic of unstable lines for $j = 0$, $k = 3$.   Unstable lines for $j = 0$, $k = 4$.

Theorem 2. *Assume that $k \geq 3$. Then the following six lines are unstable for the vector bundle $T_{\mathcal{A}}$:*

lines of $\mathcal{A}$:	$x = (k+j)z,$	$y = (k+j)z,$	$y + x = -jz,$
lines not of $\mathcal{A}$:	$x = -(j+1)z,$	$y = -(j+1)z,$	$y + x = (k+j+1)z.$

If $k \geq 4$, then there are no other unstable lines for $T_{\mathcal{A}}$. If $k = 3$ these lines are tangent to the smooth conic

$$C_j := \{3j^2z^2 + 12jz^2 - 4x^2 - 4xy - 4y^2 + 12xz + 12yz = 0\},$$

and a line is unstable for $T_{\mathcal{A}}$ if and only it is tangent to C_j.

Conics C_j of unstable lines for $k = 3$, $j = 0, 1, 2$.

Remark 1. When $k = 2$, for any j we will see that $T_{\mathcal{A}} \simeq T_{\mathbb{P}^2}(-6 - 3j)$. Hence all lines are unstable in this case, cf. §1.3.

As a corollary of Theorem 1 and 2, we prove that the shift isomorphism does not hold even for A_2 when $k \geq 3$.

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Corollary 1. *When the root system Φ is of the type A_2 the shift isomorphism (1) holds if and only if $k = 0, 1$ or 2 i.e. it does not hold for $k \geq 3$.*

Let us go back to more general irreducible crystallographic root systems. Let Φ be such a system, set $\mathcal{A} = \mathcal{A}_\Phi^{[-j, k+j]}$ and call *graded Betti numbers* of $T_\mathcal{A}$ the integers $\beta_{i,j}$ appearing in a minimal graded free resolution:

$$0 \to \oplus_i S(-i - d)^{\beta_{d,i}} \to \cdots \to \oplus_i S(-i)^{\beta_{0,i}} \to D_0(\mathcal{A}) \to 0.$$

Our results together with experimental computation with the computer algebra package Macaulay2 ([8]) suggest the following.

Conjecture 1. Let $k \geq 1$, $j \geq 0$, $\mathcal{A} = \mathcal{A}_\Phi^{[-j, k+j]}$ and $\mathcal{B} = \mathcal{A}_\Phi^{[-j-1, k+j+1]}$.
  i) The graded Betti numbers of $T_\mathcal{A}$ and $T_\mathcal{B}(\eta)$ are the same.
 ii) The graded Betti numbers of $T_\mathcal{A}^\vee(-\eta(k + 2j + 1))$ and $T_\mathcal{A}$ are the same.
iii) The projective dimension of $T_\mathcal{A}$ is $\min(m - 1, k - 1)$.
 iv) The sheaf $T_\mathcal{A}$ has a linear resolution if $\Phi = A_m$.

Linear resolution here means that $\beta_{j,i} = 0$ for all j except for a single $i = i_0$. The projective dimension is the greatest j with $\beta_{j,i} \neq 0$ for some i.

Theorem 1 proves parts (i) and (iv) for $\Phi = A_2$, while (ii) and (iii) are true for any root system of rank 2, so the above conjecture holds for $\Phi = A_2$.

Remark 2. Part (ii) of the above conjecture refines the shifted dual isomorphism problem. Indeed, a computer-aided argument shows that the isomorphism (2) does not exist in general, even for $\Phi = A_3$, cf. §4.

The paper is organized as follows. In §1 we introduce several results and definitions which will be used to prove Theorems 1 and 2. In Theorem 3 we propose a new criterion to determine the splitting type of $\mathcal{A}^{[-j, k+j]}$ on a line. In §2 we prove Theorem 1. In §3 we show Theorem 2. In §4 we give some computer-aided determination of jumping lines and unstable hyperplanes.

1. PRELIMINARIES

In this section let us recall several results on arrangements of hyperplanes and the related algebraic geometry.

1.1. Vector bundles on the projective plane.
In this subsection we review some results and definitions on rank two vector bundles over the projective plane. The main reference here is [9]. We write the Chern classes of a sheaf E as integers (c_1, c_2), meaning $c_1(E) = c_1 H$ and $c_2(E) = c_2 H^2$, H being the class of a line and H^2 the class of a point in $\mathbb{P}^2$.

1.1.1. *Splitting type of bundles.* Let E be a rank two vector bundle on $\mathbb{P}^2$. First of all, recall that the restriction of any vector bundle E to any projective line $L \subset \mathbb{P}^2$ splits as a direct sum of line bundles.

If E has rank 2, there are two integers $a_{1,L} \leq a_{2,L}$ such that:

$$E|_L \simeq \mathcal{O}_L(a_{1,L}) \oplus \mathcal{O}_L(a_{2,L}), \qquad \text{with } a_{1,L} + a_{2,L} = c_1(E).$$

We call $(a_{1,L}, a_{2,L})$ the *splitting type* of E with respect to L.

By the semi-continuity theorem, these integers do not change when the line L is chosen a non-empty Zariski open set $U \subset \check{\mathbb{P}}^2$. This means that there exist two integers a_1 and a_2 such that on any line $L \in U$, the restricted

bundle $E|_L = \mathcal{O}_L(a_1) \oplus \mathcal{O}_L(a_2)$. Moreover, $|a_1 - a_2| = \min_{L \in \check{\mathbb{P}}^2}\{|a_{1,L} - a_{2,L}|\}$. The couple (a_1, a_2) is called the *generic splitting type* of E.

1.1.2. *Stability of bundles and jumping lines.* Let E be a vector bundle of rank 2 on $\mathbb{P}^2$. We can tensor E by $\mathcal{O}_{\mathbb{P}^2}(t)$ in such a way that $c_1(E(t)) \in \{-1, 0\}$. We call $E(t)$ the normalized twist of E. Then E is *stable* if and only if $H^0(\mathbb{P}^2, E(t)) = 0$. If $c_1(E(t)) = 0$, and $H^0(\mathbb{P}^2, E(t-1)) = 0$ then E is *semistable* and E is *strictly semistable* if it is semistable, but not stable.

By the Grauert-Mülich theorem, if E is stable (or semi-stable), then the generic splitting type of E verifies $|a_1 - a_2| \leq 1$. This leads to the notion of jumping lines.

Definition 2. A line L such that $E|_L = \mathcal{O}_L(a_1 - k) \oplus \mathcal{O}_L(a_2 + k)$, where $k > 0$ and (a_1, a_2) with $a_1 \leq a_2$ is the generic splitting type of a rank-two semi-stable vector bundle E, is a *jumping line of order k* of E. The scheme of jumping lines lives in $\check{\mathbb{P}}^2$ and it is denoted by $S(E)$. A jumping line of order $k > 1$ is a singular point in $S(E)$.

1.2. **Steiner bundles and unstable lines.** Let $i \geq 2$ and let us consider a Steiner bundle F_i:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{i-2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}^i \longrightarrow F_i \longrightarrow 0.$$

Then one can construct a new Steiner bundle from F_i in two different ways.

1.2.1. *By reduction.* Assume that $i \geq 4$ and that the line L is unstable for F_i. Then the kernel F_{i-1} of the map $F_i \twoheadrightarrow \mathcal{O}_L$ is again a Steiner bundle with resolution:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{i-3} \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{i-1} \longrightarrow F_{i-1} \longrightarrow 0.$$

The proof is done in [11, Proposition 2.1]. Moreover in the same proposition it is proved that $W(F_i) \setminus \{L\} \subset W(F_{i-1})$.

1.2.2. *By extension.* Let $H \subset \mathbb{P}^2$ be any line. A non-trivial extension

$$0 \longrightarrow F_i \longrightarrow F \longrightarrow \mathcal{O}_H \longrightarrow 0$$

is a also a Steiner bundle with resolution:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{i-1} \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{i+1} \longrightarrow F \longrightarrow 0.$$

Indeed, the surjection $\mathcal{O}_{\mathbb{P}^2} \twoheadrightarrow \mathcal{O}_H$ lifts to $\mathcal{O}_{\mathbb{P}^2} \to F$ because $H^1(\mathbb{P}^2, F_i) = 0$. Combining this with the surjection $\mathcal{O}_{\mathbb{P}^2}^i \twoheadrightarrow F_i$ we obtain the required epimorphism $\mathcal{O}_{\mathbb{P}^2}^{i+1} \twoheadrightarrow F$, whose kernel is an extension of $\mathcal{O}_{\mathbb{P}^2}^{i-2}(-1)$ by $\mathcal{O}_{\mathbb{P}^2}(-1)$, and therefore precisely $\mathcal{O}_{\mathbb{P}^2}^{i-1}(-1)$.

Moreover, when $i \geq 3$ the line H is unstable for F and again according to [11, Proposition 2.1] we know that $W(F) \setminus \{H\} \subset W(F_i)$. We will use these two constructions later on.

1.3. **Behaviour of first Steiner bundles.** We give here a quick overview of jumping and unstable lines for Steiner for low i.

$i = 2$. In this case we have $F_2 \simeq \mathcal{O}_{\mathbb{P}^2}^2$.

$i = 3$. It is well-known that $F_3 \simeq T_{\mathbb{P}^2}(-1)$. All lines are unstable and none of them is a jumping line.

$i = 4$. Unstable and jumping lines coincide this time, i.e. $S(F_4) = W(F_4)$. By [4], $W(F_4)$ is a smooth conic in $\check{\mathbb{P}}^2$, and the unstable lines of F_4 are the tangent lines to the dual conic.

$i = 5$. Also this time we have $S(F_5) = W(F_5)$. The scheme $W(F_5)$ is either finite of length 6 or consist of a smooth conic in $\check{\mathbb{P}}^2$, see [4].

$i \geq 6$. Unstable lines do not always exist in this range. When they do, they are jumping lines of maximal order, namely the splitting type on an unstable line H is $(0, i - 2)$. The scheme $W(F_5)$ is either finite of length $\leq i + 1$ or consist of a smooth conic in $\check{\mathbb{P}}^2$, see [4, 11].

1.4. **Line arrangements and vector bundles.** Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ and let be H a line of $\mathcal{A}$. Define $n = |\mathcal{A}|$ and the restricted arrangement of points $\mathcal{A}^H := \{K \cap H \mid K \in \mathcal{A}, \ K \neq H\}$. Set $h := |\mathcal{A}^H|$.

Let $t_{\mathcal{A},H,i}$ be the number of points with multiplicity i on H. The "number of triple points" on H is:

$$t_{\mathcal{A},H} = \sum_{i \geq 3}(i - 2)t_{\mathcal{A},H,i}.$$

We recall the following result, which is often used in the rest of this article.

Proposition 1 ([7], Proposiont 5.1). *There is an exact sequence*

$$0 \to T_{\mathcal{A}} \to T_{\mathcal{A} \setminus \{H\}} \to \mathcal{O}_H(-t_{\mathcal{A},H}) \to 0.$$

Lemma 1. *We have $t_{\mathcal{A},H} = n - 1 - h$.*

Proof. It is clear that $h = n - 1$ when there are only double points on H. When there is a point of multiplicity $i \geq 3$ on H, it is necessary to remove $i - 2$ to $n - 1$ to compute h. More generally, this procedure gives $h = n - 1 - \sum_{i \geq 3}(i - 2)t_{\mathcal{A},H,i}$ which means $t_{\mathcal{A},H} = n - 1 - h$. $\square$

Proposition 1 gives the following criterion to determine the splitting type.

Theorem 3. *Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ with $n := |\mathcal{A}|$. Let $H \in \mathcal{A}$ and L be a line not in $\mathcal{A}$. Define $h := |\mathcal{A}^H|$ and $\ell := |\{L \cap K \mid K \in \mathcal{A}\}|$. Let $d_1 \leq d_2$ and $e_1 \leq e_2$ be integers such that:*

$$T_{\mathcal{A}}|_H \simeq \mathcal{O}_H(-d_1) \oplus \mathcal{O}_H(-d_2) \qquad \text{and} \qquad T_{\mathcal{A}}|_L \simeq \mathcal{O}_L(-e_1) \oplus \mathcal{O}_L(-e_2).$$

Then we have:

i) if $n - h \leq \lceil \frac{n-1}{2} \rceil$, then $(d_1, d_2) = (n - h, h - 1)$,

ii) if $n - \ell \geq \lceil \frac{n}{2} \rceil$, then $(e_1, e_2) = (n - \ell, \ell - 1)$.

Proof. The assertion (i) is known (see [13] for example) but not the assertion (ii). Since we can give the same kind of proof to these two statements by using Proposition 1, we prove both of them.

For (i), by Proposition 1, there is an exact sequence

$$0 \longrightarrow T_{\mathcal{A}} \longrightarrow T_{\mathcal{A} \setminus \{H\}} \longrightarrow \mathcal{O}_H(-t_{\mathcal{A},H}) \longrightarrow 0,$$

where $t_{\mathcal{A},H} = n-1-h$. Take the dual of this sequence and tensor it by $\mathcal{O}_H$ to obtain a surjection

$$T_{\mathcal{A}}^{\vee}|_H \longrightarrow \mathcal{O}_H(t_{\mathcal{A},H}+1) \longrightarrow 0.$$

Since $T_{\mathcal{A}}^{\vee}|_H \simeq \mathcal{O}_H(d_1) \oplus \mathcal{O}_H(d_2)$, the surjection requires $t_{\mathcal{A},H}+1 = n-h = d_1$ or $t_{\mathcal{A},H}+1 = n-h \geq d_2$. If $n-h \leq \lceil\frac{n-1}{2}\rceil$, then $n-h \leq d_2$ and therefore $n-h = d_1$ or $n-h = d_2$. In both cases we get $(d_1, d_2) = (n-h, h-1)$.

Let us now prove (ii). Consider the exact sequence of Proposition 1:

$$0 \longrightarrow T_{\mathcal{B}} \longrightarrow T_{\mathcal{A}} \longrightarrow \mathcal{O}_L(-t_{\mathcal{B},L}) \longrightarrow 0,$$

where $\mathcal{B} := \mathcal{A} \cup \{L\}$ and $t_{\mathcal{B},L} = n-\ell$. Restricting this onto L, we have

$$T_{\mathcal{A}}|_L \simeq \mathcal{O}_L(-e_1) \oplus \mathcal{O}_L(-e_2) \longrightarrow \mathcal{O}_L(-n+\ell) \longrightarrow 0.$$

The surjection requires $n-\ell \leq e_1$ or $n-\ell = e_2$. If $n-\ell \geq \lceil\frac{n}{2}\rceil$, then $n-\ell \geq e_1$ and therefore $n-\ell = e_1$ or $n-\ell = e_2$. In both cases we obtain $(e_1, e_2) = (n-\ell, \ell-1)$.

2. Resolution of the logarithmic bundle

Here we prove Theorem 1. We construct the arrangement $\mathcal{A}$ in three steps, starting from a grid of horizontal and vertical lines, and adding two series of diagonal lines, in such a way that the resolution of the logarithmic bundle remains under control.
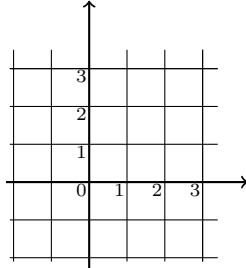
2.1. Starting from the grid. Let us start by defining the lines:

$$X_i = \{x = iz\}, \qquad Y_i = \{y = iz\}, \qquad H_{\infty} = \{z = 0\}.$$

The *grid* arrangement consists of the line at infinity and $k+2j+1$ "parallel" lines X_i and Y_i:

$$\mathcal{A}_0 = H_{\infty} \cup \bigcup_{i=-j}^{k+j} (X_i \cup Y_i) = \{z \prod_{i=-j}^{k+j} (x-iz)(y-iz) = 0\}.$$
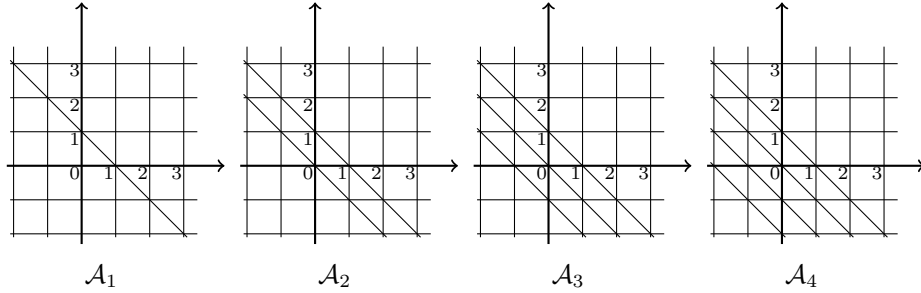
It is well-known (or we may apply [7, Proposition 3.3] to show) that $\mathcal{A}_0$ is free and we have $T_{\mathcal{A}_0} \simeq \mathcal{O}_{\mathbb{P}^2}(-k-2j-1)^2$.

Grid arrangement $\mathcal{A}_0$ for $j = 2$, $k = 1$.

We will now add diagonal lines. For $i = -j, \ldots, k+j+1$ we let:

$$H_i := \{y + x = (k-i+1)z\}.$$

Nested arrangements $\mathcal{A}_i$ for $j = 2$, $k = 1$.

2.2. Adding diagonal inner lines.

Now we add the diagonal inner lines, namely the lines H_i of $\mathcal{A}$, for $i = 1, \ldots, k+j+1$. In other words, starting from the grid, we add diagonal lines lying k integral steps above the origin and proceed downwards, in such a way that the total number of triple points along each new line is constant. Indeed, the decreasing number of affine triple points on H_i is compensated by the increasing multiplicity at infinity.

Now define, the nested arrangements $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_{k+j+1}$ as follows:

$$\mathcal{A}_i = \mathcal{A}_0 \cup H_1 \cup \cdots \cup H_i.$$

For $i = 1, \ldots, k + j + 1$, we compute

$$|\mathcal{A}_i| - 1 - |(\mathcal{A}_i \cup \{H_{i+1}\})^{H_{i+1}}| = k + 2j + 1.$$

In other words, $t_{\mathcal{A}_i, H_{i+1}} = k + 2j + 1$ by Lemma 1. Hence by Proposition 1 there is an exact sequence:

(3) $$0 \longrightarrow T_{\mathcal{A}_{i+1}} \longrightarrow T_{\mathcal{A}_i} \longrightarrow \mathcal{O}_{H_{i+1}}(-k - 2j - 1) \longrightarrow 0.$$

For $i = 1, \ldots, k + j + 1$, let:

$$F_i := T_{\mathcal{A}_i}(k + 2j + i).$$

Then $c_1(F_i) = i - 2$ and it is easy to check that:

$$F_1 \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1), \qquad F_2 \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \qquad \text{and} \qquad F_3 \simeq T_{\mathbb{P}^2}(-1).$$

The sequence of F_i constructed in this way corresponds to the "extension step" explained in §1.2. Indeed, using that $E^\vee \simeq E(-c_1(E))$ for a rank-2 bundle E, the dual exact sequence of (3) gives the following extension for $2 \le i \le k + j$:

$$0 \longrightarrow F_i \longrightarrow F_{i+1} \longrightarrow \mathcal{O}_{H_{i+1}} \longrightarrow 0.$$

Then, for $2 \le i \le k + j$, by §1.2.2 the line H_{i+1} is unstable for F_{i+1}, and F_{i+1} is a Steiner bundle with resolution:
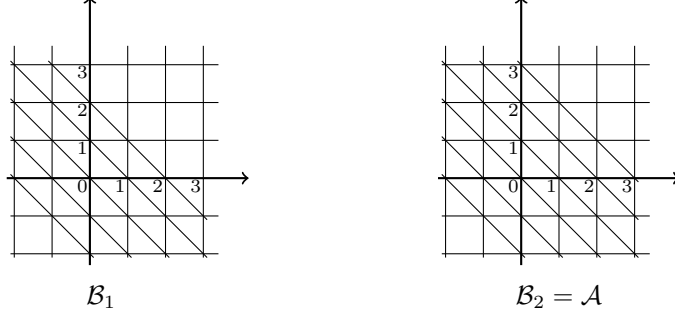
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{i-1} \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{i+1} \longrightarrow F_{i+1} \longrightarrow 0.$$

2.3. Diagonal outer lines.

Now we add the remaining diagonal lines of $\mathcal{A}$, which we call "outer". We start from H_0 (i.e., the line lying right above H_1) and go upwards, i.e. we add $H_{-1}, \ldots, H_{1-j}$. In other words, for $i = 1, \ldots, j$, we define the nested arrangements:

$$\mathcal{B}_i = \mathcal{A}_{k+j+1} \cup H_0 \cup \cdots \cup H_{1-i}.$$

Therefore $\mathcal{A} = \mathcal{B}_j$. We fix the following notation:

$$E_0 = F_{k+j+1} = T_{\mathcal{A}_{k+j+1}}(2k+3j+1) \qquad \text{and} \qquad E_i = T_{\mathcal{B}_i}(2k+3j+1).$$

$\mathcal{B}_1$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\mathcal{B}_2 = \mathcal{A}$

Arrangements $\mathcal{B}_i$ for $j = 2$, $k = 1$.

For $i = 1, \ldots, j$, it is easy to compute $t_{\mathcal{B}_i, H_{1-i}} = 2k + 3j + 1$. By Theorem 3, we have thus the exact sequences:

$$0 \longrightarrow E_i \longrightarrow E_{i-1} \longrightarrow \mathcal{O}_{H_{1-i}} \longrightarrow 0.$$

The line H_{1-i} is then unstable for E_{i-1}. So the "reduction step" recalled in §1.2.1, implies that E_i is a Steiner bundle, for $i = 1, \ldots, j$, with resolution of the form:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{k+j-1-i}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{k+j+1-i} \longrightarrow E_i \longrightarrow 0.$$

The case $i = j$ completes the proof of Theorem 1. $\qquad\qquad\qquad$ $\square$

Remark 3. If we extend the definition of H_{1-i} and $\mathcal{B}_i$ to the range $i = j+1, \ldots, k+j-2$, the equality $t_{\mathcal{B}_i, H_{1-i}} = 2k + 3j + 1$ remains valid. Applying repeatedly the reduction step §1.2.1, we get $E_{k+j-2} \simeq T_{\mathbb{P}^2}(-1)$. Also, $W(E_{k+j-3})$ is a smooth conic $C \subset \check{\mathbb{P}}^2$. Also, we will see that E_{k+j-4} has exactly six jumping lines, and that only five of them lie on C. We will use these supplementary bundles in the next section.

3. Unstable lines of the logarithmic bundle

Let us fix again the notation as in the introduction:

$$\mathcal{A} = \mathcal{A}_{A_2}^{[-j,k+j]}.$$

We have just verified that $T_{\mathcal{A}}$ is a Steiner bundle. We can now investigate its splitting type in more detail. The following is known, see [12] for example.

Proposition 2. *The splitting type of $\mathcal{A}$ at infinity is:*

$$T_{\mathcal{A}}(2k+3j+1)|_{H_\infty} \simeq \mathcal{O}_{H_\infty}(\lceil \tfrac{k-1}{2} \rceil) \oplus \mathcal{O}_{H_\infty}(\lfloor \tfrac{k+1}{2} \rfloor).$$

In other words, the splitting type of $T_{\mathcal{A}}$ onto H_∞ is generic. Now let us look for jumping lines of $T_{\mathcal{A}}$. Recall that $c_1(T_{\mathcal{A}}(2k+1+3j)) = k-1$. Then a line $L \subset \mathbb{P}^2$ is a jumping line of $T_{\mathcal{A}}$ if and only if:

$$H^0(\mathbb{P}^2, T_{\mathcal{A}}(2k+3j+1-s)|_L) \neq 0, \qquad \text{for } 2s \geq k+1,$$

or equivalently, by Serre duality, if and only if:

$$H^1(\mathbb{P}^2, T_{\mathcal{A}}(k + 3j + s)|_L) \neq 0, \qquad \text{for } 2s \geq k + 1.$$

The jumping lines are unstable lines when $s = k - 1$.

Proposition 3. *Let $0 \leq 2s \leq k - 1$ and let H be one of the following lines:*

lines of $\mathcal{A}$:	$X_{k+j-s},$	$Y_{k+j-s},$	$H_{k+j+1-s},$
lines outside $\mathcal{A}$:	$X_{-(j+s+1)},$	$Y_{-(j+s+1)},$	$H_{-(j+s)}.$

Then $T_{\mathcal{A}}(2k + 3j + 1)|_H \simeq \mathcal{O}_H(k - 1 - s) \oplus \mathcal{O}_H(s).$

Proof. Let us prove the statement for the lines of $\mathcal{A}$. It is easy to check that $t_{\mathcal{A},H} = k + 3j + 1 + s$. Hence applying Theorem 3, part (i), we know that:

$$(4) \qquad T_{\mathcal{A}}|_H \simeq \mathcal{O}_H(-2k - 3j - 1 + s) \oplus \mathcal{O}_H(-k - 3j - 2 - s).$$

Let us check the lines outside $\mathcal{A}$. It is easy to see that $t_{\mathcal{A} \cup \{H\}, H} = 2k + 3j + 1 - s$. Hence applying Theorem 3, part (ii), we get again (4). $\square$

Corollary 2. *Let $\kappa \geq 0$ and $s \geq 0$ be integers. Then the set of jumping lines of order $\kappa + s - j$ of $T_{\mathcal{A}}$ contains:*

$X_{2\kappa+s},$	$Y_{2\kappa+s},$	$H_{k+s+1},$
$X_{(-2j+s+1)},$	$Y_{(-2j+s+1)},$	$H_{k+s-2\kappa-2j}.$

In particular, the following lines are unstable for $T_{\mathcal{A}}$:

$X_{k+j},$	$Y_{k+j},$	$H_{k+j+1},$
$X_{-(j+1)},$	$Y_{-(j+1)},$	$H_{-j}.$

Remark 4. These lines do not cover the whole set of jumping lines of any given order, even when the first Chern class is odd (in which case some of these sets are presumably finite). These lines are determined by the arrangement, but it seems very difficult to determine all of them. We comment on this shortly in one example, see §4.

Proof of Theorem 2. In order to prove Theorem 2, we have to show that the six lines appearing in Corollary 2 are tangent to C_j for $k = 3$ and that these six lines form the set of all unstable lines of $T_{\mathcal{A}}$ for $k \geq 4$. The first fact is clear from the picture. To prove the second one, we will use the step-by-step construction like in the proof of Theorem 1 whereby keeping the same notation.

Recall that H_{i+1} is an unstable line for the Steiner bundle F_{i+1}. Now computing the number of triple points and using Theorem 3 we can see that the following 5 lines are also unstable for F_{i+1}:

$$(5) \qquad \begin{array}{ccc} X_{k+j}, & Y_{k+j}, & H_0, \\ X_{-(j+1)}, & Y_{-(j+1)}. & \end{array}$$

Let us call $C_{k,j}$ the conic tangent to these last 5 lines. The line H_4 is tangent to $C_{k,j}$, indeed $W(F_4)$ consist of all lines tangent to $C_{k,j}$. Since H_5 is parallel both to H_4 and H_0, it is not tangent to $C_{k,j}$. Then, $W(F_5)$ consists of the 5 lines of (5), plus H_5. In other words:

$$W(F_5) \setminus \{H_5\} \subset W(F_{i+1}).$$

Applying the procedure of §1.2.2 repeatedly, we see that:

$$W(F_{i+1}) \setminus \{H_{i+1}\} \subset W(F_i),$$

for $i = 5, \ldots, k + j$. Using Proposition 3, we see that the line H_i is not unstable for F_{i+1}. Then we have more precisely, for $i = 5, \ldots, k + j$,

$$W(F_{i+1}) \setminus \{H_{i+1}\} \subset W(F_i) \setminus \{H_i\}.$$

The line H_{k+j+1} is of course not unstable for F_5. Then putting all these inclusions together, we find:

$$W(F_5) \setminus \{H_5\} \subset W(F_{k+j+1}) \setminus \{H_{k+j+1}\} \subset W(F_5) \setminus \{H_5\}.$$

We have thus proved:

$$W(F_{k+j+1}) = \{H_{k+j+1}\} \cup W(F_5) \setminus \{H_5\}.$$

Now we consider the "outer" construction. Computing the number of triple points and using Theorem 3 we can see that H_{-i} and the following 5 lines are unstable for $E_i = T_{\mathcal{B}_i}(2k + 3j + 1)$:

(6)
$$\begin{array}{ccc} X_{k+j}, & Y_{k+j}, & H_{k+j+1}, \\ X_{-(j+1)}, & Y_{-(j+1)}. & \end{array}$$

This happens for $i = 1, \ldots, j$, but also for $i = j + 1, \ldots, k + j - 2$, as it was observed in Remark 3. Let us call $\Gamma_{k,j}$ the conic tangent to these last 5 lines. The line H_{-k-j+3} is tangent to $\Gamma_{k,j}$, as $W(E_{k+j-3})$ consists of all lines tangent to $\Gamma_{k,j}$. Since H_{-k-j+4} is parallel both to H_{-k-j+3} and H_0, it is not tangent to $\Gamma_{k,j}$. Therefore, $W(E_{k+j-4})$ consists of the 5 lines of (6), and of the extra line H_{-k-j+4}.

In other words:

$$W(E_{k+j-4}) \setminus \{H_{-k-j+4}\} \subset W(E_j).$$

Applying the procedure of §1.2.2 repeatedly, we see that:

$$W(E_i) \setminus \{H_{-i}\} \subset W(E_{i+1}),$$

for $i = 1, \ldots, k + j - 4$. Using Proposition 3, we see that the line H_{-i-1} is not unstable for E_i. Then we have more precisely, for $i = 1, \ldots, k + j - 4$,

$$W(E_i) \setminus \{H_{-i}\} \subset W(E_{i+1}) \setminus \{H_{-i-1}\}$$

The line H_{-j} is of course not unstable for E_{k+j-4}. Then putting all these inclusions together, we find:

$$W(E_{k+j-4}) \setminus \{H_{-k-j+4}\} \subset W(E_j) \setminus \{H_{-j}\} \subset W(E_{k+j-4}) \setminus \{H_{-k-j+4}\}.$$

This proves:

$$W(E_j) = \{H_{-j}\} \cup W(E_{k+j-4}) \setminus \{H_{-k-j+4}\}.$$

The proof of the theorem is now complete. □

Proof of Corollary 1. When $k = 0, 1, 2$, the isomorphism (1) is shown in [14] and [1]. Assume that $k \geq 3$. Then Theorem 2 shows that the unstable lines vary when j varies, which completes the proof. □

4. Experimental results

We outline here some experimental results concerning logarithmic bundles associated with root systems of type A_m. First, we give a first estimate on the behaviour of the set of jumping lines for $m = 2$. Then, we provide a counter-example to the dual shift isomorphism in case $m = 3$.

4.1. Jumping lines.

An interesting question is to determine the set of all jumping lines of $T_{\mathcal{A}}$ when $\mathcal{A} = \mathcal{A}_{A_2}^{[-j,k+j]}$ and not only the unstable ones. The first open case is $j = 0$ and $k = 6$, so that the arrangement $\mathcal{A}$ consists of 22 lines, and $T_{\mathcal{A}}$ has the resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^5 \to \mathcal{O}_{\mathbb{P}^2}^7 \to T_{\mathcal{A}}(13) \to 0.$$

The set $S(T_{\mathcal{A}})$ of jumping lines is the support in $\check{\mathbb{P}}^2$ of the cokernel of a map:

$$(\mathrm{Sym}^2(\Omega_{\check{\mathbb{P}}^2}(1)))^5 \to \Omega_{\check{\mathbb{P}}^2}(1)^7.$$

Combining with the Euler sequence and taking syzygies, $S(T_{\mathcal{A}})$ is seen to be the locus cut by the 8×8 minors of a matrix of the form:

$$\mathcal{O}_{\check{\mathbb{P}}^2}^9(-1) \to \mathcal{O}_{\check{\mathbb{P}}^2}^8.$$

We may determine this matrix (defined over $\mathbb{Q}$) explicitly with Macaulay2, and find that that $S(T_{\mathcal{A}})$ has length 36. It contains 6 triple points, one for each unstable line, and 6 smooth points corresponding to the 6 jumping lines of Corollary 2. Moreover it contains 12 (smooth) points with irrational coordinates which are a complete intersection of a quartic and a cubic. These curves both have rational (or integral) coefficients, and can also be determined explicitly. For instance the cubic is:

$$62x^3 - 90x^2y - 90xy^2 + 62y^3 + x^2z - 109xyz + y^2z - 18xz^2 - 18yz^2 - 3z^3.$$

4.2. Failure of dual shift isomorphism.

In $\mathbb{P}^m$ with $m \geq 3$ the notion of unstable line for a Steiner sheaf E lying in:

$$0 \to \mathcal{O}_{\mathbb{P}^m}(-1)^\ell \to \mathcal{O}_{\mathbb{P}^m}^{\ell+m} \to E \to 0$$

is replaced by the idea of *unstable hyperplane*, namely H is so if:

$$H^{m-1}(H, E|_H(-m)) \neq 0.$$

Let $\mathcal{A} = \mathcal{A}_{A_3}^{[0,2]}$. With Macaulay2 we get the resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-8)^3 \to \mathcal{O}_{\mathbb{P}^3}(-7)^6 \to T_{\mathcal{A}} \to 0,$$

and check that $T' = T_{\mathcal{A}}^\vee(-12)$ has the same Betti numbers of $T_{\mathcal{A}}$, i.e. 6 generators of degree 7 and 3 linear syzygies. This confirms Conjecture 1.

The sets of unstable planes of $E = T_{\mathcal{A}}(7)$ and $E' = T'(7)$ are computed as loci cut by maximal minors, respectively, of matrices N and N':

$$N, N' : \mathcal{O}_{\check{\mathbb{P}}^3}(-1)^{18} \to \mathcal{O}_{\check{\mathbb{P}}^3}^6.$$

Macaulay2 here says that, although both N and N' have a cokernel of length 7 consisting of 7 smooth points (so that E and E' both have 7 unstable distinct planes), the underlying sets of planes are different, actually only 4 planes are common to these sets. So E and E' cannot be isomorphic. This shows that the dual shift isomorphism (2) does not take place in general.

Note that, for instance for $\Phi = A_4$, one can choose k and j in such a way that $T_{\mathcal{A}}$ and $T_{\mathcal{A}}^{\vee}(-\eta(k + 2j + 1))$ do not even have the same number of unstable hyperplanes.

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